Contact 3-manifolds with Reeb flow invariant characteristic Jacobi operator

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Abstract We study contact Riemannian 3-manifolds whose characteristic Jacobi operator is invariant under Reeb flows.

Keywords Contact manifolds · characteristic Jacobi operator · 3-dimensional Lie groups

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Introduction

In a contact manifold $(M, \eta)$, the Reeb vector field $\xi$ of $\eta$ generates a contact transformation, since $\mathcal{L}_\xi \eta = 0$. Here $\mathcal{L}_\xi$ is the Lie differentiation by $\xi$. An associated Riemannian metric $g$ to $(M, \eta)$ is said to be a $K$-contact metric if the Reeb flow is isometric with respect to it, that is, $\xi$ is a Killing vector field with respect to $g$. A contact manifold $(M, \eta)$ equipped with a $K$-contact metric is called a $K$-contact manifold.

Sasakian manifolds are defined as contact Riemannian manifolds which satisfy a normality condition. One can see that Sasakian manifolds are $K$-contact, but converse does not hold.

In case $\dim M = 3$, $K$-contact manifolds are Sasakian. In 3-dimensional contact Riemannian geometry, it is known that the local symmetry (parallelism of the Riemannian curvature $\mathcal{R}$, i.e., $\nabla \mathcal{R} = 0$) is a strong restriction for associated metrics. In fact, Blair and Sharma showed that locally symmetric contact Riemannian 3-manifolds are of constant curvature 1 or 0 [5]. Note that local symmetry is equivalent to the
parallelism of the Ricci operator \( S \). Thus such a contact Riemannian 3-manifold is locally isomorphic to the unit 3-sphere \( S^3 \) or the universal covering of the Euclidean motion group \( E(2) \). It should be remarked that \( E(2) \) is identified with the unit tangent sphere bundle \( T_1 \mathbb{E}^2 \) of the Euclidean plane \( \mathbb{E}^2 \).

In the previous paper [8], the first named author investigated a milder condition on the Ricci operator \( S \). He classified contact Riemannian 3-manifold whose Ricci operator \( S \) is invariant under the Reeb flow.

**Theorem 0.1** ([8]) Let \( M \) be a contact Riemannian 3-manifold. Then \( M \) satisfies \( \mathcal{L}_\xi S = 0 \) if and only if \( M \) is Sasakian or locally isomorphic to \( SU(2) \), \( SL_2 \mathbb{R} \) or \( E(2) \) equipped with a left invariant contact Riemannian structure.

On the other hand, in contact Riemannian geometry the self-adjoint operator \( \ell = R(\cdot, \xi)\xi \) plays an important role. In this paper we study Reeb flow invariance of the characteristic Jacobi operator \( \ell \) on contact Riemannian 3-manifolds. We will show that the Reeb flow invariance \( \mathcal{L}_\xi \ell = 0 \) of \( \ell \) is weaker than \( \mathcal{L}_\xi S = 0 \). The main result of this paper is the following theorem (Theorem 4.8):

**Main Theorem** Let \( M \) be a contact Riemannian 3-manifold. Then \( M \) satisfies \( S\xi = \sigma\xi \) for some function \( \sigma \) and \( \mathcal{L}_\xi \ell = 0 \) if and only if \( M \) is Sasakian or locally isomorphic to \( SU(2) \), \( SL_2 \mathbb{R} \) or \( E(2) \) equipped with a left invariant contact Riemannian structure.

In the main theorem, the condition \( S\xi = \sigma\xi \) cannot be removed. In fact, there exist non-unimodular Lie groups equipped with left invariant contact Riemannian structure which satisfy \( \mathcal{L}_\xi \ell = 0 \) but not \( S\xi = \sigma\xi \).

Throughout this paper, all manifolds are assumed to be smooth and connected.

**1 Preliminaries**

1.1. Let \((M, g)\) be a Riemannian manifold with Riemannian curvature

\[
R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - [X, Y], \quad X, Y \in \mathfrak{X}(M),
\]

where \( \mathfrak{X}(M) \) denotes the Lie algebra of all vector fields on \( M \). Then \((M, g)\) is said to be pseudo-symmetric if there exists a function \( L \) such that

\[
R(X, Y) \cdot R = L(X \wedge Y) \cdot R
\]

holds for all \( X \) and \( Y \in \mathfrak{X}(M) \). The curvature-like tensor field \( (X \wedge Y) \) is defined by

\[
(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y.
\]

In particular, a pseudo-symmetric Riemannian manifold is called a pseudo-symmetric space of constant type if \( L \) is constant.

On a Riemannian 3-manifold \((M, g)\), the Riemannian curvature \( R \) is described by the Ricci tensor field \( Ric \) and corresponding Ricci operator \( S \) by

\[
R(X, Y)Z = Ric(Y, Z)X - Ric(Z, X)Y + g(Y, Z)SX - g(Z, X)SY - \frac{\tau}{2}(X \wedge Y)Z
\]

for all vector fields \( X, Y \) and \( Z \) on \( M \). Here \( \tau = \text{tr} S \) is the scalar curvature. From this, the following characterizations of pseudo-symmetry is deduced.
**Proposition 1.1** A Riemannian 3-manifold is a pseudo-symmetric space of constant type with $R(X, Y) \cdot R = L(X \wedge Y) \cdot R$ if and only if the principal Ricci curvatures (eigenvalues of the Ricci tensor) locally satisfy the following relations (up to numeration):

$$\rho_1 = \rho_2, \quad \rho_3 = 2L.$$

**Proposition 1.2** A Riemannian 3-manifold $(M, g)$ is pseudo-symmetric if and only if it is quasi-Einstein. This means that there exists a one-form $\omega$ such that the Ricci tensor field $\text{Ric}$ has the form:

$$\text{Ric} = a g + b \omega \otimes \omega.$$

Here $a$ and $b$ are functions.

## 2 Contact Riemannian 3-manifolds

2.1. A 3-dimensional manifold $M$ is said to be a contact manifold if it admits a 1-form $\eta$ satisfying $\eta \wedge (d\eta) \neq 0$. Such a 1-form is called a contact form. On a contact 3-manifold $(M, \eta)$, there is a unique vector field $\xi$ such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. The vector field $\xi$ is called the characteristic vector field of $(M, \eta)$. The characteristic vector field $\xi$ is also called the Reeb vector field of $(M, \eta)$. Moreover $(M, \eta)$ admits a Riemannian metric $g$, an endomorphism field $\varphi$ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 = -I + \eta \otimes \xi, \quad (2.1)$$

for all vector fields $X$ and $Y$ on $M$. From (2.1) it follows that

$$\varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M). \quad (2.2)$$

A contact 3-manifold $(M, \eta)$ equipped with the structure tensor $(\varphi, \xi, g)$ satisfying (2.2) is called a contact Riemannian 3-manifold and denoted by $M = (M, \varphi, \xi, \eta, g)$. 

2.2. Given a contact Riemannian 3-manifold $M$, following Blair [1], we define an endomorphism field $h$ on $M$ by $h = \mathcal{L}_\xi \varphi/2$. Then we observe that $h$ is self-adjoint with respect to $g$ and satisfies

$$h\xi = 0, \quad h\varphi = -\varphi h, \quad (2.3)$$

$$\nabla_X \xi = -\varphi X - \varphi hX, \quad X \in \mathfrak{X}(M). \quad (2.4)$$

Here $\nabla$ is the Levi-Civita connection of $g$. From (2.3)–(2.4), we see that each trajectory of $\xi$ is a geodesic. Along trajectories of $\xi$, the characteristic Jacobi operator $\ell$ defined by $\ell(X) = R(X, \xi)\xi$ is a self-adjoint endomorphism field, that is, $g(\ell X, Y) = g(X, \ell Y)$. We have

$$\text{trace } \ell = \text{Ric } (\xi, \xi) = 2 - \text{trace } (h^2), \quad (2.5)$$

$$\nabla_\xi h = \varphi - \varphi \ell - \varphi h^2, \quad (2.6)$$

$$g(R(X, Y)\xi, Z) = g((\nabla_Z \varphi)X, Y) + g((\nabla_Y \varphi)X - (\nabla_X \varphi)hY, Z) \quad (2.7)$$

for all $X, Y, Z \in \mathfrak{X}(M)$. 
2.3. A contact Riemannian 3-manifold is said to be $K$-contact if $\xi$ is a Killing vector field. It is easy to see that a contact Riemannian 3-manifold is $K$-contact if and only if $h = 0$.

On the other hand, a contact Riemannian 3-manifold $M$ is said to be a Sasakian 3-manifold if it is normal. Here a contact Riemannian 3-manifold $M$ is said to be normal if it satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$. A Sasakian manifold is characterized by a condition

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in \mathfrak{X}(M). \tag{2.8}$$

It is also known that a contact Riemannian 3-manifold is Sasakian if and only if it satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad X, Y \in \mathfrak{X}(M). \tag{2.9}$$

2.4. It was proved in [22] that a contact Riemannian 3-manifold always satisfies

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{2.10}$$

Comparing (2.8) and (2.10), one can see that a contact Riemannian 3-manifold is Sasakian if and only if $M$ is $K$-contact.

From (2.7) and (2.10) we have

$$R(X, Y)\xi = \eta(Y)(X + hX) - \eta(X)(Y + hY) + \varphi((\nabla_Y h)X - (\nabla_X h)Y) \tag{2.11}$$

for all $X, Y \in \mathfrak{X}(M)$.

2.5. The curvature characterization (2.9) of Sasakian manifolds motivates the following definition.

**Definition 2.1** ([17]) A contact Riemannian 3-manifold $M$ is said to be a contact generalized $(\kappa, \mu, \nu)$-space if its Riemannian curvature $R$ satisfies

$$R(X, Y)\xi = (\kappa I + \mu h + \nu \varphi h)\{\eta(Y)X - \eta(X)Y\} \tag{2.12}$$

for all $X, Y \in \mathfrak{X}(M)$. Here $\kappa$, $\mu$, and $\nu$ are smooth functions. When $\kappa$, $\mu$ and $\nu$ are constants, then a contact generalized $(\kappa, \mu, \nu)$-space is called a contact $(\kappa, \mu, \nu)$-space. A contact $(\kappa, 0)$-space is called a generalized $(\kappa, \mu)$-space.

**Definition 2.2** ([3]) Let $M$ be a contact generalized $(\kappa, \mu)$-space. If both the functions $\kappa$ and $\mu$ are constants, then $M$ is called a contact $(\kappa, \mu)$-space. A contact generalized $(\kappa, \mu)$-space is said to be proper if $|d\kappa|^2 + |d\mu|^2 \neq 0$.

One can see that Sasakian manifolds are contact $(\kappa, \mu)$-spaces with $\kappa = 1$ and $h = 0$.

2.6. To close this section, here we prove the following fundamental fact.

**Lemma 2.3** A contact Riemannian 3-manifold $M$ is Sasakian if and only if $\ell = -\varphi^2$.

**Proof.** Assume that $M$ is Sasakian, then from (2.11), we get $\ell = -\varphi^2$. Conversely, if we assume that $\ell = -\varphi^2$ then from (2.5), trace $(h^2) = 0$. Since $h$ is self-adjoint with respect to $g$, $h$ vanishes identically on $M$, that is, $M$ is Sasakian. \qed

**Remark 2.1** In [4], it is shown that every contact Riemannian 3-manifold satisfying $S\varphi = \varphi S$ has the characteristic Jacobi operator of the form $\ell = -\varphi^2 - h^2$. Moreover such a space satisfies $\nabla_\ell h = 0$.

For more details on contact Riemannian geometry, we refer to [1].
3 Contact 3-manifolds with Reeb-flow invariance

Here we recall the following lemma.

**Lemma 3.1 ([16],[7])** Let $M$ be a contact Riemannian 3-manifold. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ such that:

$$he_1 = \lambda e_1, \ e_2 = \varphi e_1, \ e_3 = \xi.$$  

With respect to $\mathcal{E}$, the Levi-Civita connection $\nabla$ is given by

\[
\begin{align*}
\nabla_{e_1}e_1 &= be_2, & \nabla_{e_1}e_2 &= -be_1 + (1 + \lambda)\xi, & \nabla_{e_1}\xi &= -(1 + \lambda)e_2, \\
\nabla_{e_2}e_1 &= -ce_2 + (\lambda - 1)\xi, & \nabla_{e_2}e_2 &= ce_1, & \nabla_{e_2}\xi &= (1 - \lambda)e_1, \\
\nabla_{\xi}e_1 &= \alpha e_2, & \nabla_{\xi}e_2 &= -\alpha e_1, & \nabla_{\xi}\xi &= 0.
\end{align*}
\]

The Ricci operator $S$ is given by

\[
\begin{align*}
Se_1 &= \rho_{11} e_1 + \xi(\lambda)e_2 + (2b\lambda - e_2(\lambda))\xi, \\
Se_2 &= \xi(\lambda)e_1 + \rho_{22} e_2 + (2c\lambda - e_1(\lambda))\xi, \\
S\xi &= (2b\lambda - e_2(\lambda))e_1 + (2c\lambda - e_1(\lambda))e_2 + 2(1 - \lambda^2)\xi,
\end{align*}
\]

where

\[
\rho_{11} = \frac{\tau}{2} + \lambda^2 - 2\alpha\lambda - 1, \quad \rho_{22} = \frac{\tau}{2} + \lambda^2 + 2\alpha\lambda - 1.
\]

Here $\tau$ is the scalar curvature of $M$.

From Lemma 3.1, one can deduce the following results.

**Proposition 3.2** (cf. [12, Proposition 5.1], [14, Proposition 2.4]) On a contact Riemannian 3-manifold with local orthonormal frame field $\mathcal{E}$ as in Lemma 3.1, $S\varphi = \varphi S$ holds if and only if

\[
2b\lambda - e_2(\lambda) = 0, \quad 2c\lambda - e_1(\lambda) = 0, \quad \xi(\lambda) = 0, \quad \rho_{11} = \rho_{22}.
\]

**Proposition 3.3** (cf.[12, Proposition 5.2]) Let $M$ be a contact Riemannian 3-manifold with local orthonormal frame field $\mathcal{E}$ as in Lemma 3.1. Then $\rho_{11} = \rho_{22}$ if and only if $\alpha = 0$ or $M$ is Sasakian.

Hence we obtain

**Corollary 3.4** Let $M$ be a contact Riemannian 3-manifold with local orthonormal frame field $\mathcal{E}$ as in Lemma 3.1. Then $S\varphi = \varphi S$ holds if and only if $M$ is Sasakian or

\[
\alpha = 0, \quad 2b\lambda - e_2(\lambda) = 0, \quad 2c\lambda - e_1(\lambda) = 0, \quad \xi(\lambda) = 0.
\]

In particular when $b = c = 0$, $S\varphi = \varphi S$ holds if and only if $\alpha = 0$ and $\lambda$ is constant.
4 Proof of the main theorem

4.1 The condition $S\xi = \sigma\xi$

First we recall the notion of an $H$-contact manifold.

Let $(M, \varphi, \xi, \eta, g)$ be a contact Riemannian 3-manifold with unit tangent sphere bundle $T^1M$. Denote by $X_1(M)$ the space of all sections of $T^1M$. Then $M$ is said to be an $H$-contact manifold if its Reeb vector field $\xi$ is a critical point of the energy functional restricted to $X_1(M)$.

In [21], Perrone proved the following fact.

**Theorem 4.1 ([21])** A contact Riemannian 3-manifold $M$ is $H$-contact if and only if $\xi$ is an eigenvector field of $S$.

Moreover we know the following two facts.

**Theorem 4.2 ([17])** Let $M$ be a contact Riemannian 3-manifold. If $M$ is a contact generalized $(\kappa, \mu, \nu)$-space then $M$ is an $H$-contact manifold. Conversely, if $M$ is $H$-contact, then $M$ satisfies the generalized $(\kappa, \mu, \nu)$-condition on an open dense subset of $M$.

**Proposition 4.3 ([17])** Let $M$ be a non-Sasakian 3-dimensional contact generalized $(\kappa, \mu, \nu)$-space. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3 = \xi\}$ as in Lemma 3.1 such that

$$he_1 = \lambda e_1, \quad he_2 = -\lambda e_2, \quad e_2 = \varphi e_1,$$

where $\lambda = \sqrt{1 - \kappa}$. The Ricci operator $S$ is given by

$$S = AI + B\eta \otimes \xi + \mu h + \nu \varphi h$$

with

$$A = \frac{1}{2}(\tau - 2\kappa) = \frac{\tau}{2} - (1 - \lambda^2),$$

$$B = \frac{1}{2}(6\kappa - \tau) = -\frac{\tau}{2} + 3(1 - \lambda^2),$$

$$\mu = -2\alpha, \quad \nu = \frac{\xi(\lambda)}{\lambda}.$$

More specifically we have

$$Se_1 = \frac{1}{2}(\tau - 2\kappa + 2\mu \sqrt{1 - \kappa})e_1 + \nu \sqrt{1 - \kappa}e_2,$$

$$Se_2 = \nu \sqrt{1 - \kappa}e_1 + \frac{1}{2}(\tau - 2\kappa - 2\mu \sqrt{1 - \kappa})e_2,$$

$$Se_3 = 2\kappa e_3.$$

4.2. Now we start to prove our main theorem. Since we know that Sasakian 3-manifolds satisfy $S\xi = \sigma\xi$ and $\mathcal{L}_\xi \ell = 0$, it suffices to consider non-Sasakian contact Riemannian 3-manifolds.
Let $M$ be a non-Sasakian contact Riemannian 3-manifold satisfying $S \xi = \sigma \xi$ for some function $\sigma$. Then as we have seen before, $M$ is locally a contact generalized $(\kappa, \mu, \nu)$-space with $\sigma = 2\kappa$. In this case $\ell$ is computed as

$$\ell(X) = R(X, \xi)\xi = (\kappa I + \mu h + \nu \varphi h)(X - \eta(X)\xi).$$

$$= (1 - \lambda^2)(X - \eta(X)\xi) - 2\alpha h X + \frac{1}{\lambda} \xi(\lambda) \varphi h X.$$

Hence we get

$$\ell(e_1) = (1 - \lambda^2 - 2\alpha \lambda) e_1 + \xi(\lambda) e_2, \quad \ell(e_2) = \xi(\lambda) e_1 + (1 - \lambda^2 + 2\alpha \lambda) e_2.$$

On the other hand, by using the definition of Ricci operator and the formula $S \xi = \sigma \xi$ we have

$$\ell(X) = \rho(\xi, \xi) X - \rho(\xi, X)\xi + SX - \eta(X) S\xi - \frac{\tau}{2}(X - \eta(X)\xi)$$

$$= \sigma X - \sigma \eta(X)\xi + SX - \sigma \eta(X)\xi - \frac{\tau}{2}(X - \eta(X)\xi)$$

$$= SX + \left(\sigma - \frac{\tau}{2}\right) X - \left(2\sigma - \frac{\tau}{2}\right) \eta(X)\xi.$$ 

From this equation we again get

$$\ell(e_1) = (1 - \lambda^2 - 2\alpha \lambda) e_1 + \xi(\lambda) e_2, \quad \ell(e_2) = \xi(\lambda) e_1 + (1 - \lambda^2 + 2\alpha \lambda) e_2.$$

Note that on contact generalized $(\kappa, \mu, \nu)$-spaces we have

$$\ell = -\kappa \varphi^2 + \mu h + \nu \varphi h,$$

$$\ell \varphi - \varphi \ell = 2\mu h \varphi + 2\nu h.$$

Let us compute $L_\xi \ell$. First we observe that

$$[\xi, e_1] = (1 + \lambda + \alpha) e_2, \quad [\xi, e_2] = -(1 - \lambda + \alpha) e_1.$$

We put

$$\ell(e_i) = \ell_{i1} e_1 + \ell_{i2} e_2, \quad i = 1, 2.$$

Then

$$(L_\xi \ell) e_1 = [\xi, \ell(e_1)] - \ell[\xi, e_1] = \ell_{11}[\xi, e_1] + \ell_{12}[\xi, e_2] - (1 + \lambda + \alpha) \ell(e_2)$$

$$= \ell_{11}(1 + \lambda + \alpha) e_2 - \ell_{12}(1 - \lambda + \alpha) e_1$$

$$= -2\xi(\lambda)(1 + \alpha) e_1 + (\ell_{11} - \ell_{22})(1 + \lambda + \alpha) e_2$$

$$= -2\xi(\lambda)(1 + \alpha) e_1 - 4\alpha \lambda e_2.$$

Since we assumed that $M$ is non-Sasakian, i.e., $\lambda \neq 0$, $M$ satisfies $(L_\xi \ell) e_1 = 0$ if and only if

$$\xi(\lambda) = 0, \quad \alpha = 0.$$

In a similar way we have $(L_\xi \ell) e_2 = 0$ if and only if $\xi(\lambda) = 0$ and $\alpha = 0$. Hence $M$ is a contact generalized $(\kappa, 0)$-space. In this case $\rho_{11} = \rho_{22}$, so $M$ is pseudo-symmetric. Moreover since we assume that $\rho_{31} = \rho_{32} = 0$, we get

$$2b\lambda - e_2(\lambda) = 2c\lambda - e_1(\lambda) = 0.$$

Hence $M$ satisfies $S \varphi = \varphi S$. Thus we obtain
Lemma 4.4 Let $M$ be an $H$-contact 3-manifold. If $M$ satisfies $\mathcal{L}_\xi \ell = 0$, then $M$ is a generalized $(\kappa,0)$-space. Moreover $M$ satisfies $S\varphi = \varphi S$.

As we shall see in the next subsection, the condition $S\varphi = \varphi S$ implies that $\kappa$ is constant.

4.2 The condition $S\varphi = \varphi S$

Here we recall the classification of contact Riemannian 3-manifolds satisfying $S\varphi = \varphi S$.

Theorem 4.5 ([4]) Let $M$ be a contact Riemannian 3-manifold. Then the following three conditions are mutually equivalent:

1. $M$ is $\eta$-Einstein, that is, $S = a\text{Id} + b\eta \otimes \xi$ for some functions $a$ and $b$.
2. $S\varphi = \varphi S$;
3. $M$ is a contact $(\kappa,0)$-space, that is, $M$ satisfies $R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\}$ for all vector fields $X$ and $Y$ on $M$. Here $\kappa \leq 1$ is a constant.

Theorem 4.6 ([4]) Let $M$ be a contact Riemannian 3-manifold. Then $M$ satisfies $S\varphi = \varphi S$ if and only if $M$ is either

1. a Sasakian 3-manifold,
2. a flat contact Riemannian 3-manifold, or
3. a non-Sasakian contact Riemannian space form of constant holomorphic sectional curvature $-\kappa$ and constant $\xi$-sectional curvature $\kappa < 1$.

From Proposition 1.2, one can see that $\eta$-Einstein contact Riemannian 3-manifolds are pseudo-symmetric (see [9], [12]).

We have seen that if a contact Riemannian 3-manifold $M$ satisfies $S\xi = \sigma \xi$ and $\mathcal{L}_\xi \ell = 0$, then $M$ is a generalized $(\kappa,\mu,\nu)$-space. Moreover since $S\varphi = \varphi S$, $M$ is $\eta$-Einstein and hence it is pseudo-symmetric. Moreover contact Riemannian 3-manifolds satisfying $S\varphi = \varphi S$ are pseudo-symmetric spaces of constant type (see [9]).

4.4 Now we return to non-Sasakian contact Riemannian 3-manifold $M$ satisfying $S\xi = \sigma \xi$ and $\mathcal{L}_\xi \ell = 0$. Then combining Lemma 4.4 and Theorem 4.5, $M$ is a contact $(\kappa,0)$-space. Moreover $\lambda = \sqrt{1 - \kappa^2}$ is constant. Hence from Proposition 3.2, we have $[\xi,e_1] = (1 + \lambda)e_2$, $[e_1,e_2] = 2e_3$, $[e_2,\xi] = (1 - \lambda)e_1$.

Thus $M$ admits a local structure of a Lie group (see also [2]). Put $(c_1, c_2, c_3) := (1 - \lambda, 1 + \lambda, 2)$. Then we have the following classification.

- $\kappa > 0 \iff \lambda^2 < 1$: In this case, the signature of $(c_1, c_2, c_3)$ is $(+,+,+)$. Hence the possible Lie algebra is $\mathfrak{su}(2)$.
- $\kappa = 0 \iff \lambda^2 = 1$: In this case the signature of $(c_1, c_2, c_3)$ is $(0,+,+)$ or $(+,0,+)$. The Lie algebra is $\mathfrak{e}(2)$ and $M$ is flat.
- $\kappa < 0 \iff \lambda^2 > 1$: In this case, the signature of $(c_1, c_2, c_3)$ is $(-,+,+)$ or $(+,-,+)$. Hence the Lie algebra is $\mathfrak{sl}_2\mathbb{R}$.  

Note that $\ell$ is given by $\ell(e_1) = \ell_1 e_1$, $\ell(e_2) = \ell_2 e_2$ with
\[
\ell_1 = \frac{1}{4} (c_1 - c_2)(c_1 + 3c_2 - 4) + 1 = 1 - \lambda^2, \\
\ell_2 = \frac{1}{4} (c_1 - c_2)(3c_1 + c_2 - 4) + 1 = 1 - \lambda^2.
\]
Thus we obtain the following theorem.

**Theorem 4.7** Let $M$ be an $H$-contact $3$-manifold. If $M$ satisfies $\mathcal{L}_\xi \ell = 0$, then $M$ is Sasakian or a homogeneous contact metric $3$-manifold, locally isomorphic to $SU(2)$, $SL_2\mathbb{R}$ or $E(2)$.

As a result, we obtain the following main result of this paper:

**Theorem 4.8** Let $M$ be a contact Riemannian $3$-manifold. Then $S\xi = \sigma \xi$ for some function $\sigma$ and at the same time $\mathcal{L}_\xi \ell = 0$ if and only if $M$ is Sasakian or a homogeneous contact Riemannian $3$-manifold, locally isomorphic to $SU(2)$ or $SL_2\mathbb{R}$ or $E(2)$.

As we have mentioned in Introduction, we can not remove the assumption $H$-contact in our main theorem.

Let $G$ be a $3$-dimensional non-unimodular Lie group equipped with a left invariant contact Riemannian structure. Then there exists an orthonormal basis $\{e_1, e_2 = \phi e_1, e_3 = \xi\}$ of the Lie algebra $\mathfrak{g}$ of $G$ such that
\[
[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = -\gamma e_2. \tag{4.1}
\]
The Levi-Civita connection of $G$ is given by the following table:

**Proposition 4.9** ([20, p. 251])
\[
\begin{align*}
\nabla_{e_1} e_1 &= 0, \quad \nabla_{e_2} e_1 = -\alpha e_2 - \frac{1}{2}(\gamma + 2)e_3, \quad \nabla_{e_3} e_1 = \frac{1}{2}(\gamma + 2)e_2, \\
\nabla_{e_2} e_1 &= -\alpha e_2 - \frac{1}{2}(\gamma + 2)e_3, \quad \nabla_{e_2} e_2 = \alpha e_1, \quad \nabla_{e_2} e_3 = \frac{1}{2}(\gamma + 2)e_1, \\
\nabla_{e_3} e_1 &= -\frac{1}{2}(\gamma + 2)e_2, \quad \nabla_{e_3} e_2 = \frac{1}{2}(\gamma + 2)e_1, \quad \nabla_{e_3} e_3 = 0.
\end{align*}
\]

Using this table, we get
\[
S\xi = -\alpha \gamma e_2 + \left(2 - \frac{\gamma^2}{2}\right)\xi
\]
and
\[
\ell(e_1) = \ell_1 e_1, \quad \ell(e_2) = \ell_2 e_2,
\]
where
\[
\ell_1 = -\frac{1}{4}(\gamma + 2)(3\gamma - 2), \quad \ell_2 = \frac{1}{4}(\gamma + 2).
\]

**Proposition 4.10** Let $G$ be a $3$-dimensional non-unimodular Lie group equipped with a left invariant contact Riemannian structure. Suppose that $G$ satisfies $\mathcal{L}_\xi \ell = 0$. Then the Lie algebra $\mathfrak{g}$ is generated by the commutation relation (4.1) with $\gamma = 0$ (Sasakian) or $\gamma = -2$. In particular $\ell = 0$ if and only if $\gamma = -2$. 
Proof. We compute

\[(\mathcal{L}_\xi \ell)e_1 = \mathcal{L}_\xi (\ell(e_1)) - \ell(\mathcal{L}_\xi e_1) = [\xi, \ell(e_1)] - \ell([\xi, e_1])\]

\[= \frac{1}{4}(\gamma + 2)(3\gamma - 2)\gamma e_2 + \frac{\gamma}{4}(\gamma + 2)^2 e_2\]

and

\[(\mathcal{L}_\xi \ell)e_2 = \mathcal{L}_\xi (\ell(e_2)) - \ell(\mathcal{L}_\xi e_2) = [\xi, \ell(e_2)] - \ell([\xi, e_2]) = 0.\]

Since \((\mathcal{L}_\xi \ell)\xi = 0\), we have the required result. \(\square\)

From this proposition, we see that the non-unimodular Lie group with \(\gamma = -2\) satisfies \(\mathcal{L}_\xi \ell = 0\) but \(S\xi \neq \sigma\xi\). In fact, \(S\xi = 2\alpha e_2 \neq 0\).

In our previous works, we studied pseudo-symmetry of contact Riemannian 3-manifolds \([9,10,12]\). In \([15]\), the following result was obtained.

**Theorem 4.11** ([15]) Let \(M\) be contact Riemannian 3-manifold which is pseudo-symmetric and satisfies \(S\xi = \sigma\xi\) for some function \(\sigma\) such that \(\xi(\sigma) = 0\). Then there exists at most six open subsets of \(M\) for which their union is an open and dense subset inside of the closure of \(M\) and each of them as an open submanifold of \(M\) is either

- a Sasakian manifold,
- flat,
- locally isometric to one of the Lie groups \(SU(2)\) or \(SL_2\mathbb{R}\) equipped with a left invariant contact Riemannian structure,
- a pseudo-symmetric space of constant type with constant scalar curvature \(\tau = 2(1 - \lambda^2 - 2\alpha)\),
- a semi K-contact manifold with \(L = -3\alpha^2 + 4\alpha\),
- a semi K-contact manifold with \(L = \alpha^2\).

**Theorem 4.12** ([15]) Let \(M\) be a pseudo-symmetric contact Riemannian 3-manifold of constant type such that \(S\xi = \sigma\xi\), where \(\sigma\) is a smooth function on \(M\). Then \(\sigma\) is constant. If \(M\) is also complete then it is either a Sasakian manifold (meaning \(tr\ell = 2\)) or locally isometric to one of the following Lie groups equipped with a left invariant metric: \(SU(2)\), \(SL_2\mathbb{R}\), \(E(2)\) or \(E(1,1)\).

**Theorem 4.13** ([15]) A 3-dimensional generalized \((\kappa, \mu, \nu)\)-space which is a pseudo-symmetric space of constant type is either a Sasakian manifold or a contact \((\kappa, \mu)\)-space. In the second case, if \(M\) is also complete, then it is locally isometric to one of the following Lie group with a left invariant metric: \(SU(2)\), \(SL_2\mathbb{R}\), \(E(2)\) or \(E(1,1)\).

Comparing these with the main theorem of this paper, we notice that the Minkowski motion group \(E(1,1)\) is an interesting example. In fact

- \(E(1,1)\) is pseudo-symmetric,
- \(E(1,1)\) is \((\kappa, \mu)\)-space, hence \(S\xi = \sigma\xi\)
- but not \(\mathcal{L}_\xi \ell = 0\).

The contact Riemannian structure on \(E(1,1)\) is given explicitly in our previous paper \([9]\).
Definition 4.14 ([13]) Let $M$ be a contact Riemannian 3-manifold. Let $h = \lambda h_+ - \lambda h_-$ be the spectral decomposition of $h$ on $U = \{ p \in M \mid h_p \neq 0 \}$. If $\nabla_{h_+ h_-} X = [\xi, h_+ X]$ holds for all vector field $X$ on $M$ and all points of an open subset $W$ of $U$, and if $h = 0$ on the points of $M$ that do not belong to $W$, then the manifold is said to be a semi $K$-contact manifold.

Remark 4.1 Koufogiorgos and Tsichlias [18] studied contact Riemannian 3-manifolds with $\ell = 0$. In our previous paper [11], we studied model spaces for the class of contact Riemannian 3-manifolds with vanishing $\ell$ and constant norm $|S\xi|$. Obviously the condition $\ell = 0$ is a special case of $\mathcal{L}_\xi \ell = 0$. In this paper we study contact Riemannian 3-manifolds satisfying both $S\xi = \sigma \xi$ and $\mathcal{L}_\xi \ell = 0$. However $S\xi = \sigma \xi$ can not occur in the subcase $\ell = 0$ except the flat case. See the following example due to Koufogiorgos and Tsichlias.

Example 4.1 [18] On the Cartesian 3-space $\mathbb{R}^3(x, y, z)$, define a contact 1-form $\eta$ by
$$\eta = dx + 2ye^{-z}dz.$$ 
Next define a frame field $\{e_1, e_2, e_3\}$ by
$$e_1 = -2y \frac{\partial}{\partial x} + (2x - ye^z) \frac{\partial}{\partial y} + e^z \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial x}.$$ 
Then we define a Riemannian metric $g$ by the condition $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. One can see that $g$ is an associated metric to $\eta$. As usual, the endomorphism field $\varphi$ is defined by $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$ and $\varphi e_3 = 0$. The Reeb vector field is $\xi = e_3$. Direct computation shows that $\ell = 0$, $\tau = 0$. The components of the Ricci operator are $\rho_{23} = \rho_{32} = 2e^z$. Other components are zero. In particular $S\xi = 2e^z e_2$. Hence $S\xi$ is not parallel to $\xi$. Note that $M$ is neither homogeneous nor flat.

This example also shows that we can not remove the assumption $S\xi = \sigma \xi$ in the main result of this paper.

Boeckx, Chun and the first named author of the present paper investigated unit tangent sphere bundles with $\eta$-parallel $\ell$ in [6]. In [19], K. Panagiotidou and P. J. Xenos studied (3-dimensional) real hypersurfaces in $\mathbb{C}P_2$ and $\mathbb{C}H_2$ with pseudo-parallel $\ell$.

These studies lead to the following problems:
- Classify unit tangent sphere bundles with semi-parallel $\ell$, i.e., $R \cdot \ell = 0$.
- Classify contact Riemannian 3-manifolds with semi-parallel $\ell$ or more generally pseudo-parallel $\ell$, i.e.,
$$R(X, Y) \cdot \ell = L(X \wedge Y) \cdot \ell$$
for some function $L$.

Remark 4.2 On the universal covering $\tilde{E}(2)$ of the Euclidean motion group $E(2) = SO(2) \times \mathbb{R}^2$, the standard contact form associated to the flat metric $(dx^2 + dy^2 + dz^2)/4$ is given by
$$\eta = \frac{1}{2}(\cos z \, dx + \sin z \, dy).$$
One can see that this contact structure is invariant under the action of discrete subgroup $\Gamma = 2\pi \mathbb{Z}^3$ of $E(2)$. Hence we obtain the flat torus $T^3 = E(2)/\Gamma$ equipped with left invariant contact Riemannian structure. This provide us a compact example of a homogeneous contact Riemannian 3-manifold with vanishing $\ell$. 
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