ON THE SECTIONAL CURVATURE OF DESZCZ

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. A scalar valued curvature invariant is constructed which in general depends on two tangent planes at a point. This invariant, which is called the sectional curvature of Deszcz, can be geometrically interpreted in terms of the parallelogramoids of Levi-Civita and isotropy of this invariant with respect to both planes characterises the pseudo-symmetric spaces.

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1. Introduction. One of the most natural kinds of transformations to perform on a Riemannian manifold \((M, g)\) which takes into account something essential with respect to the underlying differential structure, i.e. the co-ordinate patches, and, of course, also something essential with respect to the geometrical structure, i.e. the Riemannian connection, is the parallel transport around co-ordinate parallelograms, and around infinitesimal co-ordinate parallelograms for that matter, \(g\) after all being conceived as an infinitesimal measure of lengths on \(M\). No doubt, the simplest objects to move around ”this way” are the vectors; the symmetry of this operation was studied by Schouten who doing so obtained the geometrical interpretation of the Riemann-Christoffel curvature tensor \(R\) of \((M, g)\), which nowadays mostly even serves as its definition. Parallely, Levy-Civita introduced the parallel transport of vectors on Riemannian manifolds to define his parallelogramoids in terms of which he succeeded to give a beautiful geometrical interpretation of the Riemann or sectional curvature \(K\) of \((M, g)\). The next
simplest objects to move around "this way" are likely to be precisely then sectional curvatures. That is what the authors have been doing for some time now and on some of the results obtained along this way this paper will report.

2. The sectional curvature of a plane. Let \((M^n, g)\) be an \(n\)-dimensional manifold with Riemannian metric \(g\). Denote the Levi-Civita connection by \(\nabla\) and its related Riemann-Christoffel \((1,3)\)-curvature tensor by \(R\). The \((0,4)\)-curvature tensor \(R\) is related to the \((1,3)\)-curvature tensor by

\[
R(X, Y, U, V) = g(R(X, Y)U, V).
\]

The endomorphisms \(V \wedge g W\) and \(R(V, W)\) of the Lie algebra of vector fields \(\mathfrak{X}(M)\) of \(M\) are defined by

\[
(V \wedge g W)Z = g(W, Z)V - g(V, Z)W,
\]

and

\[
R(V, W)Z = \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V, W]} Z.
\]

Here and in the following, vectors will be denoted by lower case letters, while vector fields will be denoted by capital letters. As is well-known, and which goes back to Schouten [11], \(R(\vec{x}, \vec{y})\vec{z}\) measures the second order change of a vector \(\vec{z} \in T_p M\) at \(p \in M\) after parallel transport around an infinitesimal co-ordinate parallelogram \(\mathcal{P}\) cornered at \(p\) with sides of parameter changes \(\Delta x\) and \(\Delta y\) and with tangent vectors \(\vec{x}\) and \(\vec{y}\) at \(p\) to the \(x\)- and \(y\)-sides of \(\mathcal{P}\), namely,

\[
\vec{z}^\ast = \vec{z} + [R(\vec{x}, \vec{y})\vec{z}] \Delta x \Delta y + O^2(\Delta x, \Delta y).
\]

A vector \((\vec{x} \wedge g \vec{y})\vec{z}\) can be geometrically interpreted as follows. Assume that \(\vec{x}, \vec{y} \in T_p M\) are orthonormal and choose vectors \(\vec{e}_3, \ldots, \vec{e}_n\) so that \(\{\vec{x}, \vec{y}, \vec{e}_3, \ldots, \vec{e}_n\}\) is an orthonormal basis of \(T_p M\). Then, \(\vec{z} \in T_p M\) can be decomposed as

\[
\vec{z} = g(\vec{z}, \vec{x})\vec{x} + g(\vec{z}, \vec{y})\vec{y} + \sum_{i=3}^{n} g(\vec{z}, \vec{e}_i)\vec{e}_i.
\]

By rotating the projection of \(\vec{z}\) onto the plane \(\vec{x} \wedge \vec{y}\), spanned by \(\vec{x}\) and \(\vec{y}\), over an infinitesimal angle \(\Delta \phi\), while keeping the projection of \(\vec{z}\) onto the \((n - 2)\)-plane spanned by \(\vec{e}_3, \ldots, \vec{e}_n\) fixed, a new vector \(\tilde{z}\) is obtained, namely,
\[ \tilde{z} = \bar{z} + [g(\bar{z}, \bar{y}) \bar{x} - g(\bar{z}, \bar{x}) \bar{y}] \Delta \varphi + \mathcal{O}^{\geq 2}(\Delta \varphi). \]

Thus, the vector \((\bar{x} \wedge_{\bar{y}} \bar{y}) \bar{z}\) measures the first order change of the vector \(\bar{z}\) after such an infinitesimal rotation of \(\bar{z}\) in the plane \(\bar{x} \wedge \bar{y}\) at the point \(p\). Therefore, it seems natural to consider \((\bar{x} \wedge_{\bar{y}} \bar{y}) \bar{z}\) as some kind of normalisation for \(\mathcal{R}(\bar{x}, \bar{y}) \bar{z}\), which leads to the following definition.
**Definition 1.** At any point \( p \in M \), let \( \pi = \vec{v} \wedge \vec{w} \) be any plane tangent to \( M \) at \( p \), spanned by two of its vectors \( \vec{v} \) and \( \vec{w} \). Then, the real number

\[
K(p, \pi) = \frac{g(R(\vec{v}, \vec{w})\vec{w}, \vec{v})}{g((\vec{v} \wedge g \vec{w})\vec{w}, \vec{v})}
\]

only depends on the point \( p \) and on the plane \( \pi \) and is called the sectional curvature of \( M \) at \( p \) for the plane section \( \pi \).

As shown by Cartan, the knowledge of the full curvature tensor \( R \) is equivalent to the knowledge of the sectional curvatures \( K \). A Riemannian manifold is said to be a space of constant curvature \( c \) when all its sectional curvatures \( K(p, \pi) \) are equal to \( c \), i.e., when these curvatures are independent of both the points \( p \) and the planes \( \pi \). By Schur’s theorem, for \( n > 2 \), it suffices for this to hold that at all points \( p \) the sectional curvatures \( K(p, \pi) \) are independent of the planes \( \pi \) at \( p \). The spaces of constant curvature \( c \) are characterised by their \((0, 4)\)-curvature tensor \( R \) being given by \( R = c G \), where the \((0, 4)\)-tensor \( G \) is defined as

\[
G(X, Y, U, V) = g((X \wedge Y)U, V).
\]

A geometrical interpretation of the sectional curvature of a plane \( \pi \), spanned by the vectors \( \vec{v} \) and \( \vec{w} \), at a point \( p \) in terms of lengths of geodesics was given by Levi-Civita using his so-called parallelogramoids as follows (see e.g. [4, 10]). Consider through \( p \) the geodesic \( \alpha \) with tangent \( \vec{v} \) and let \( q \) be the point on this geodesic at an infinitesimal distance \( A \) from \( p \). Denote by \( \vec{w}^* \) the vector obtained after parallel transport of \( \vec{w} \) from \( p \) to \( q \) along \( \alpha \). Then, through \( p \) and \( q \) consider the geodesics \( \beta_p \) and \( \beta_q \) with tangents \( \vec{w} \) and \( \vec{w}^* \), respectively. Fix on them the points \( p \) and \( q \), respectively, at the same infinitesimal distance \( B \) from \( p \) and \( q \), respectively. The parallelogramoid cornered at \( p \) with sides tangent to \( \vec{v} \) and \( \vec{w} \) is then completed by the geodesic \( \alpha \) through \( p \) and \( q \). Let \( A' \) be the geodesic distance between \( p \) and \( q \). Levi-Civita showed that, in first order approximation, the sectional curvature of the plane \( \pi = \vec{v} \wedge \vec{w} \) can be expressed as

\[
K(p, \pi) \approx \frac{A^2 - A'^2}{(AB \sin \phi)^2},
\]

whereby \( \phi \) is the angle between the vectors \( \vec{v} \) and \( \vec{w} \). In particular, let \( \vec{v} \) and \( \vec{w} \) be orthonormal vectors at \( p \in M \). Consider the Levi-Civita squaroid based on \( \vec{v} \) and \( \vec{w} \) with side \( \varepsilon \), i.e., the parallelogramoid for which \( A = B = \varepsilon \). Then, when \( \varepsilon' \) is the length of the closing geodesic, the sectional curvature
$K(p, \pi)$ is given by

$$K(p, \pi) \approx \frac{\varepsilon^2 - \varepsilon'^2}{\varepsilon^4}.$$  

3. The sectional curvature of Deszcz of two planes. Consider the $(0,6)$-tensor $R \cdot R$, obtained by the action of the curvature operator $\mathcal{R}(X,Y)$ on the $(0,4)$-curvature tensor $R$,

$$(R \cdot R)(X_1, X_2, X_3, X_4; X, Y) := (\mathcal{R}(X,Y) \cdot R)(X_1, X_2, X_3, X_4)$$

$$= -R\left( \mathcal{R}(X,Y)X_1, X_2, X_3, X_4 \right)$$

$$-R\left( X_1, \mathcal{R}(X,Y)X_2, X_3, X_4 \right)$$

$$-R\left( X_1, X_2, \mathcal{R}(X,Y)X_3, X_4 \right)$$

$$-R\left( X_1, X_2, X_3, \mathcal{R}(X,Y)X_4 \right),$$

whereby $X_1, X_2, X_3, X_4, X, Y \in \mathfrak{X}(M)$. A Riemannian manifold $M$ is said to be *semi-symmetric* when the tensor $R \cdot R$ vanishes, i.e., $R \cdot R = 0$. Consider any two linearly independent vectors $\vec{v}$ and $\vec{w}$ at any point $p$ of $M$ and any co-ordinate parallelogram $\mathcal{P}$ cornered at $p$ with sides of lengths $\Delta x$ and $\Delta y$ tangent to the linearly independent vectors $\vec{x}$ and $\vec{y}$ at $p$. Then, by parallel transport of $\vec{v}$ and $\vec{w}$ around $\mathcal{P}$ we obtain the vectors $\vec{v}^* =$
\[ \vec{v} + [R(\vec{x}, \vec{y})\vec{v}] \Delta \vec{x} \Delta \vec{y} + \mathcal{O}^2(\Delta \vec{x}, \Delta \vec{y}) \] and \[ \vec{w}^* = \vec{w} + [R(\vec{x}, \vec{y})\vec{w}] \Delta \vec{x} \Delta \vec{y} + \mathcal{O}^2(\Delta \vec{x}, \Delta \vec{y}), \] so that

\[
R(\vec{v}^*, \vec{w}^*, \vec{w}^*, \vec{v}^*) = R(\vec{v}, \vec{w}, \vec{w}, \vec{v}) - [(R \cdot R)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})] \Delta \vec{x} \Delta \vec{y} + \mathcal{O}^2(\Delta \vec{x}, \Delta \vec{y}).
\]

In particular, since the Levi-Civita connection is metrical, this shows that for orthonormal vectors \( \vec{v} \) and \( \vec{w} \), in approximation up to second order,

\[
K(p, \pi^*) \approx K(p, \pi) + [(R \cdot R)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})] \Delta \vec{x} \Delta \vec{y}.
\]

Thus, the \((0,6)\)-tensor \( R \cdot R \) of \( M \) measures the change in sectional curvature at any point \( p \) for any plane \( \pi \) under parallel transport of \( \pi \) around any infinitesimal co-ordinate parallelogram \( \mathcal{P} \) cornered at \( p \) \[5\]. As a consequence it follows that a Riemannian manifold \( M \) is semi-symmetric if and only if its sectional curvature function \( K(p, \pi) \) is invariant, up to second order, under parallel transport of any plane \( \pi \) at any point \( p \) of \( M \) around any infinitesimal co-ordinate parallelogram cornered at \( p \).

Probably the simplest \((0,6)\)-tensor on an \((n \geq 3)\)-dimensional Riemannian manifold having the same symmetry properties as \( R \cdot R \) is the Tachibana tensor \( Q(g, R) \), defined by

\[
Q(g, R)(X_1, X_2, X_3, X_4; X, Y) := -((X \wedge Y) \cdot R)(X_1, X_2, X_3, X_4)
= R((X \wedge Y)X_1, X_2, X_3, X_4)
+ R(X_1, (X \wedge Y)X_2, X_3, X_4)
+ R(X_1, X_2, (X \wedge Y)X_3, X_4)
+ R(X_1, X_2, X_3, (X \wedge Y)X_4).
\]

A classical result states that the vanishing of this tensor, i.e., \( Q(g, R) = 0 \), is a necessary and sufficient condition for \( M \) to be of constant curvature. Using the above geometrical interpretation of \((\vec{x} \wedge \vec{y})\vec{z}^n\), a geometrical meaning of the components \( Q(g, R)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) \) of the Tachibana tensor can be obtained as follows. Let \( \{\vec{x}, \vec{y}, \vec{e}_3, \ldots, \vec{e}_n\} \) be an orthonormal basis of \( T_pM \) and consider orthonormal vectors \( \vec{v}, \vec{w} \in T_pM \). The vectors \( \vec{v} \) and \( \vec{w} \) are the vectors obtained after an infinitesimal rotation of the projection of \( \vec{v} \) and \( \vec{w} \) in the plane \( \vec{x} \wedge \vec{y} \), namely \( \vec{v} = \vec{v} + [(\vec{x} \wedge \vec{y})\vec{v}] \Delta \varphi + \mathcal{O}^{2}(\Delta \varphi) \), and \( \vec{w} = \vec{w} + [(\vec{x} \wedge \vec{y})\vec{w}] \Delta \varphi + \mathcal{O}^{2}(\Delta \varphi) \). Comparing the sectional curvatures of the planes \( \pi = \vec{v} \wedge \vec{w} \) and \( \tilde{\pi} = \vec{v} \wedge \vec{w} \), we find

\[
K(p, \tilde{\pi}) = K(p, \pi) + [Q(g, R)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})] \Delta \varphi + \mathcal{O}^{2}(\Delta \varphi).
\]
Thus, the components $Q(g, R)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})$ measure the change of sectional curvature $K(p, \pi)$ under an operation involving infinitesimal rotations performed at the point $p$, without leaving this point, in contrast to the components $(R \cdot R)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})$, which measure the change of sectional curvature $K(p, \pi)$ after the movement of the plane $\pi$ in an infinitesimal neighbourhood of the point $p$. It seems therefore natural to consider the components $Q(g, R)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})$ as some kind of normalisation for the components $(R \cdot R)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})$.

**Definition 2.** Let $(M^n, g)$ be an $n(\geq 3)$-dimensional Riemannian manifold which is not of constant curvature and denote by $\mathcal{U}$ the set of points where the Tachibana tensor $Q(g, R)$ is not identically zero, i.e.,

$$\mathcal{U} = \{ x \in M \mid Q(g, R)_x \neq 0 \}.$$ 

Then, at a point $p \in \mathcal{U}$, a plane $\pi = \vec{v} \wedge \vec{w} \subset T_p M$ is said to be curvature-dependent with respect to a plane $\pi = \vec{x} \wedge \vec{y} \subset T_p M$ if $Q(g, R)(\vec{v}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) \neq 0$.

This definition is independent of the choice of bases for $\pi$ and $\pi$. In analogy with Definition 1 we propose the following.

**Definition 3.** At a point $p \in \mathcal{U} \subset M$, let the tangent plane $\pi = \vec{v} \wedge \vec{w}$ be curvature-dependent with respect to $\pi = \vec{x} \wedge \vec{y}$. Then, the sectional curvature of Deszcz $L(p, \pi, \pi)$ of the plane $\pi$ with respect to $\pi$ at $p$ is the scalar

$$L(p, \pi, \pi) = \frac{(R \cdot R)(\vec{v}, \vec{w}, \vec{v}; \vec{x}, \vec{y})}{Q(g, R)(\vec{v}, \vec{w}, \vec{v}; \vec{x}, \vec{y})}.$$

This definition is again independent of the choice of bases for the tangent planes $\pi$ and $\pi$. Analogously to the result of Cartan concerning the determination of the Riemann curvature tensor through the sectional curvatures, one can show that at any point $p \in \mathcal{U}$, the tensor $R \cdot R$ is completely determined by the knowledge of the sectional curvatures of Deszcz $L(p, \pi, \pi)$ of curvature-dependent planes $\pi, \pi \subset T_p M$.

The analogy between the sectional curvature of a plane and the sectional curvature of Deszcz of two curvature-dependent planes goes further, in the sense that a geometrical interpretation of the sectional curvature of Deszcz can be given in terms of the squaroids of Levi-Civita as follows. At a point $p \in M$, consider two planes $\pi = \vec{v} \wedge \vec{w}$ and $\pi = \vec{x} \wedge \vec{y}$ and parallel
transport the vectors $\vec{v}$ and $\vec{w}$ around the infinitesimal co-ordinate parallelogram formed by the tangents $\vec{x}$ and $\vec{y}$ at $p$. We construct the two squaroids starting from the vectors $\vec{v}, \vec{w}$ and $\vec{v}^*, \vec{w}^*$, respectively, with equal sides $\varepsilon$. In general, the lengths of the closing geodesics, $\varepsilon'$ and $\varepsilon^*$, will be different. We find, up to second order with respect to the sides $\Delta x$ and $\Delta y$ of the co-ordinate parallelogram that
\[
(R \cdot R)(\vec{v}, \vec{w}, \vec{v}, \vec{x}, \vec{y}) \approx \frac{(\varepsilon^*)^2 - (\varepsilon')^2}{\varepsilon^4} \frac{1}{\Delta x \Delta y}.
\]
Let $\tilde{v}, \tilde{w}$ be the vectors which are obtained after an infinitesimal rotation as before from the vectors $\vec{v}, \vec{w}$ with respect to the plane $\pi = \vec{x} \wedge \vec{y}$, and construct for the plane $\tilde{v} \wedge \tilde{w}$ the squaroid of Levi-Civita, with the side $\varepsilon$. Denote the lengths of the completing geodesics by $\tilde{\varepsilon}'$. We find, with respect to the angle $\Delta \varphi$ of infinitesimal rotation, that
\[
Q(g, R)(\tilde{v}, \tilde{w}, \tilde{v}, \tilde{x}, \tilde{y}) \approx \frac{(\tilde{\varepsilon}')^2 - (\varepsilon')^2}{\varepsilon^4} \frac{1}{\Delta \varphi}.
\]
Thus, calibrating the changes of the Riemann sectional curvatures under parallel translation ($*$) around a parallelogram $P$ with infinitesimal parameter growths $\Delta x$ and $\Delta y$ by the changes of the same curvatures under rotation ($\sim$) over an infinitesimal angle $\Delta \varphi = \Delta x \Delta y$ with respect to $\pi$, we find the following approximate geometrical expression in terms of the squaroids of Levi-Civita of sides $\varepsilon$, for the sectional curvature of Deszcz $L$,
\[
L(p, \pi, \bar{\pi}) \approx \frac{(\varepsilon^*)^2 - (\varepsilon')^2}{(\tilde{\varepsilon}')^2 - (\varepsilon')^2}.
\]
In the particular case that, at a point $p \in U \subset M$, the sectional curvature of Deszcz $L(p, \pi, \bar{\pi})$ is independent of the planes $\pi$ and $\bar{\pi}$, the manifold $M$ is said to be pseudo-symmetric in the sense of Deszcz at $p$. If the manifold $M$ is pseudo-symmetric at all points of $U \subset M$, the manifold $M$ is said to be pseudo-symmetric in the sense of Deszcz. In this case, there holds that $R \cdot R = L Q(g, R)$. We observe that there does not hold a strict analog of the theorem of Schur in the case of pseudo-symmetric manifolds, i.e., there are many examples of pseudo-symmetric manifolds with non-constant sectional curvature of Deszcz. However, a partial analog does hold in the sense that if the sectional curvatures of Deszcz $L(p, \pi, \bar{\pi})$ at $p \in U$ are independent of $\pi$, i.e., $L(p, \pi, \bar{\pi}) = L(p, \bar{\pi})$ for every tangent plane $\pi$ which is
curvature-dependent with respect to $\pi$, then the Riemannian manifold $M$ is pseudo-symmetric at $p$.

Following Kowalski and Sekizawa, a pseudo-symmetric space for which the sectional curvature of Deszcz is constant is said to be \textit{pseudo-symmetric of constant type}. The three-dimensional Riemannian pseudo-symmetric spaces of constant type are obtained in [8, 9]. For example, the eight three-dimensional Thurston metrics have either constant sectional curvature $K$ equal to 0, 1 or $-1$, or have constant sectional curvature of Deszcz $L$ equal to 0, 1 or $-1$ [1].

That the pseudo-symmetric spaces are natural generalisations of the spaces of constant curvature can be seen from both intrinsic and extrinsic points of view. Extrinsically, it was shown by Deszcz [3] that the extrinsic spheres $M^n$, i.e., totally umbilical submanifolds with parallel mean curvature vector, of semi-symmetric spaces $\tilde{M}^{n+m}$ are pseudo-symmetric, which extends the result that ordinary spheres in Euclidean space are of constant curvature. And similar to the fact that the extrinsic spheres $M^n$ in spheres $S^{n+m}$ are themselves of constant curvature, the extrinsic spheres $M^n$ in pseudo-symmetric spaces $\tilde{M}^{n+m}$ are also pseudo-symmetric. From an intrinsic point of view, pseudo-symmetry appears in the study of geodesic mappings. If a Riemannian manifold $M^n$ admits a geodesic mapping onto a locally flat Riemannian space $\tilde{M}^n$, then $M^n$ itself must be a space of constant curvature. Further, if a space $M^n$ admits a geodesic mapping onto a space $\tilde{M}^n$ of constant curvature, then $M^n$ must itself have constant curvature. Accordingly, results of Mikesh and Venzi [12] and Defever and Deszcz [2] learn that when a Riemannian manifold $M^n$ admits a geodesic mapping onto a semi-symmetric manifold $\tilde{M}^n$, then $M^n$ must be pseudo-symmetric, and when a manifold $M^n$ admits a geodesic mapping onto a pseudo-symmetric manifold, then $M^n$ itself must also be pseudo-symmetric. So, in some sense, the extension of space-symmetry along these lines terminates with the pseudo-symmetry of Deszcz.

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