PHYSICAL FIELDS, SOLITON SYSTEMS AND KAWAGUCHI SPACE

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. Nonlinear physical fields are discussed based on the Zermelo condition in the Kawaguchi space or the higher-order space. This higher-order approach is related to the geometrical theory of the nonlinear dynamical systems called the KCC-theory. In the nonlinear fields, taking into account a scale transformation of the fields, nonlinear field equations called soliton equations are obtained from the Zermelo condition with the Lagrangian. Moreover, the general form for the soliton equation (the Lax equation) can be expressed in the Kawaguchi space.

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Key words: Nonlinear physical field; Soliton system; Kawaguchi space; Higher-order space; Zermelo condition; Nonlinear dynamical system; KCC-theory.

1. Introduction. The nonlinear physical fields called the soliton systems are commonly found in various physical fields (e.g., fluid mechanics [13], plasma physics [46], and geophysics [10, 31, 39, 44]). Mathematically, the soliton equations can be expressed by the Lax equation [29] and the soliton systems can be regarded as dynamical systems on infinite dimensional Grassmann manifold (Sato theory [43]). From a view point of geometry, it has been investigated relations between the soliton equations and the differential geometry of surfaces [8, 42] and curves [9, 11, 12]. Topologically, it has been shown that the solutions of the soliton equations are related to the extended objects defined as a behaving macroscopic objects in a quantum ordered state [32]. Moreover, in the nonlinear dynamical systems (e.g. the
Lorenz model [30] in meteorology and the Rikitake system [41] in geomagnetism, the behavior of the system can be studied based on a geometric theory (the KCC-theory [6, 7, 27]). However, these nonlinear physical fields have been studied individually based on different model equations. On the other hand, it has been shown that the physical field [17] can be geometrically described in a generalized Finsler space (the Kawaguchi space [19, 20] or the higher-order space [36, 38]). For example, the Klein-Gordon equation and the Dirac equation in quantum field have been discussed from a view point of the Kawaguchi space [23, 24, 25, 26]. Therefore, the nonlinear physical fields can be expressed uniquely in terms of the Kawaguchi space.

In this study, we apply the geometric approach to the soliton systems. In section 2, definition of the Kawaguchi space and the geometrical study of physical fields are briefly reviewed. In section 3, at first, we discuss a relation between the KCC-theory for the nonlinear dynamical systems and the Kawaguchi space. Then, the soliton systems as the nonlinear physical fields are studied based on the theory of the Kawaguchi space. The Lagrangian is determined for the soliton equation as the Euler-Lagrange equation. Finally, a geometrical expression of the Lax equation in the higher-order space is shown.

2. Geometrical background. In this section, the geometrical background of the Kawaguchi space or higher-order space is introduced based on the notations [21]. Throughout this paper, Einstein’s summation convention is used. Moreover, Latin and Greek indices run from 1 to $n$.

2.1. Definition of the Kawaguchi space. Let $X_n$ be an $n$-dimensional manifold. The local coordinate on $X_n$ is denoted by $(x^i)$. When a curve on $X_n$ is given by a parametric form $x^i = x^i(t)$, a set of values $(x^{(0)}i, x^{(1)}i, \ldots, x^{(M)}i)$ along the curve represents a line element of order $M$, where the parameter $t$ is a time and $x^{(0)}i = x^i, x^{(1)}i = dx^i/dt, \ldots, x^{(M)}i = d^Mx^i/dt^M$. The metric space in which an arc length $s$ along the curve is given by the integral

$$s = \int F(x^i, x^{(1)}i, x^{(2)}i, \ldots, x^{(M)}i) dt$$

is called the Kawaguchi space [19, 20] or higher-order space [36, 38] of order $M$, where the function $F$ is the Lagrangian. Especially, when $M = 1$, the
higher-order space is called the Finsler space \([33]\). The arc length should not be altered by a change of time \(t\). Therefore, the following invariant condition (the Zermelo condition) has to be hold \([21]\):

\[
\Delta_1 F \equiv \sum_{k=1}^{M} k x^{(k)} i F_{(k)i} = F, \tag{2}
\]

\[
\Delta_N F \equiv \sum_{k=N}^{M} \frac{k}{N} x^{(k-N+1)i} F_{(k)i} = 0, \tag{3}
\]

where \(N = 2, 3, \ldots, M\) and \(F_{(k)i} = \partial F/\partial x^{(k)i}\). Then, under the Zermelo condition, the Euler-Lagrange equation is obtained from the vanishing of variation of the arc length \([21, 45]\):

\[
E_i \equiv \sum_{k=0}^{M} (-1)^k (F_{(k)i})^{(k)} = 0. \tag{4}
\]

The covariant vector \(E_i\) called the Euler’s vector \([20, 21, 45]\) determines a motion of particles in the Kawaguchi space. The characteristic conceptions of the higher-order space are given by the intrinsic vector:

\[
\varepsilon^z_i \equiv \frac{1}{F} \sum_{\mathcal{L}=z}^{M} E^z_{\mathcal{L}} A_{\mathcal{L}-z+1}^z, \quad (z = 0, 1, \ldots M), \tag{5}
\]

where \(E^z_{\mathcal{L}}\) denotes the generalized Euler’s vector (Synge’s vector \([45]\)),

\[
E^z_{\mathcal{L}} \equiv \sum_{k=\mathcal{L}}^{M} (-1)^k \left(\frac{k}{\mathcal{L}}\right) (F_{(k)i})^{(k-\mathcal{L})} \tag{6}
\]

and the coefficients are determined by

\[
A^0_1 = 1, \quad A^\lambda_\mu = \frac{dA^\lambda_{\mu-1}}{dt} + A^\lambda_{\mu-1} F \quad (\lambda, \mu = 0, 1, \ldots, M), \tag{7}
\]

\[
A^\omega_1 = F^{(\omega-1)} = \frac{d^\omega-1 F}{dt^\omega-1}, \quad A^\omega_0 = 0, \quad A^0_\theta = 0 \quad (\omega = 1, \ldots, M; \theta = 2, \ldots, M). \tag{8}
\]

From the intrinsic vector, the metric tensor in the Kawaguchi space is defined by

\[
g_{ij} \equiv MF^{2M-1} F_{(M)i(M)j} + \varepsilon^M_i \varepsilon^M_j + \varepsilon^1_i \varepsilon^1_j. \tag{9}
\]
In order to consider a physical application, it has been proposed the simplified metric tensor [15]:

\begin{align}
  g_{ij} &= \frac{1}{2}(G_i G_j + G_j G_i), \\
  G_i &= \varepsilon^1_i + i\varepsilon^M_i \quad \text{and} \quad G_i^* = \varepsilon^1_i - i\varepsilon^M_i,
\end{align}

where the inner product is defined by \( G_i G_j = G_i^* \cdot G_j \) and the coefficient \( i \) of the \( \varepsilon^M_i \) is the imaginary number (not index). Therefore, the Lagrangian gives the vector \( G_i \) which represents a certain state quantity of the field.

### 2.2. The Kawaguchi space and quantum physical fields.

In physical field theory, the quantum field has been geometrically studied by means of the Kawaguchi space based on the theory of the statistical observation [23, 24, 25, 26]. In the following, let us review the geometric approach.

In the statistical observation theory, the differential, \( dx \), is regarded as the disturbance of \( x \), \( \Delta x = x - x_0 \), where \( x_0 \) is the average of the disturbance. Then, the higher-order differential \( \partial^{(k)} = \partial/\partial x^{(k)} \) is expressed by the zero-order differential \( \partial^{(0)} = \partial/\partial x \) [25]:

\begin{equation}
  x^{(1)} \partial^{(1)} = \frac{\Delta x}{\Delta t} \frac{\partial}{\partial \left( \frac{\Delta x}{\Delta t} \right)} = \frac{x - x_0}{\Delta t} \frac{\partial}{\partial \left( \frac{x - x_0}{\Delta t} \right)} = x \frac{\partial}{\partial x} = x^{(0)} \partial^{(0)},
\end{equation}

where \( \Delta t \) is a time difference. From this relation (12), the Zermelo condition can be rewritten as:

\begin{equation}
  \left( x^i \frac{\partial}{\partial x^i} + \Lambda \right) F = F,
\end{equation}

where \( \Lambda \) is the higher-order term \( (k \geq 1) \) of the Zermelo operator \( \Delta_1 \) in equation (2). From a viewpoint of the nonlocal field theory by means of the higher-order space [15], the Lagrangian connects with the spinor \( \phi \) through a relation \( \phi = a^i G_i \), where \( a^i \) is the Pauli matrix [4]. Hence, from the above Zermelo condition (13), one can obtain the Dirac equation in the quantum field theory [23, 24, 25, 26]:

\begin{equation}
  \gamma^j \left( \frac{1}{i} \partial_j - \frac{e}{\hbar c} \mu_j \right) \phi = \rho \phi,
\end{equation}
where the $\gamma^j$ is the Dirac matrix, $\rho$ is the mass of the particle, $h$ is the Planck’s constant, $c$ is the speed of light and $i$ in equation (14) is the imaginary number (not index). Here, the higher-order term $\Lambda$ is regarded as the external field like the electromagnetic field potential $e_{\mu j}$ [24]. Based on this approach, the theory of fields in the higher-order space quantum field has been discussed geometrically.

From a view point of the wave geometry [34, 35], the metric is defined by $d\tau\phi = (g_i dx^i)\phi$, where $d\tau = g_i dx^i$ is a Riemannian linearized arc length and $g_i$ is the Dirac matrix. Instead of $d\tau\phi = (g_i dx^i)\phi$, when the wave geometrical metric is defined by $ds\phi = (G_i dx^i)\phi$, a wave geometrical field theory in the higher-order space can be considered [15, 16]. Therefore, the wave geometry can be also linked with the higher-order space [14, 15, 16].

3. Geometrical descriptions of nonlinear physical fields

3.1. Nonlinear dynamical systems in the Kawaguchi space.

Geometrical structures of the nonlinear dynamical system are given by the theory of the 1-parameter Kawaguchi space of order 1 (Finsler or Lagrange space [37]). In this case, the Lagrangian in the arc length (1) is expressed by $F = F(x^i, y^i)$, where $y^i = dx^i/dt$. The metric tensor (9) is given by $g_{ij} = \partial^2 F^2 / 2\partial y^j \partial y^i$. Then, the Euler-Lagrange equation (4) can be reduced to

\begin{equation}
\frac{\partial F}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial F}{\partial y^i} \right) = 0.
\end{equation}

This equation is equivalent to the following system of the second order differential equations [37]:

\begin{equation}
\frac{d^2 x^i}{dt^2} + 2H^i(x^j, y^j) = 0, \quad i, j = 1, \ldots, n,
\end{equation}

where the smooth function $H^i$ is defined by

\begin{equation}
H^i = \frac{1}{4} g^{ij} \left( \frac{\partial^2 F^2}{\partial y^j \partial x^h} y^h - \frac{\partial F^2}{\partial x^j} \right).
\end{equation}

Geometric theory of this second order system is named by Kosambi [27], Cartan [6] and Chern [7] (the KCC-theory). It is known that under a smooth coordinate transformation $\tilde{t} = t, \tilde{x} = x$ ($x^1, x^2, \ldots, x^n$), five geometrical
invariants of this system can be obtained [2]. This KCC-theory has been applied to the nonlinear dynamical systems of the ecological field and the geomagnetic field [3,47,48].

Now, let us show that the geometrical invariant can be represented in the Kawaguchi space. The second invariant determines the Jacobi stability of the nonlinear dynamical system when the trajectory \( x^i(t) \) of the system (16) is varied into nearby ones according to \( \tilde{x}^i(t) = x^i(t) + \xi^i(t) \), where \( \xi^i = v(t)\zeta^i \) is a Jacobi field and \( \zeta^i \) is the unit normal vector field along the trajectory. This second invariant is related to the flag curvature \( K \) [5]:

\[
\left( \frac{d^2}{dt^2} + K \right) v = 0.
\]

On the other hand, in the Kawaguchi space, by multiplying the Zermelo condition (2) by the 1-parameter Zermelo operator \( \Delta_1 \), we have

\[
\{ (\Delta_1)^2 - 1 \} F = 0.
\]

Here, as a state quantity \( G_i \) in (11), we take the deviation from a normal state of the dynamical system. Then, the deviation \( v(t) \) of the trajectory can be replaced by the Lagrangian \( F \). Therefore, the equation of deviation (18) can be induced from the higher-order equation (19). Hence, from the general stand point, the behaviors of the nonlinear dynamical systems can be described by the theory of the higher-order geometry.

### 3.2. Lagrangian and Zermelo condition in the 2-parameter Kawaguchi space.

At first, we set following two conditions for the nonlinear physical field.

(a) We consider 2-parameter space because the field is expressed by at least two parameters, position \( x \) and time \( t \), i.e. \( (t^\alpha) = (t^1, t^2) \equiv (x, t) \).

(b) For the simplification, the physical field is described by only one function, i.e. the dimension of the configuration space is assumed as one; \( (\psi^j) = (\psi^1) \equiv \psi(x, t) \).

With respect to these conditions (a) and (b), the 2-parameter Kawaguchi space is defined by [22]:

\[
S = \int \int F(\psi, \psi_\alpha, \psi_{\alpha\beta}, \cdots) dt^1 dt^2.
\]
In the 2-parameter Kawaguchi space, the Lagrangian $F$ is consisted of the function $\psi$ and its differentials $\psi_\alpha, \psi_{\alpha\beta}, \cdots$, where $\psi_\alpha = \partial \psi / \partial t^\alpha$ and $\psi_{\alpha\beta} = \partial^2 \psi / \partial t^\alpha \partial t^\beta$. Therefore, the Zermelo condition is also changed to $\Delta'_1 F = F$, where the operator $\Delta'_1$ is

$$\Delta'_1 \equiv \sum_{r=1}^{M} r\psi_{\alpha(r)} \frac{\partial}{\partial \psi_{\alpha(r)}}. \tag{21}$$

Here, we put $\psi_{\alpha(r)} = \partial^r \psi / \partial t^{\alpha_1} \cdots \partial t^{\alpha_r}$ and $\bar{\partial} / \partial \psi \propto \partial / \partial \psi \tag{22}$ [22]. This is the generalization of one of the 1-parameter Zermelo condition. As well as 1-parameter case, the 2-parameter Zermelo operator (21) is rewritten as follows:

$$\Delta'_1 = \psi \frac{\partial}{\partial \psi} + \Lambda', \tag{22}$$

where the $\Lambda'$ is the higher-order term in the 2-parameter Kawaguchi space. From the operator (22), the Zermelo condition $\Delta'_1 F = F$ can be regarded as an eigen equation for the Lagrangian $F$:

$$\psi \frac{\partial}{\partial \psi} F = (1 - \Lambda') F. \tag{23}$$

Thus, the Lagrangian can take an exponential function: $F = e^{h(\psi, \psi_\alpha, \psi_{\alpha\beta}, \cdots)}$. Hence, the Euler-Lagrange equation for a field equation is given by the Zermelo condition with this exponential Lagrangian.

### 3.3. Application of the Kawaguchi space to the soliton systems.

Based on the theory of 2-parameter Kawaguchi space, we consider the integrable nonlinear field equation called the soliton equation. At first, from the operator (22), the Zermelo condition of order three is written by

$$\psi \frac{\partial F}{\partial \psi} + \psi_\alpha \frac{\partial F}{\partial \psi_\alpha} + \psi_{\alpha\beta} \frac{\partial F}{\partial \psi_{\alpha\beta}} + \psi_{\alpha\beta\gamma} \frac{\partial F}{\partial \psi_{\alpha\beta\gamma}} = F. \tag{24}$$

Since this Zermelo condition is an eigen equation, the Lagrangian is the eigen function and can take the following form:

$$F = \sum_{\Gamma=0}^{3} e^{A_{\Gamma} \psi_{\Gamma}}, \tag{25}$$
where we put $A^0 \psi_0 \equiv A(\psi)$, $A^1 \psi_1 \equiv A^\alpha(\psi)\psi_\alpha$, $A^2 \psi_2 \equiv A^{\alpha\beta}\psi_{\alpha\beta}$ and $A^3 \psi_3 \equiv A^{\alpha\beta\gamma}\psi_{\alpha\beta\gamma}$. The $A^{\alpha\beta\gamma}$ and $A^{\alpha\beta}$ are constants. From the Zermelo condition (24) with the Lagrangian (25), the nonlinear field equation as the Euler-Lagrange equation of order three is derived. Moreover, it is known that the nonlinear field equation called the soliton equation is invariant under the space-time transformations and the scale transformations [40, 42]. Geometrically, these invariances for the soliton equation correspond to the conformal invariance [18] of the Lagrangian. Hence, the Zermelo condition with the conformal invariant Lagrangian can lead to the soliton equation.

**Example 1 (modified KdV equation).** Let us consider the soliton equation $\psi_t - 6\psi^2 \psi_x + \psi_{xxx} = 0$. This is a modified KdV equation (mKdV equation) which is one of general forms of the KdV equation. The mKdV equation is invariant under the scale transformation, $x \to x' = \lambda x$, $t \to t' = \lambda^3 t$ and $\psi \to \psi' = \lambda^{-1} \psi$ [40, 42]. For the mKdV equation, we put the coefficients of the Lagrangian (25) as

(26) $A^{\alpha\beta\gamma} = \begin{cases} 1 & (\alpha = \beta = \gamma = 1) \\ 0 & \text{otherwise} \end{cases}$, $A^{\alpha\beta} = 0,$

$A^\alpha = \begin{cases} -2\psi^2 & (\alpha = 1) \\ 1 & (\alpha = 2) \end{cases}$, $A = \ln \psi.$

Then, under the scale transformation, the Lagrangian $F$ with these coefficients is changed into $\bar{F} = e^\sigma F$, where

(27) $\sigma = \log \lambda^{-1} + (\lambda^{-4} - 1)(A^{\alpha\beta\gamma}\psi_{\alpha\beta\gamma} + A^{\alpha\beta}\psi_{\alpha\beta} + A^{\alpha}\psi_\alpha).$

Therefore, the Lagrangian $F$ is conformally invariant with respect to the scale transformation of the mKdV equation. Then the Zermelo condition (24) with this conformally invariant Lagrangian leads to the mKdV equation in the Kawaguchi space.

**Example 2 (sine-Gordon equation).** We consider the sine-Gordon equation: $\psi_{xt} = \sin \psi$. The sine-Gordon equation is invariant under the scale transformation, $x \to x' = \lambda x$, $t \to t' = \lambda^{-1} t$ and $\psi \to \psi' = \psi$ [40, 42]. For the sine-Gordon equation, we put the coefficients as

$A^{\alpha\beta\gamma} = 0$, $A^{\alpha\beta} = \begin{cases} 1 & \text{for } A^{12} = A^{21} \\ 0 & \text{otherwise} \end{cases}$, $A^\alpha = 0$, $A = H(\psi) + \ln \psi,$
where $\mathcal{H}$ is a sine integral:

\begin{equation}
\mathcal{H} = \int_0^\varphi \sin \frac{\chi}{\chi} d\chi,
\end{equation}

where $\chi = \chi(x, t)$. Under the scale transformation, the Lagrangian $F$ with the coefficients is changed into $\bar{F} = F$. Therefore, the Lagrangian of the sine-Gordon equation is invariant under the transformation. In this case, the sine-Gordon equation is derived from the Zermelo condition with the Lagrangian.

### 3.4. A relation between the Lax equation and the Kawaguchi space

It is known that soliton equations can be generally expressed by the Lax equation [29]. In this section, we show a relation between this general expression and the Kawaguchi space.

The Lax equation is given by the form [29]:

\begin{equation}
L_t + [L, B] = 0,
\end{equation}

where operators $L$ and $B$ are called the Lax pair and defined by [1, 28]:

\begin{equation}
L = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & -iq \\ 0 & 0 \end{pmatrix}, \quad B = \frac{q_t}{2q} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{equation}

The $q$ is a smooth function of $x$ and $t$. The $i$ is the imaginary number (not index). Using the operators $L$ and $B$ in (30), the Lax equation (29) can be expressed by the function $q$:

\begin{equation}
qq_{xt} - q_x q_t = 0.
\end{equation}

It has been discussed a Finslerian approach to the soliton systems [28], i.e. the Lax equation relates to the Euler-Lagrange equation of order one when the function $q$ takes the form:

\begin{equation}
q = e^{\int \sqrt{T}d\psi},
\end{equation}

where $T = T(\psi)$. Moreover, this function can be rewritten as:

\begin{equation}
q = e^{\int \Xi(\psi, \psi_x)dx + \int \Pi(\psi, \psi_t)dt},
\end{equation}

where $\Xi$ and $\Pi$ are differential forms.
where $\Xi = \sqrt{T(\psi)}\psi_2$ and $\Pi = \sqrt{T(\psi)}\psi_t$. In stead of the first order functions $\Xi$ and $\Pi$, we introduce more general form of $q$ which depends on higher-order functions $X$ and $Y$:

$$q = e^{\left\{ \int X(\psi,\psi_\alpha,\psi_\alpha^\beta)dx + \int Y(\psi,\psi_\alpha,\psi_\alpha^\beta)dt + w\psi \right\}}, \tag{33}$$

where $X$ and $Y$ are

$$X = f(\psi)\psi + a\psi_t + b\psi_{tt} + c\psi_{xt}, \tag{34}$$

$$Y = g(\psi)\psi + \hat{a}\psi_x + \hat{b}\psi_{xx} + \hat{c}\psi_{xt}. \tag{35}$$

The coefficients $a, \hat{a}, b, \hat{b}, c, \hat{c}$ and $w$ are constants. Then, it can verify that the Lax equation (31) with the function $q$ (33) is equivalent to the Zermelo condition (24) with the Lagrangian (25) for a relation of coefficients:

$$A_{111} = \hat{b}, A_{222} = b, A_{112} = A_{121} = A_{211} = 1/9, A_{122} = A_{212} = A_{221} = 1/9, A_{11} = \hat{a}, A_{12} = A_{21} = w/4, A_{22} = a, A^1 = g, A^2 = f \text{ and } A = \log \psi. \text{ Thus,}$$

geometrically, the Lax equation can be described in the framework of the higher-order geometry.

4. Conclusions. The nonlinear physical field can be uniquely studied by means of the Kawaguchi space or the higher-order space. In the geometrical theory of nonlinear dynamical systems (the KCC-theory), the second invariant is represented by the Zermelo operator. Under the 2-parameter Zermelo condition with a Lagrangian and a conformal transformation, the soliton equation as the Euler-Lagrange equation can be obtained. In this case, the conformal invariance of the Lagrangian corresponds to the scale invariance of the soliton equation. Moreover, general form for the soliton equation (the Lax equation) can be expressed by the Kawaguchi space.

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