A VANISHING THEOREM FOR VERTICAL TENSOR FIELDS ON COMPLEX FINSLER BUNDLES

BY

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Abstract. Using the Bochner technique for the vertical Laplacian associated to a complex Finsler bundle \((E, L)\) over a hermitian manifold \((M, g)\), we obtain a vanishing theorem for holomorphic vertical tensor fields with compact support on the total space of \((E, L)\).

Mathematics Subject Classification 2000: 53B40, 53B21, 32L20.

Key words: complex Finsler bundles, vertical Laplacian, vanishing theorems.

1. Introduction and preliminaries notions

The importance of Bochner technique, initiated in \([4, 5, 6]\), in geometrical study both in Riemannian and Kählerian manifolds, is without question, and for instance, were discussed in details in \([8, 12, 16, 19, 20, 22]\). The study of existence of holomorphic sections on complex Finsler bundles was initiated in \([13]\). There is obtained an important theorem which states that a holomorphic vector bundle \(E\) with a strongly pseudoconvex complex Finsler metric \(F\) over a compact hermitian manifold \((M, g)\) admits no holomorphic sections if a mean curvature is negative, (see also \([3, 10]\) and \([14]\)). Different from \([13, 14]\), using the Bochner technique for the horizontal Laplacian associated to a strongly Kähler-Finsler manifold, recently in \([24]\), is obtained a vanishing theorem of holomorphic forms on strongly Kähler-Finsler manifolds. This theorem generalizes the classical vanishing theorems of holomorphic tensor fields on Kähler manifolds \([5, 6]\). Also, in \([21]\), are obtained some vanishing theorems of Bochner and Kodaira type for some horizontal forms on strongly Kähler-Finsler manifolds.
In a previous paper [11], we have proved that the vertical Laplacian associated to a complex Finsler bundle \((E, L)\) over a hermitian manifold \((M, g)\) has similar properties as the horizontal Laplacian associated to a strongly Kähler-Finsler manifold. Thus, following the same ideas from [23, 24], in the present paper we obtain a vanishing theorem of Bochner type for holomorphic vertical tensor fields with compact support on the total space of \((E, L)\).

Let us begin our study with a short review of the geometry of the total space of a complex Finsler bundle. For more, see [1, 2, 3, 17].

Let \(\pi : E \to M\) be a holomorphic vector bundle of rank \(m \geq 2\) over a complex manifold \(M\) of \(\text{dim}_{\mathbb{C}} M = n\). Let us consider \(U = \{U_\alpha\}\) an open covering set of \(M\) and \((z^k)\), \(k = 1, \ldots, n\) the local complex coordinates in a chart \((U, \varphi)\) and \(s_U = \{s_a\}, a = 1, \ldots, m\) a local holomorphic frame for the sections of \(E\) over \(U\). The covering \(\{U, s_U\}, U \in U\) induces a complex coordinate system \(\mathbf{u} = (z^k, \eta^a)\) on \(\pi_1^{-1}(U)\), where \(s = \eta^a s_a\) is a holomorphic section on \(E_z = \pi^{-1}(z)\). If we denote by \(g_{UV} : U \cap V \to GL(m, \mathbb{C})\) the transition functions, then in \(z \in U \cap V\), \(g_{UV}(z)\) has a local representation by the complex matrix \(M^a_b(z)\) and if \((z'^k, \eta'^a)\) are the complex coordinates in \(\pi^{-1}(V)\), the transition laws of these coordinates are:

\[
(1.1) \quad z'^k = z'^k(z), \quad \eta'^a = M^a_b(z) \eta^b,
\]

where \(M^a_b(z), a, b = 1, \ldots, m\) are holomorphic functions and \(\det M^a_b \neq 0\).

As we already know, the total space \(E\) has a structure of \((m + n)\)-dimensional complex manifold because the transition functions \(M^a_b(z)\) are holomorphic. Consider the complexified tangent bundle \(T^C E = T' E \oplus T'' E\), where \(T' E\) and \(T'' E = \overline{T'E}\) are the holomorphic and antiholomorphic tangent bundles. The vertical holomorphic tangent bundle \(V E = \ker \pi_*\) is the relative tangent bundle of the holomorphic projection \(\pi\) and a local frame field on \(V_u E\) is \(\{\frac{\partial}{\partial z^k}\}, a = 1, \ldots, m\). The vertical distribution \(V_u E\) is isomorphic to the sections module of \(E\) in \(u\).

A subbundle \(H E\) of \(T' E\) complementary to \(V E\), i.e. \(T' E = V E \oplus H E\), is called a complex nonlinear connection on \(E\), briefly c.n.c. A local base for the horizontal distribution \(H_u E\), called adapted for the c.n.c. is \(\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^a_k \frac{\partial}{\partial \eta^a}\}, k = 1, \ldots, n\), where \(N^a_k(z, \eta)\) are the coefficients of the c.n.c.

As in the case of general theory of vector bundles see [15], we notice that the existence of a c.n.c. is an important ingredient in the "linearization" of geometry of the total space of a holomorphic vector bundle.
In the following we consider the abbreviate notations: $\partial_k = \frac{\partial}{\partial z^k}$; $\delta_k = \frac{\partial}{\partial \zeta^k}$. The adapted frames denoted by $\{\delta_k := \frac{\delta}{\delta z^k}\}$ and $\{\partial_k := \frac{\partial}{\partial \zeta^k}\}$, for the distributions $\mathcal{V}E$ and $\mathcal{H}E$ are obtained respectively by conjugation. The adapted coframes are locally given by $\{dz^k\}$, $\{\delta \eta^a = d\eta^a + N^a_k dz^k\}$, $\{d\zeta^k\}$ and $\{\delta \eta^a = d\eta^a + N^a_k d\zeta^k\}$, respectively.

A strictly pseudoconvex complex Finsler structure on $E$, is a positive real valued smooth function $F^2 = L : E \to \mathbb{R}_+ \cup \{0\}$, which satisfies the following conditions:

(i) $L$ is smooth on $E - \{\text{zero section}\}$;
(ii) $L(z, \eta) \geq 0$ and $L(z, \eta) = 0$ if and only if $\eta = 0$;
(iii) $L(z, \lambda \eta) = |\lambda|^2 L(z, \eta)$ for any $\lambda \in \mathbb{C}$;
(iv) $(h^a_b) = (\delta_a \delta_b (L))$ (the complex Levi matrix) is positive defined and determines a hermitian metric tensor on the fibers of vertical bundle $VE$.

**Definition 1.1** ([13]). The pair $(E, L)$ is called a complex Finsler bundle.

According to [2], [17], a c.n.c. related only to the complex Finsler structure $L$ is the Chern-Finsler c.n.c., locally given by

\begin{equation}
(1.2)
\end{equation}

\[ C^F_k = h^a_a \partial_k \delta_a (L). \]

We also identify the holomorphic local frame fields $s = \{s_1, \ldots, s_m\}$ of $E$, with the one of the pull-back bundle $\tilde{E} := \pi^* E$ which is isomorphic to $VE$ by $\delta_a \leftrightarrow \pi^* s_a := s_a$. Then, $\tilde{E}$ admits a hermitian metric $h_L$ defined by

\begin{equation}
(1.3)
\end{equation}

\[ h_L(Z, W) = h_{ab} Z^a \overline{W^b} \text{ for any } Z = Z^a s_a, W^a s_a \in \Gamma(\tilde{E}). \]

Let $\nabla : \Gamma(\tilde{E}) \to A^1(\tilde{E})$ be the hermitian connection of the bundle $(\tilde{E}, h_L)$, i.e. $\nabla = \nabla' + \nabla''$ is the unique connection on the bundle $(\tilde{E}, h_L)$ satisfying the conditions

\begin{equation}
(1.4)
\end{equation}

\[ \nabla'' = d''; \quad dh_L(Z, W) = h_L(\nabla Z, W) + h_L(Z, \nabla W), \quad \forall Z, W \in \Gamma(\tilde{E}). \]

**Definition 1.2** ([1]). The hermitian connection $\nabla$ on $(\tilde{E}, h_L)$ is called the Chern-Finsler linear connection of the bundle $(E, L)$. 
The \((1,0)\)-connection form \(\omega = (\omega^a_b)\) of \(\nabla\), with respect to a holomorphic frame \(s = \{s_a\}, a = 1, \ldots, m\), is defined by
\[
\nabla s_b = \omega^a_b \otimes s_a, \quad \omega^a_b = \Gamma^a_{bk} dz^k + C^a_{bc} d\eta^c,
\]
where the local coefficients of the connection are given by
\[
\Gamma^a_{bk} = h^a_{ca} \partial_k (h_{bc}), \quad C^a_{bc} = h^a_{da} \partial_c (h_{bd}).
\]
Using the adapted frames and coframes with respect to the Chern-Finsler c.n.c., the \((1,0)\)-connection form \(\omega^a_b\) can be rewritten by
\[
\omega^a_b = L^a_{bk} dz^k + C^a_{bc} \delta \eta^c, \quad L^a_{bk} = h^a_{ca} \delta_k (h_{bc}).
\]
We notice that the vertical coefficients \(C^a_{bc}\) satisfy the symmetry relation \(C^a_{bc} = C^a_{cb}\) and it is easy to check that \(C^a_{ab} = \partial_b (\ln h)\) where \(h = \det(h_{ab})\).

The \((1,1)\)-curvature form \(R = (R^a_b)\) of \(\nabla\) is locally given by
\[
R^a_b = d'' \omega^a_b.
\]
Using the decomposition \(d'' = d''^h + d''^v + \partial_1 + \partial_2\), see [9], [18], a straightforward calculus in (1.8) leads to the following decomposition of the curvature
\[
R = R^{\bar{h}\bar{b}} + R^{h\bar{b}} + R^{\bar{v}\bar{b}} + R^{v\bar{b}},
\]
i.e. in a horizontal component \(R^{\bar{h}\bar{b}}\), in two mixed components \(R^{h\bar{b}}, R^{\bar{v}\bar{b}}\) and a vertical component \(R^{v\bar{b}}\). Also, we notice that with respect to a holomorphic local frame field \(s = \{s_a\}\) the vertical part of the curvature is given by
\[
R^{v\bar{b}} s_b = (S^a_{b,\bar{c}} \delta \eta^c \wedge \delta \eta^d) s_a,
\]
where \(S^a_{b,\bar{c}} = - \partial_\bar{c} (C^a_{bc})\) and it will be important in our study.

2. Vertical Ricci identities

In this section, we briefly recall the vertical covariant derivatives with respect to the Chern-Finsler connection for contravariant and covariant vertical tensor fields. Also, some vertical Ricci identities are established.
If $T_{ApBq}(z, \eta)$ are the components of a contravariant vertical complex tensor field of type $(p, q)$ on $E$, where $A_p = (a_1 \ldots a_p)$ and $B_q = (b_1 \ldots b_q)$, then its vertical covariant derivatives with respect to the Chern-Finsler connection are given by

\[(2.1) \quad \nabla_{\partial a} T_{ApBq} = \dot{\partial}_a (T_{ApBq}) + \sum_{k=1}^{p} T_{a_1 \ldots a_{k-1} \lambda a_{k+1} \ldots a_p Bq} C_{\lambda a}^{a_k},\]

\[(2.2) \quad \nabla_{\partial \pi} T_{ApBq} = \dot{\partial}_{\pi} (T_{ApBq}) + \sum_{k=1}^{q} T_{Apb_1 \ldots b_{k-1} \lambda b_{k+1} \ldots b_q Bq} C_{Bq}^{b_k}.\]

If $T_{ApBq}(z, \eta)$ are the components of a covariant vertical complex tensor field of type $(p, q)$ on $E$, then its vertical covariant derivatives with respect to the Chern-Finsler connection are given by

\[(2.3) \quad \nabla_{\partial a} T_{ApBq} = \dot{\partial}_a (T_{ApBq}) - \sum_{k=1}^{p} T_{a_1 \ldots a_{k-1} \lambda a_{k+1} \ldots a_p Bq} C_{\lambda a}^{a_k},\]

\[(2.4) \quad \nabla_{\partial \pi} T_{ApBq} = \dot{\partial}_{\pi} (T_{ApBq}) - \sum_{k=1}^{q} T_{Apb_1 \ldots b_{k-1} \lambda b_{k+1} \ldots b_q Bq} C_{Bq}^{b_k}.\]

We remark that if we combine the formulas above, we get the vertical covariant derivatives for mixed contravariant and covariant vertical complex tensor fields. Also, it is easy to check

\[(2.5) \quad \nabla_{\partial c} h_{\alpha \bar{\beta}} = \nabla_{\partial \alpha} h_{\bar{\beta} \alpha} = \nabla_{\partial \alpha} h_{\bar{\beta} \alpha} = \nabla_{\partial \alpha} h_{\bar{\beta} \alpha} = 0.\]

Now, by using the above vertical covariant derivatives with respect to Chern-Finsler connection, a straightforward calculus leads to

**Proposition 2.1.** Let $(E, L)$ be a complex Finsler bundle, $X^a$ the components of a contravariant vertical complex tensor field $X$ on $E$ and $\varphi_a$ the components of a covariant vertical complex tensor field $\varphi$ on $E$. Then

\[\left[\nabla_{\partial \alpha}, \nabla_{\partial \beta}\right] X^a = X^d S_{d,a,\beta}^a, \quad \left[\nabla_{\partial \alpha}, \nabla_{\partial \beta}\right] X^\pi = -X^d S_{d,a,\beta}^\pi,\]

\[\left[\nabla_{\partial \alpha}, \nabla_{\partial \beta}\right] \varphi_a = -\varphi_d S_{d,a,\beta}^a, \quad \left[\nabla_{\partial \alpha}, \nabla_{\partial \beta}\right] \varphi^\pi = \varphi_d S_{d,a,\beta}^{\pi}.\]

If we denote by $S_{a_5,a_6} = h_{a_5} S_{a_5 a_6}^{a_7}$, then, by using the homogeneity conditions of complex Finsler structure $L$, namely

\[\dot{\partial}_c (h_{a \bar{\beta}}) \eta^c = 0, \quad \dot{\partial}_c (h_{a \bar{\beta}}) \eta^a = 0, \quad \dot{\partial}_{\pi} (h_{a \bar{\beta}}) \eta^b = 0, \quad \dot{\partial}_{\pi} (h_{a \bar{\beta}}) \eta^c = 0,\]

we obtain
Proposition 2.2. Let \((E, L)\) be a complex Finsler bundle. Then

\[
(S_{\bar{a},\bar{c}}^a - S_{\bar{b},\bar{a}}^c)\eta^a \bar{\eta}^b = 0, \tag{2.6}
\]
\[
S_{\bar{a}}^c \bar{a} - S_{\bar{c}, \bar{a}}^a = 0, \tag{2.7}
\]
\[
(S_{\bar{a}, \bar{c}}^a - S_{\bar{d}, a\bar{d}})\eta^a \bar{\eta}^b \bar{\eta}^c \bar{\eta}^d = 0. \tag{2.8}
\]

Definition 2.1. The vertical generalized Ricci tensor field of \((E, L)\) is locally defined by

\[
S_{a\bar{d}} = S_{\bar{a}, \bar{d}}^b = -\partial_a (C_{ab})^b. \tag{2.9}
\]

We also have

\[
S_{a\bar{d}} = h^{ac} S_{a\bar{c}, \bar{a}}^d = -\partial_a (\ln h). \tag{2.10}
\]

3. A vanishing theorem

Let us suppose that the base manifold of the complex Finsler bundle \((E, L)\) is a hermitian manifold \((M, g)\). Then, due to [7], in natural manner we can consider the following hermitian metric structure of Sasaki type on \(E\)

\[
G = g_{\bar{z}}(z) dz^j \otimes d\bar{z}^k + h_{ab}(z, \eta) \delta \eta^a \otimes \delta \bar{\eta}^b, \tag{3.1}
\]

where \(g_{\bar{z}}(z)\) is the hermitian metric on the base manifold \(M\), \(h_{ab}(z, \eta)\) is the fundamental metric tensor defined by the complex Finsler structure \(L\) and the adapted coframes are considered with respect to the Chern-Finsler c.n.c.

In [11], is proved that with respect to the hermitian metric structure from (3.1) the vertical complex Laplacians for smooth functions on \(E\) are given by

\[
\Box' f = \frac{1}{h} \partial_a (hh^{\bar{a}} \partial_{\bar{a}} f) \quad \text{and} \quad \Box'' f = \frac{1}{h} \partial_{\bar{a}} (hh^{\bar{a}} \partial_{\bar{a}} f). \tag{3.2}
\]

Remark 3.1. In terms of the vertical covariant derivatives with respect to the Chern-Finsler linear connection, we have

\[
\Box' f = h^{ba} \nabla_{\partial_a} \nabla_{\partial_b} f. \tag{3.3}
\]
Also, if \( f \) is a smooth function with compact support on \( E \) then, we have

\[
\int_E \Box^v f dV_E = 0,
\]

where \( dV_E \) is the volume form associated to the hermitian structure \( G \) on \( E \), see for details [11].

Now, let \( X^{A_p}_{B_q}(z, \eta) \) be the components of a mixed vertical complex tensor field \( X \) with compact support on \( E \) of contravariant valency \( p \) and of covariant valency \( q \). In the following we denote by \( ||X||^2 \) the local scalar product of this tensor with itself with respect to the natural inner product induced by the hermitian metric structure \( G \) on \( E \), that is

\[
||X||^2 = h_{A_pC_p}h^{D_qB_q}X^{A_p}_{B_q}X^{C_p}_{D_q}
\]

where \( h_{A_pC_p} = h_{a_1\bar{a}_1}\ldots h_{a_p\bar{a}_p} \) and \( h^{D_qB_q} = h_{\bar{d}_1b_1}\ldots h_{\bar{d}_q b_q} \).

If \( X^{A_p}_{B_q}(z, \eta) \) are holomorphic functions of the fiber coordinates \( (\eta^a) \), then

\[
\nabla_{\frac{\partial}{\partial \eta}} X^{A_p}_{B_q} = 0.
\]

We have

**Proposition 3.1.** Let \((E, L)\) be a complex Finsler bundle over a hermitian manifold \((M, g)\). Then

\[
\Box^v ||X||^2 = ||\nabla^v X||^2 + G^v(X),
\]

where we have denoted \( ||\nabla^v X||^2 = h^{5a}h_{A_pC_p}h^{D_qB_q}(\nabla_{\frac{\partial}{\partial \eta}} X^{A_p}_{B_q})(\nabla_{\frac{\partial}{\partial \eta}} X^{C_p}_{D_q}) \),

\[
G^v(X) = -\sum_{k=1}^{p} S_{\lambda_k}^{\bar{X}_{C_p}} X^{D_q}_{C_p} X^{\bar{X}_{\bar{d}_1k+1\ldots \bar{d}_q}} + \sum_{k=1}^{q} S_{\lambda_k}^{\bar{X}_{D_q}} X^{C_p}_{D_q} X^{\bar{X}_{\bar{d}_1k+1\ldots \bar{d}_q}},
\]

and \( X^{D_q}_{C_p} = h_{A_pC_p}h^{D_qB_q}X^{A_p}_{B_q} \), \( S_{\lambda_k}^{\bar{X}_{C_p}} = h_{5a}S_{\lambda_k}^{\bar{X}_{\bar{d}_1k+1\ldots \bar{d}_q}} \).

**Proof.** We have

\[
\Box^v ||X||^2 = h^{5a} \nabla_{\frac{\partial}{\partial \eta}} \nabla_{\frac{\partial}{\partial \eta}} (h_{A_pC_p}h^{D_qB_q}X^{A_p}_{B_q}X^{C_p}_{D_q})
\]

\[
= h^{5a}h_{A_pC_p}h^{D_qB_q} \nabla_{\frac{\partial}{\partial \eta}} (X^{A_p}_{B_q} \nabla_{\frac{\partial}{\partial \eta}} X^{C_p}_{D_q})
\]

\[
= h^{5a}h_{A_pC_p}h^{D_qB_q} \nabla_{\frac{\partial}{\partial \eta}} X^{A_p}_{B_q} \nabla_{\frac{\partial}{\partial \eta}} X^{C_p}_{D_q} + X^{A_p}_{B_q} \nabla_{\frac{\partial}{\partial \eta}} X^{C_p}_{D_q}
\]

\[
+ \sum_{k=1}^{p} X^{\bar{X}_{C_p}} X^{\bar{X}_{\bar{d}_1k+1\ldots \bar{d}_q}} C_{\lambda_k}^{\bar{X}_{\bar{d}_1k+1\ldots \bar{d}_q}} B_{\lambda_k}^{\bar{X}} = \sum_{k=1}^{q} X^{\bar{X}_{D_q}} X^{\bar{X}_{\bar{d}_1k+1\ldots \bar{d}_q}} C_{\lambda_k}^{\bar{X}_{\bar{d}_1k+1\ldots \bar{d}_q}} B_{\lambda_k}^{\bar{X}}
\]
Let $b \in L^2_{\text{loc}}(X, \mathbb{R})$. If $X = \sum_{a} X^a \frac{\partial}{\partial x^a}$, then

$$\nabla_{\partial b} X^a = 0.$$ 

which completes the proof. \hfill \Box

**Theorem 3.1.** Let $(E, L)$ be a complex Finsler bundle over a hermitian manifold $(M, g)$ and $X^A_{B_q}(z, \eta)$ be the components of a mixed vertical complex tensor field with compact support on $E$. If $X^A_{B_q}$ are holomorphic functions of fiber coordinates $(\eta^a)$ and satisfy the inequality $\Re G^v(X) \geq 0$ then $G^v(X) = 0$ and $\nabla_{\partial a} X^A_{B_q} = 0$ at every point $(z, \eta) \in E$.

Finally, we give some particularizations of the above theorem.

**Proposition 3.2.** Let $(E, L)$ be a complex Finsler bundle over a hermitian manifold $(M, g)$ and $X^a(z, \eta)$ the components of a contravariant vertical complex tensor field with compact support on $E$. Then

(i) If $X^a$ are holomorphic functions of $(\eta^a)$ coordinates and $\Re S^a_{b} X^a X^b \leq 0$ then $\nabla_{\partial b} X^a = 0$;

(ii) If $X^a$ are holomorphic functions of $(\eta^a)$ coordinates and $\Re S^a_{b} X^a X^b < 0$ then $X^a = 0$.

**Proposition 3.3.** Let $(E, L)$ be a complex Finsler bundle over a hermitian manifold $(M, g)$ and $\varphi_a(z, \eta)$ the components of a covariant vertical complex tensor field with compact support on $E$. If we denote by $S^a_{b} = h^{c\bar{d}} h^p_{a\bar{q}} S^c_{b} \mathcal{E}_{c\bar{d}p \bar{q}}$, then

(i) If $\varphi_a$ are holomorphic functions of $(\eta^a)$ coordinates and $\Re S^a_{b} \varphi_a \overline{\varphi_b} \geq 0$ then $\nabla_{\partial b} \varphi_a = 0$;

(ii) If $\varphi_a$ are holomorphic functions of $(\eta^a)$ coordinates and $\Re S^a_{b} \varphi_a \overline{\varphi_b} > 0$ then $\varphi_a = 0$.

**Remark 3.2.** If the complex Finsler structure comes from a hermitian structure on $E$, that is $L(z, \eta) = h_{ab}(z) \eta^a \eta^b$, then $C^u_{bc} = 0$ and so $G^v(X) = 0$. In this case, if $X^A_{B_q}(z, \eta)$ are the components of a mixed vertical complex
tensor field with compact support on $E$ and if $X^{A_p}_{B_q}$ are holomorphic functions of $(\eta^a)$ coordinates, then $\nabla_{\partial_b} X^{A_p}_{B_q} = \partial_b X^{A_p}_{B_q} = 0$, so $X^{A_p}_{B_q} = X^{A_p}_{B_q}(z)$. But on the other hand, if this happens, then obviously $X^{A_p}_{B_q}$ doesn’t have compact in $E$. This contradicts the assumption that $X^{A_p}_{B_q}$ are compactly supported in $E$, so $X^{A_p}_{B_q} = 0$.

Acknowledgments. The author would like to thank the anonymous referee for his/her suggestions and comments that helped us improve this article.

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Revised: 21.II.2011