ON SOME ASPECTS OF THE $J$-SPECTRAL FACTORIZATION

BY

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Abstract. The $J$-spectral factorization problem naturally arises in control theory and it plays an important role in $H_\infty$-control, linear quadratic optimal control, Hankel norm approximation problem. Characterization of solution for control problems is sometimes given using the $J$-spectral factor(s). The paper has the nature of a survey article. We review the band method version of the Grassmannian approach for solving strictly contractive extension problems and the $J$-spectral factorization approach for solving the suboptimal Nehari problem in the setting of the Wiener algebra on the imaginary axis. The new (and modest) contributions is to clarify the connections between the two approaches.

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1. Introduction

Let $G$ be a bounded transfer function, and assume that $\sigma$ is larger than the biggest singular value of the Hankel operator $H_G$. Roughly speaking, the suboptimal Nehari problem (SNP) is the following: Find a matrix-valued function $K$ without poles in the closed right half-plane such that $\|G + K\|_\infty \leq \sigma$, where $\| \cdot \|_\infty$ denotes the $L_\infty$-norm. Note that the above is the frequency domain formulation for the SNP as used in the systems and control community.

An alternative formulation (the Grassmannian approach) for the SNP, using strictly contractive extensions, is known in the literature. Given any $k$ in a $C^*$-algebra, find $f$, a strictly contractive extension for $k$ (see Section 2 for the precise definition). The methods for solving a variety of extension problems has evolved into increasing levels of sophistication over the
past two decades (see [3] for some recent results and a short overview). The article [7] enhances the Grassmannian version of the band method to handle the Nehari-Takagi problem rather than merely the Nehari problem. Remark that SNP is a particular case of the Hankel-norm approximation problem (HNAP) presented in [2]. The SNP (and HNAP) has been studied extensively in the literature ([1], [5], [4], [11]).

Two approaches to solving the SNP are reviewed in this paper: the Grassmannian approach considered in [2] (extended in [7]) and the $J$-spectral factorization approach taken in [9]. In the more general setting (as compared to the one in [2]) presented in [7], one has a family of problems (indexed by the nonnegative integers $l$) and in the special case when $l = 0$, one gets the strictly contractive extension problem of [2]). Thus, the setting in [7] can be thought of as a more refined version of [2]). Since a $J$-spectral factor for $W$ given in (3.10) always exists, the situation for the Wiener algebra is a simplified version of the general result in $C^*$-algebras. The relations (results and assumption) between the Grassmannian approach and the $J$-spectral factorization approach are analyzed. As a consequence one may remark that the $J$-spectral factorization approach is a reasonably accessible step on the path towards understanding the Grassmannian approach since it eliminates some conditions necessary for the Grassmannian approach. Moreover, using this explanatory note, control applications with transfer functions arising from multi-input multi-output systems as well as time varying and time-delay finite-dimensional systems may benefit from the abstract Grassmannian framework for solving strictly contractive extension problems.

The paper consists of four sections, the introduction being the first. In the second section the band method version of the Grassmannian approach for solving strictly contractive extension problems is reviewed. The $J$-spectral factorization approach for solving the suboptimal Nehari problem in the setting of the Wiener algebra on the imaginary axis is reviewed in the third section. In the final section, again in the Wiener algebra setting, the author it is shown how the $J$-spectral factorization approach can be put into a Grassmannian setting and vice versa.

2. The Grassmannian approach to the Nehari problem

This section follows the set up used in [2]. Given a unital $C^*$-algebra $\mathcal{R}$ with unit $e$, a subalgebra $\mathcal{N}$ of $\mathcal{R}$, a subalgebra $\mathcal{N}_+$ of $\mathcal{N}$, and a linear
submanifold $N_1$ of $N$. Assume that the unit $e$ of $R$ is in $N_+$ and $N_1$ is a right module over $N_+$, i.e.,

\[(2.1) \quad e \in N_+, \quad N_1 N_+ \subset N_1.\]

Furthermore, we fix an element $k \in N$. The above notation will be kept fixed throughout this section. Note that we do not require $N_1$ to be a subset of $N_+$.

An element $g \in R$ is said to be \textit{positive definite} in $R$ (notation $g >_R 0$) if $g = a^* a$ for some invertible element $a \in R$. We call $g \in R$ \textit{strictly contractive} if $e - g^* g$ is positive definite or, equivalently, if $\|g\|_R < 1$, where $\| \cdot \|_R$ denotes the norm on $R$.

We call $f \in R$ a \textit{strictly contractive extension} of $k$ if

\[(2.2) \quad k - f \in N_1 \text{ and } \|f\|_R < 1.\]

Notice that strictly contractive extensions $f$ of $k$ are elements in $N$.

In [2] linear fractional representations of all strictly contractive extensions of $k$ were derived. An additional connection between $N_+$ and $R$ was used:

**Axiom 1.** If $g \in N_+$ and $\|g\|_R < 1$, then $(e - g)^{-1} \in N_+$.

Further, consider a $2 \times 2$ block matrix $\Theta$ with entries in $R$,

\[(2.3) \quad \left( \begin{array}{cc} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{array} \right) \in R^{2 \times 2}.\]

One can see $\Theta$ as the coefficient matrix of a linear fractional map, i.e.

\[(2.4) \quad F_\Theta(h) := (\Theta_{11} h + \Theta_{12})/(\Theta_{21} h + \Theta_{22})^{-1},\]

where $h$ is a free parameter. One shall consider coefficient matrices $\Theta$ satisfying the following additional conditions:

\begin{align*}
(CM1) \quad & \Theta \left( \begin{array}{c} N_1 \\ N_+ \end{array} \right) = \left( \begin{array}{cc} e & k \\ 0 & e \end{array} \right) \left( \begin{array}{c} N_1 \\ N_+ \end{array} \right), \\
(CM2) \quad & \Theta_{22} \text{ is invertible and } \Theta_{22}^{-1} \in N_+, \\
(CM3) \quad & \Theta^* \left( \begin{array}{cc} e & 0 \\ 0 & -e \end{array} \right) \Theta = \left( \begin{array}{cc} p & 0 \\ 0 & -q \end{array} \right).
\end{align*}

Here $p$ and $q$ are positive elements in $R$. The following two lemmas are proved using only ”half” of the condition $(CM2)$ and the condition $(CM3)$. 
Lemma 2.1. Let $\Theta$ be as in (2.3) and assume that $\Theta_{22}$ is invertible in $\mathcal{R}$. Put $f = \Theta_{12}\Theta_{22}^{-1}$. If, in addition, condition $(CM3)$ holds, then
\begin{align}
\Theta_{21} &= f^*\Theta_{11}, \\
\Theta_{11}(e - ff^*)\Theta_{11} &= p.
\end{align}

Lemma 2.2. Let $\Theta$ be as in (2.3) and assume that $\Theta_{22}$ is invertible in $\mathcal{R}$. If, in addition, condition $(CM3)$ holds, then $\Theta$ is invertible in $\mathcal{R}^{2\times 2}$ if and only if $\Theta_{11}$ is invertible in $\mathcal{R}$.

The following theorems hold (see Theorem 2.4(b) and Theorem 2.2(b) in [2]).

Theorem 2.3. Assume that Axiom 1 holds. There exists $\Theta$ as in (2.3) invertible in $\mathcal{R}^{2\times 2}$, satisfying the conditions $(CM1)$, $(CM2)$ and $(CM3)$ if and only if $k$ has a strictly contractive extension $f$ with the following properties
\begin{enumerate}
\item[(P1)] $e - ff^* = u^{-1}qu$ with $u, v \in N_+$,
\item[(P2)] $e - f f^* = v^{-1}pv$ with $v \in \mathcal{R}$ and $p^{-1}v^*N_1 = N_1$,
\item[(P3)] $(e - f f^*)^{-1}f^*N_1 \subset N_+$.
\end{enumerate}
In fact, if $\Theta$ is invertible in $\mathcal{R}^{2\times 2}$ and it satisfies $(CM1)$, $(CM2)$ and $(CM3)$, then
\begin{equation}
f = \Theta_{12}\Theta_{22}^{-1}
\end{equation}
is a strictly contractive extension of $k$ and the statements (P1), (P2) and (P3) hold with $u = \Theta_{22}$ and $v = \Theta_{11}$.

Conversely, if $f$ is a strictly contractive extension of $k$ satisfying (P1), (P2) and (P3), then $\Theta = \begin{bmatrix}
v & f^*u \\
v^*f & u
\end{bmatrix}$ is invertible in $\mathcal{R}^{2\times 2}$ and $\Theta$ satisfies $(CM1)$, $(CM2)$ and $(CM3)$.

Theorem 2.4. Assume that Axiom 1 holds. If the block matrix $\Theta$ defined in (2.3) satisfies the conditions $(CM1)$, $(CM2)$ and $(CM3)$, then each strictly contractive extension $f$ of $k$ is of the form
\begin{equation}
f = \mathcal{F}_\Theta(h),
\end{equation}
where the free parameter $h$ is an arbitrary element in $\mathbb{N}_1$ such that $q - h^* ph$ is positive definite in $\mathbb{R}$. Moreover, the map $\mathcal{F}_q$ provides one-to-one correspondence between all such $h$ and all strictly contractive extension $f$ of $k$.

3. A $J$-spectral factorization approach to the Nehari problem

This section follows the set up used in [10]. For the Wiener algebra on the imaginary axis a frequency domain solution for the SNP was obtained. The approach is via $J$-spectral factorization, and it uses the concept of equalizing vectors and frequency-domain techniques.

3.1. The Wiener algebra on the imaginary axis

We use the notation $\mathcal{L}_1(\mathbb{R})$ for the Banach space of Lebesgue integrable functions on the real line. The Banach space $\mathcal{L}_1(\mathbb{R})$ is a commutative Banach algebra with convolution "$*$" as multiplication, and it does not have a unit [6]. Let $C(\hat{\mathbb{C}}_0)$ be the algebra of continuous functions on the one point compactified imaginary axis $\hat{\mathbb{C}}_0 := \mathbb{C}_0 \cup \{\infty\}$. If $f \in \mathcal{L}_1(\mathbb{R})$, then the Laplace transform of $f$ is the function $\hat{f} \in C(\hat{\mathbb{C}}_0)$ such that $\lim_{j \to 1} f(s) = 0$, defined by $\hat{f}(s) := \int_{-\infty}^{\infty} e^{-st} f(t) dt$.

The Wiener algebra on the imaginary axis $\hat{\mathcal{W}}(\hat{\mathbb{C}}_0)$, or simply $\hat{\mathcal{W}}$, is the algebra of functions in $C(\hat{\mathbb{C}}_0)$ of the form $g = \hat{f} + c$, where $f \in \mathcal{L}_1(\mathbb{R})$ and $c$ is a complex constant. The norm of the element $g$ is defined by $\|g\|_{\hat{\mathcal{W}}(\hat{\mathbb{C}}_0)} := |c| + \int_{-\infty}^{\infty} |f(t)| dt$. The algebra of rational functions is dense in $\hat{\mathcal{W}}(\hat{\mathbb{C}}_0)$. Transfer functions of delay systems are included in $\hat{\mathcal{W}}(\hat{\mathbb{C}}_0)$.

Any $g \in \hat{\mathcal{W}}(\hat{\mathbb{C}}_0)$ can be decomposed as $g = \hat{f} + c = g_- + c + g_+$, where $g_-(s) = \int_{-\infty}^{0} e^{-st} f(t) dt$ and $g_+(s) = \int_{0}^{\infty} e^{-st} f(t) dt$. If $\hat{A}$ denotes the causal Wiener algebra on the imaginary axis $\hat{A} := \{g \in \hat{\mathcal{W}} \mid g_- = 0\}$ and $\hat{A}_0^\ast$ the anti-causal Wiener algebra on the imaginary axis $\hat{A}_0^\ast := \{g \in \hat{\mathcal{W}} \mid g_+ = 0, c = 0\}$, then $\hat{\mathcal{W}} = \hat{A}_0^\ast \oplus \hat{A}$, where "$\oplus$" denotes the direct sum. Note that the condition $c = 0$ in the definition of $\hat{A}_0^\ast$ is the same with $g(\infty) = 0$.

In what follows, for any Banach algebra $\mathcal{B}$, consider matrices with entries in $\mathcal{B}$. Endow $\mathcal{B}^{m \times m}$ with the usual matrix operations and the maximum norm. Then $\mathcal{B}^{m \times m}$ is also a Banach algebra. If $\mathcal{B}$ has a unit $e$, then $\mathcal{B}^{m \times m}$ also has a unit, namely the $m \times m$ matrix $I$ with $e$ on the diagonal and zeros elsewhere. The precise definition of the norm on $\mathcal{B}^{m \times m}$ is not so important, and for some cases there are other more natural norms. An important result
says that in a unital Banach algebra, an element with its entries commuting one with the other is invertible if and only if its determinant is invertible.

3.2. The $J$-spectral factorization in $\hat{W}(\hat{C}_0)$

In this section we are interested in the $J$-spectral factorization of matrix-valued functions in the Wiener algebra. Consider the signature matrix

$$J_{m_+,m_-} = \begin{bmatrix} I_{m_+} & 0 \\ 0 & -I_{m_-} \end{bmatrix},$$

where $m_-, m_+ \in \mathbb{N}$. Sometimes simply use $J$ without indices.

**Definition 3.1.** Let $Z = Z^* \in \hat{W}^{n \times n}$. Then $Z$ has a $J$-spectral factorization if there exists an $X \in \hat{A}^{n \times n}$ such that

1. $X^{-1}$ exists and $X^{-1} \in \hat{A}^{n \times n}$,
2. $Z(s) = X^*(s)JX(s)$ for all $s \in \hat{C}_0$.

Such a $X$ is called a $J$-spectral factor of $Z$.

Necessary and sufficient conditions for the existence of a $J$-spectral factorization in the Wiener algebra can be found in [9]. The $J$-spectral factor is unique up to a multiplication with a $J$-unitary constant matrix $Q$.

3.3. A $J$-spectral factorization approach to SNP

For the Wiener algebra on the imaginary axis a simple frequency domain solution for the SNP was obtained in [10]. In this subsection some of the main results and other important auxiliary results are presented.

Denote $G(s)^\sim := [G(-\bar{s})]^*$, where $\bar{s}$ is the complex conjugate of $s$. Note that $G(s)^\sim = G(s)^*$ on the imaginary axis. A first theorem provides the existence of some $J$-spectral factorization [8, Theorem 3.2].

**Theorem 3.2.** Let $G \in \hat{W}^{k \times m}$ be a matrix-valued function of a complex variable such that $G^\sim \in \hat{A}^{m \times k}$ and $\sigma$ a positive real number which satisfies $\sigma_{l+1} < \sigma < \sigma_1$ or $\sigma_1 < \sigma$. Then there exists a $(k+m) \times (k+m)$-matrix-valued function $\Lambda \in \hat{A}$ such that $W(s)$, defined by

$$W(s) = \begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix}^* J_{\sigma,k,m} \begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix},$$

has the $J_{k,m}$-spectral factorization

$$W(s) = \Lambda(s)^\sim J_{k,m} \Lambda(s).$$
Moreover, if \( G \) is strictly proper, then \( \Lambda \) can be chosen such that

\[
\lim_{|s| \to \infty} \Lambda(s) = \begin{bmatrix} I_k & 0 \\ 0 & \sigma I_m \end{bmatrix}.
\]

Further, a solution for the SNP can be obtained.

**Lemma 3.3.** Let \( G \) be such that \( G^\sim \in \mathring{\mathcal{A}}_{0}^{k \times m} \). Then \( \Lambda_{11} \) is invertible in \( \mathcal{W} \).

**Lemma 3.4.** Let \( \Lambda \) be a \( J \)-spectral factor for \( W \) such that \( \Lambda_{11} \) is invertible in \( \mathcal{W} \) and \( \Lambda_{11}^{-1} \in \mathring{\mathcal{A}} \). Then

\[
(3.12) \quad K_0 = V_1 V_2^{-1}
\]

is a solution for the SNP.

Using the fact that the SNP is trivial for \( G \in \mathring{\mathcal{A}}_{0}^{k \times m} \), one can restrict this problem, without loss of generality, to matrix-valued functions \( G \in (\mathring{\mathcal{A}}_{0}^{k \times m})^\sim \). Moreover, a parametrization of all solutions for the SNP is provided [10, Theorem 3.1].

**Theorem 3.5.** Let \( G \) be such that \( G^\sim \in \mathring{\mathcal{A}}_{0}^{k \times m} \), and \( \sigma \) a positive real number. The following statements are equivalent:

1. \( \| H_G \| < \sigma \).
2. There exists \( K(s) \in \mathring{\mathcal{A}}^{k \times m} \) such that

\[
(3.13) \quad \| G + K \|_\infty < \sigma.
\]
3. There exists \( \Lambda(s) \in \mathring{\mathcal{A}}^{(k \times m) \times (k \times m)} \) a \( J \)-spectral factor for \( W \) (defined in (3.10)) with \( \Lambda_{11}^{-1}(s) \in \mathring{\mathcal{A}}^{k \times k} \).

**Theorem 3.6.** Let \( G \) be such that \( G^\sim \in \mathring{\mathcal{A}}_{0}^{k \times m} \), and \( \sigma \) a positive real number. Suppose that any of the statements of the previous theorem holds. Then, all solutions for the SNP are parameterized by \( K(s) = X_1(s)X_2(s)^{-1} \), where

\[
(3.14) \quad \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \Lambda(s)^{-1} \begin{bmatrix} Q(s) \\ I_m \end{bmatrix},
\]

with \( Q(s) \in \mathring{\mathcal{A}}^{k \times m} \), \( \| Q \|_\infty < 1 \).
4. Connections between the Grassmannian approach and the \( J \)-spectral factorization approach

In this section only the Wiener algebra (recall that systems with delays are included) is considered, and several relations between the results stated in Section 2 and 3 are analyzed.

Let \( G \in \hat{W}^{k \times m} \) be such that \( G^\sim \in \hat{A}^{m \times k} \). From Theorem 3.2 it follows that there exists \( \Lambda \), a \( J \)-spectral factor of \( W \) defined in (3.10). That is, \( \Lambda \) is invertible over \( \hat{A} \) and the equality (3.11) holds.

Consider the following particular case in the Grassmannian approach: \( R = \mathcal{C}(\hat{C}_0) \), \( N_+ = N_1 = \hat{A} \), \( e = I \). It is easy to see that the relations in (2.1) hold and Axiom 1 is satisfied. The correspondence between the notations is:

\[
\begin{align*}
(4.15) & \quad k \leftrightarrow G, \\
(4.16) & \quad f \leftrightarrow G + K, \\
(4.17) & \quad \Theta \leftrightarrow \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \Lambda^{-1}, \\
(4.18) & \quad h \leftrightarrow Q,
\end{align*}
\]

where Theorem 3.2 was used for the existence of a \( J \)-spectral factor \( \Lambda \) of \( W \) defined in (3.10). If one writes \( \Theta \) and \( V \), the inverse of \( \Lambda \), as 2 \( \times \) 2 block matrices, the correspondence (4.17) is

\[
(4.19) \quad \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} V_{11} + GV_{21} & V_{12} + GV_{22} \\ V_{21} & V_{22} \end{bmatrix},
\]

where, \( \Lambda_{11} \) is invertible (see Lemma 3.3), \( V \) is given by

\[
(4.20) \quad \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} \Lambda_{11}^{-1} + E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix},
\]

with \( \Delta = \Lambda_{22} - \Lambda_{21}\Lambda_{11}^{-1}\Lambda_{12} \), the Schur complement of \( \Lambda_{11} \), \( E = \Lambda_{11}^{-1}\Lambda_{12} \), and \( F = \Lambda_{21}\Lambda_{11}^{-1} \).

In the same way as \( \Theta \) is the coefficient matrix of the linear fractional map \( F_\Theta(h) \) defined in (2.4), one can see \( V \) as the coefficient matrix of

\[
(4.21) \quad F_V(Q) := (V_{11}Q + V_{12})(V_{21}Q + V_{22})^{-1}.
\]

Using (4.17) and (4.18), we have that

\[
(4.22) \quad F_\Theta(h) \leftrightarrow F_V(Q) + G.
\]
One can rewrite the conditions \((CM1), (CM2)\) and \((CM3)\), satisfied by the coefficient matrix \(\Theta\) of the linear fractional map \(F(h)\) defined in (2.4), using the corresponding notations, (4.15)-(4.18), (4.19), and (4.20).

In condition \((CM3)\) we may take \(p = q = I\).

\[(WCM1)\] \[
\begin{bmatrix}
I & G \\
0 & I
\end{bmatrix} \Lambda^{-1} \begin{bmatrix}
\hat{A} \\
\hat{A}
\end{bmatrix} = \begin{bmatrix}
I & G \\
0 & I
\end{bmatrix} \begin{bmatrix}
\hat{A} \\
\hat{A}
\end{bmatrix},
\]

\[(WCM2)\] \(V_{22}^{-1} \in \hat{A},\)

\[(WCM3)\] \(\left(\begin{bmatrix}
I & G \\
0 & I
\end{bmatrix} \Lambda^{-1}\right)^* J \begin{bmatrix}
I & G \\
0 & I
\end{bmatrix} \Lambda^{-1} = J.\)

One can analyze the above conditions one by one.

\((WCM1)\): Multiplying the equality in \((WCM1)\) to the left with the matrix \[
\begin{bmatrix}
I & -G \\
0 & I
\end{bmatrix}
\]
and then with \(\Lambda\) one obtains the equalities: \(\Lambda^{-1} \begin{bmatrix}
\hat{A} \\
\hat{A}
\end{bmatrix} = \begin{bmatrix}
\hat{A} \\
\hat{A}
\end{bmatrix}\)
and \(\Lambda \begin{bmatrix}
\hat{A} \\
\hat{A}
\end{bmatrix} = \begin{bmatrix}
\hat{A} \\
\hat{A}
\end{bmatrix}\). Since \(\Lambda\) is a \(J\)-spectral factor for \(W\) defined in (3.10), than \(\Lambda, \Lambda^{-1} \in \hat{A}\). Form this one concludes that for the Wiener algebra \((WCM1)\) is always satisfied.

\((WCM2)\): It was proved that the assumption \((WCM1)\) holds. Suppose that the assumption \((WCM2)\) holds. Since \(V = \Lambda^{-1}, \Lambda_{11}^{-1} = V_{11} - V_{12}V_{22}^{-1}V_{21}\) (it can be computed), \(V, V_{22} \in \hat{A}\) conclude that \(\Lambda_{11}\) is invertible and \(\Lambda_{11}^{-1} \in \hat{A}\).

Reciprocally, assume that \(\Lambda_{11}\) is invertible and \(\Lambda_{11}^{-1} \in \hat{A}\). Since \(\Lambda = V^{-1}\), it can be computed that \(V_{22}^{-1} = \Lambda_{22} - \Lambda_{21}\Lambda_{11}^{-1}\Lambda_{12}\), and, similarly, one obtain that \((WCM2)\) holds.

\((WCM3)\): Since from Theorem 3.2 \(\Lambda\), a \(J\)-spectral factor of \(W\), defined in (3.10), always exists, than \((WCM3)\) is always satisfied.

A first conclusion is that in the Wiener algebra always exists a \(\Theta\) invertible over \(\hat{W}\) such that \((WCM1)\) and \((WCM3)\) hold. Moreover, suppose that \(\Theta_{22}\) is invertible over \(R\). From Lemma 2.2, we always have \(\Theta_{11} = V_{11} + GV_{21}\) invertible in \(W\).

**Lemma 4.1.** Consider the Wiener algebra and the correspondences defined before. Then

1. \(\Theta\) defined by (4.17) is invertible in \(\hat{W}\);

2. \(\Theta_{22}\) invertible in \(R \Rightarrow \Theta_{11}\) invertible in \(\hat{W}\);
3. The conditions \((WCM1)\) and \((WCM3)\) hold.

Further, the result stated in Theorem 2.3 is analyzed. Using Lemma 4.1, one can prove that \(\Theta\), as in (4.17), is invertible and the conditions \((WCM1)\) and \((WCM3)\) hold. Rewrite the properties \((P1)\), \((P2)\) and \((P3)\) in the notations of Section 3:

\[
\begin{align*}
(WP1) & \quad I - (G + K)^*(G + K) = w^{-*}u^{-1}, \\
(WP2) & \quad I - (G + K)(G + K)^* = v^{-*}v^{-1}, \\
(WP3) & \quad (I - (G + K)^*(G + K))^{-1}(G + K)^* \hat{A} \subset \hat{A}.
\end{align*}
\]

where \(u^\pm \in \hat{A}, v^\pm \in \hat{W}, v^* \hat{A} = \hat{A}\) and \(v^* \hat{A} = \hat{A}\).

Theorem 2.3 IF:

It is enough to consider as hypothesis that \(\Theta\) satisfies \((CM2)\). From Theorem 2.3, there exists a strictly contractive extension of \(k\) given by (2.8) such that \((WP1)\), \((WP2)\) and \((WP3)\) hold. Using the correspondence (4.19), one expresses the strictly contractive extension as provided in (2.8). This is \(f = \Theta_{21}\Theta_{22}^{-1} = (V_{12} + GV_{22})V_{22}^{-1} = V_{12}V_{22}^{-1} + G\). Using (4.16), one obtains that \(K = V_{12}V_{22}^{-1}\) is a strictly contractive extension of \(G\). This means that \(V_{12}V_{22}^{-1}\) is a solution for the SNP.

We obtained that Lemma 3.4 is proved, and also the implication "3 \(\Rightarrow\) 2" from Theorem 3.5 is proved [10]. The conditions \((WP1)\), \((WP2)\) and \((WP3)\) are supplementary information provided by the "IF" part of Theorem 2.3, compared to Theorem 3.5.

Analyze now what the above conditions mean. From \((CM2)\) we know that \(\Theta_{22}\) is invertible. Using Lemma 2.1 and Lemma 2.2, one obtains that \((WP1)\) and \((WP2)\) hold, in case one proves, as in [2], that \(v^* \hat{A} = \hat{A}\). More precisely, we have \(u = \Theta_{22} = V_{22}\) and \(v = \Theta_{11} = V_{11} + GV_{21}\). In fact, the condition \((WP1)\) is hidden in the proof of Theorem 3.5 as proved in [10].

For a proof of the condition \((WP3)\) we refer again to [2] (this is because it is a natural proof). However, it does not explicitly come into the proof of the result given in Section 3 (see [10]).

Theorem 2.3 ONLY IF:

Assume that there exists \(f\) a strictly contractive extension of \(k\), such that the statements \((P1)\), \((P2)\) and \((P3)\) hold. Consequently, the statement 2 in Theorem 3.5 is true for \(\sigma = 1\). Using the implication 2 \(\Rightarrow\) 3 from Theorem 3.5 one sees that \(\Lambda_{11}^{-1}\) exists and is in \(\hat{A}\). This means \(V_{22}^{-1}\) has the same properties (see \((WCM2)\)) so the condition \((CM2)\) is verified. The
statements \((P1), (P2)\) and \((P3)\) were not used. But one can prove that they hold! Conclude that, in case that any of the statements of Theorem 3.5 holds, then \((P1), (P2)\) and \((P3)\) hold.

**Remark 4.2.** Since a \(J\)-spectral factor for the matrix-valued function \(W\) (given in (3.10)) exists, the situation for the Wiener algebra is a simplified version of the general result in \(C^*\)-algebras (less conditions for the solvability of the SNP in terms of the \(J\)-spectral factorization). Similarly, using (4.22) and (3.14), the connection between Theorem 2.4 and Theorem 3.6 can be established. One way to generalize the \(J\)-spectral factorization follows from the above connections.

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