A CAUCHY PROBLEM ON TIME SCALES WITH APPLICATIONS

BY

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Abstract. We obtain the existence of continuous solutions for a nonlocal Cauchy problem on time scales in Banach spaces, considering non-absolutely convergent delta-integrals. As this kind of problems contains the classical cases of differential and difference equations and not only these ones, our main theorem offers a very general existence result. We then deduce, by an embedding procedure, a result for impulsive differential equations under a new kind of assumptions on the jump functions.

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1. Introduction

Although there are many analogies between the study of differential equations and that of difference equations, until two decades ago, these situations were investigated separately. In 1988, the German mathematician S. Hilger published, in his PhD Thesis and then in [25], a method to unify the continuous and the discrete cases, by introducing the such-called time scale theory. In this way, one can prove a result for dynamic problems on time scales and to deduce, in particular, results for differential or discrete problems (for a survey of papers in this direction, we refer to [6], [7] or [27]). In the present work, we obtain an existence theorem for a nonlocal Cauchy problem (for the importance of nonlocal conditions, see [8], [9]) on time scales:

\[ x^\Delta(t) = f(t, x(t)), \ a.e. \ t \in \mathbb{T}, \quad x(0) = \int_0^T b(s)x(s)\Delta s. \]
To this purpose, we apply a version of Krasnosel’skii fixed point result involving conditions expressed in terms of weak topology, presented in [3]. Note that, by considering Henstock integrals (that are non-absolute integrals), we allow to the function on the right hand side to be an oscillating function.

In the second part, by embedding impulsive problems into problems on adequate time scales, as in [16], we give an existence result for impulsive nonlocal differential equations. We emphasize that on the jump functions are made some assumptions concerning the weak topology of Banach space.

2. Notations and preliminary facts

Let us begin by presenting some preliminary definitions and notations of time scales that can be easily found in literature (see [1], [6], [7] and references therein).

A time scale \( T \) is a nonempty closed set of real numbers \( \mathbb{R} \), with the subspace topology inherited from the standard topology of \( \mathbb{R} \) (for example \( T = \mathbb{R} \), \( T = \mathbb{N} \) and \( T = q\mathbb{Z} = \{ qt : t \in \mathbb{Z} \} \), where \( q > 1 \)). For two points \( a, b \) in \( T \), we denote by \([a, b]_T = \{ t \in T : a \leq t \leq b \}\) the time scales interval.

**Definition 1.** The forward jump operator \( \sigma : T \to T \) and the backward jump operator \( \rho : T \to T \) are defined by \( \sigma(t) = \inf\{ s \in T : s > t \} \), respectively \( \rho(t) = \sup\{ s \in T : s < t \} \). Also, \( \inf\emptyset = \sup T \) (i.e. \( \sigma(M) = M \) if \( T \) has a maximum \( M \)) and \( \sup\emptyset = \inf T \) (i.e. \( \rho(m) = m \) if \( T \) has a minimum \( m \)).

A point \( t \in T \) is called right dense, right scattered, left dense, left scattered, dense, respectively isolated if \( \sigma(t) = t, \sigma(t) > t, \rho(t) = t, \rho(t) < t, \rho(t) = t = \sigma(t) \) and \( \rho(t) < t < \sigma(t) \), respectively.

Let \( X \) be a Banach space and denote by \( C(T, X) \) the space of \( X \)-valued continuous functions on \( T \) and by \( B_R \) its closed ball of radius \( R \) centered in the null element of this space, while \( \| \cdot \|_C \) stands for the supremum norm.

**Definition 2.** Let \( f : T \to X \) and \( t \in T \). Then the \( \Delta \)-derivative \( f^\Delta(t) \) is the element of \( X \) (if it exists) with the property that for any \( \varepsilon > 0 \) there exists a neighborhood of \( t \) on which \( \| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \| \leq \varepsilon|\sigma(t) - s| \).

**Remark 3.** It is not difficult to see that, in particular,

(i) \( f^\Delta = f' \) is the usual derivative if \( T = \mathbb{R} \),
(ii) $f^\Delta = \Delta f$ is the usual forward difference operator if $T = \mathbb{Z}$.

Therefore, the time scale calculus unifies (and generalizes) the treatment of differential and difference equations.

In a similar way can be defined the $\nabla$-derivative:

**Definition 4.** Let $f : T \to X$ and fix $t \in T$. Then the $\nabla$-derivative $f^\nabla(t)$ is the element of $X$ (if it exists) such that for any $\varepsilon > 0$ there exists a neighborhood of $t$ on which

$$\|f(\rho(t)) - f(s) - f^\nabla(t)\rho(t) - s\| \leq \varepsilon|\rho(t) - s|$$

(in an obvious way, all the discussion concerning $\Delta$-integrals can be retaken for $\nabla$-integrals).

We denote by $\mu_\Delta$ the Lebesgue measure on $T$ (for its definition and properties we refer the reader to [10]). For properties of Riemann delta-integral we refer to [22] and for Lebesgue integral on time scales see [5], [6], [7] or [22]. See also [14] for an interesting discussion on Cauchy problems on time scales.

As about the Henstock-type integrals, as in particular cases (see [11] in $\mathbb{R}$ or [31], [2] in $T$ for real-valued functions), we need to consider two definitions of vector-valued integrals.

Let $\delta = (\delta_L, \delta_R)$ be a $\Delta$-gauge, that is a pair of positive functions such that $\delta_L(t) > 0$ on $(a, b]$, $\delta_R(t) > 0$ and $\delta_R(t) \geq \sigma(t) - t$ on $[a, b]$. A partition $D = \{[x_{i-1}, x_i]; \xi_i, i = 1, 2, \ldots n\}$ of $[a, b]_T$ is $\delta$-fine whenever:

$$\xi_i \in [x_{i-1}, x_i] \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)], \forall 1 \leq i \leq n.$$ 

Let us recall that such a partition exists for arbitrary positive pair of functions (Cousin’s Lemma, see Lemma 1.9 in [31]).

**Definition 5** ([15]). A function $f : [a, b]_T \to X$ is Henstock-$\Delta$-integrable on $[a, b]_T$ if there exists a function $F : [a, b]_T \to X$ satisfying the following property: given $\varepsilon > 0$, there exists a $\Delta$-gauge $\delta$ on $[a, b]_T$ such that for every $\delta$-fine division $D = \{[x_{i-1}, x_i], \xi_i\}$ of $[a, b]_T$, we have

$$\left\| \sum_{i=1}^{n} f(\xi) \mu_\Delta([x_{i-1}, x_i]) - (F(x_i) - F(x_{i-1})) \right\| < \varepsilon.$$

Then denote $F(t)$ by $(H) \int_{a}^{t} f(s)\Delta s$ and call it the Henstock-$\Delta$-integral of $f$ on $[a, t]_T$. 
Definition 6 ([15]). A function \( f : [a, b]_T \to X \) is Henstock-Lebesgue-\( \Delta \)-integrable on \( [a, b]_T \) if there exists a function \( F : [a, b]_T \to X \) satisfying the following property: given \( \varepsilon > 0 \), there exists a \( \Delta \)-gauge \( \delta \) on \( [a, b]_T \) such that for every \( \delta \)-fine division \( D = \{[x_{i-1}, x_i], \xi_i\} \) of \( [a, b]_T \),
\[
\sum_{i=1}^{n} \| f(\xi_i) \Delta([x_{i-1}, x_i]) - (F(x_i) - F(x_{i-1})) \| < \varepsilon.
\]

Then \( F(t) \) is denoted by \( (HL) \int_a^t f(s) \Delta s \) and it is called the Henstock-Lebesgue-\( \Delta \)-integral of \( f \) on \( [a, t]_T \).

Although this will not be here in our attention, we must remind that the Henstock-Kurzweil-Pettis delta-integral was also considered in literature (see [15] for basic facts on it and [17], [32] for applications).

Denote by \( \mathcal{HL}([a, b]_T, X) \) the space of all HL-\( \Delta \)-integrable functions provided with the topology given by the Alexiewicz norm:
\[
\| f \|_A = \sup_{t \in [a, b]_T} \left\| (HL) \int_a^t f(s) \Delta s \right\|
\]

Obviously, by the triangle inequality, if \( f \) is Henstock-Lebesgue-\( \Delta \)-integrable it is also Henstock-\( \Delta \)-integrable. In general, the converse is not true. For real-valued functions, the two integrals are equivalent (the definition and some properties of the Henstock-Kurzweil, shortly (HK) \( \Delta \)-integral in this case can be found in [31]).

Remark 7. There is a major difference between these two notions: even in the case of \( T = \mathbb{R} \), the primitive function of the Henstock-Lebesgue-\( \Delta \)-integral is continuous and almost everywhere differentiable, while the primitive in the sense of Henstock-\( \Delta \)-integral is continuous, but in general not a.e. differentiable.

This means that, by considering solutions of differential problems in the sense of Carathéodory, we have an equivalence between differential and integral problems with Henstock-Lebesgue integral. We will thus use the Henstock-Lebesgue integral in the setting of time-scale domains too.

In the case where \( T = \mathbb{R} \), there are many studies on this subject (see Kurzweil and Schwabik [28], [29], Chew and Flordelija [12], [13], Federson and Táboas [20], Di Piazza and Satco [18] or Heikkilä, Kumpulainen and Seikkala [23], [24], for instance).
3. An existence result for Cauchy problems with nonlocal conditions on time scale

In this section, we prove an existence result for the following dynamic equation with nonlocal condition on a bounded time scale $T$ contained in the real interval $[0, T]$: 

$$ x^\Delta(t) = f(t, x(t)), \text{ a.e. } t \in T, $$

$$ x(0) = \int_0^T b(s)x(s)\Delta s. $$

The main result is obtained by applying a fixed point theorem of Krasnosel’skii type considering the weak topology, presented in [3] (Theorem 2.1):

**Theorem 8.** Let $M$ be a nonempty bounded closed convex subset of a Banach space $E$ and $A : M \to E$, $B : E \to E$ be two weakly sequentially continuous mappings satisfying:

i) $AM$ is relatively weakly compact;

ii) $B$ is a strict contraction;

iii) if $x = Bx + Ay$ and $y \in M$, it follows that $x \in M$.

Then $A + B$ has at least one fixed point in $M$.

We will make use of the following result, given in [19] (Theorem 9):

**Proposition 9.** Let $(f_n)_n$ be a bounded sequence of $C([0, 1], X)$. Then $(f_n)_n$ is convergent to $f \in C([0, 1], X)$ with respect to the weak topology of $C([0, 1], X)$ if and only if $(f_n(t))_n$ is weakly convergent to $f(t)$ for every $t \in [0, 1]$.

Let us also remind of several notions extremely useful in various convergence results, that will appear in our existence theorem (we refer to [32] and, for $T = \mathbb{R}$, to [29] or [21]):

**Definition 10.** i) A function $F : [a, b]_T \to \mathbb{R}$ is absolutely continuous in the restricted sense (shortly, $AC_*$) on $E \subset [a, b]_T$ if, for any $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that, whenever $\{(c_i, d_i)_T, 1 \leq i \leq N\}$ is a finite collection of non-overlapping intervals that have endpoints in $E$ and satisfy $\sum_{i=1}^N \mu_\Delta([c_i, d_i]_T) < \eta_\varepsilon$, one has $\sum_{i=1}^N \text{osc}(F, [c_i, d_i]_T) < \varepsilon$;
ii) $F$ is said to be generalized absolutely continuous in the restricted sense (shortly, $ACG_*$) if it is continuous and the unit interval can be written as a countable union of sets on each of which $F$ is $AC_*$;

iii) A family of real functions is uniformly $ACG_*$ if one can write the unit interval as a countable union of sets on each of which the family is uniformly $AC_*$ (i.e. the above mentioned $\eta_e$ is the same for all elements of the family).

We present now the main result of this paper.

**Theorem 11.** Let $b : \mathbb{T} \to \mathbb{R}^+$ be a continuous function and $f : \mathbb{T} \times X \to X$ satisfy:

i) for every continuous $x : \mathbb{T} \to X$, the function $t \in \mathbb{T} \mapsto f(t, x(t))$ is $HL$-$\Delta$-integrable;

ii) if $(x_n)_n$ converges to $x$ with respect to the weak topology of $C(\mathbb{T}, X)$, then $(f(\cdot, x_n(\cdot)))_n$ pointwisely weakly converges to $f(\cdot, x(\cdot))$;

iii) for every $R > 0$, the set $\{(HL) \int_0^t f(s, x(s)) \Delta s, \|x\|_C \leq R\} \subset C(\mathbb{T}, X)$ is:

iii1) equicontinuous and pointwisely relatively weakly compact;

iii2) weakly uniformly $ACG_*$, i.e. $\{(HK) \int_0^t \langle x^*, f(s, x(s)) \rangle \Delta s, \|x\|_C \leq R\}$ is uniformly $ACG_*$, for all $x^* \in X^*$;

iv) $\limsup_{R \to \infty} \left( \frac{1}{R} \sup_{\|x\|_C \leq R} \|f(\cdot, x(\cdot))\|_A \right) < \frac{1}{2}$;

v) $\|b\|_C \leq \frac{1}{2T}$.

Then the differential problem possesses global continuous solutions.

**Proof.** By hypothesis $iv)$, one can find $R_0 > 0$ such that for any $R \geq R_0$,

$$\sup_{\|x\|_C \leq R} \sup_{t \in \mathbb{T}} \left\| (HL) \int_0^t f(s, x(s)) \Delta s \right\| < \frac{R}{2}.$$  

Let $B_{R_0}$ be the closed ball of $C(\mathbb{T}, X)$ centered in the null function, of radius $R_0$, and define the operators $A : B_{R_0} \to C(\mathbb{T}, X)$ and $B : B_{R_0} \to C(\mathbb{T}, X)$ by

$$Ax(t) = (HL) \int_0^t f(s, x(s)) \Delta s$$  

$$Bx(t) = \left( \frac{1}{R} \sup_{\|x\|_C \leq R} \|f(\cdot, x(\cdot))\|_A \right) x(t).$$
respectively

\[ Bx(t) = \int_0^T b(s)x(s)\Delta s. \]

Since the primitives in the sense of \(\Delta\)-Henstock-Lebesgue integral are continuous, it follows that \(A\) is \(C(\mathbb{T}, X)\)-valued. On the other hand, let us note that the values of \(B\) are constant functions (here the function to be integrated is continuous, therefore the integral is in the sense of \(\Delta\)-Lebesgue integral).

Condition \(iii1\) implies, by Arzela-Ascoli Theorem, that the set \(\{(HL) \int_0^T f(s, x(s))\Delta s, x \in B_{R_0}\}\) is relatively weakly compact.

Let us now prove that the operator \(A\) is weakly sequentially continuous. Consider an arbitrary sequence \((x_n)_n \subset C(\mathbb{T}; X)\) weakly convergent to \(x \in C(\mathbb{T}; X)\). By hypothesis \(ii\), for every \(x^* \in X^*\), \(\langle x^*, f(\cdot, x_n(\cdot)) \rangle \to \langle x^*, f(\cdot, x(\cdot)) \rangle\) pointwisely. Since the set \(\{(HL) \int_0^T f(s, x_n(s))\Delta s, n \in \mathbb{N}\}\) is equicontinuous, the family of real-valued continuous functions on \(\mathbb{T}, \{(HK) \int_0^T \langle x^*, f(s, x_n(s)) \rangle\Delta s, n \in \mathbb{N}\}\), is equi-continuous and uniformly AC\(G^*\). As any Henstock-Lebesgue delta-integrable function is also Henstock-Kurzweil-Pettis delta-integrable (we refer the reader to [15]), it follows, by the convergence Theorem 2.14 in [32], that

\[
\langle x^*, (HL) \int_0^T f(s, x_n(s))\Delta s \rangle \to \langle x^*, (HL) \int_0^T f(s, x(s))\Delta s \rangle.
\]

Applying Proposition 9, one deduces that

\[
(HL) \int_0^T f(s, x_n(s))\Delta s \to (HL) \int_0^T f(s, x(s))\Delta s
\]

with respect to the weak topology of the space \(C(\mathbb{T}, X)\) and this gives the sequential weak continuity of the operator \(A\).

We prove now that the same is available for the operator \(B\).

Take a sequence \((x_n)_n \subset B_{R_0}\) weakly convergent to \(x \in C(\mathbb{T}, X)\) and take an arbitrary \(x^* \in B^*\), the unit ball of the topological dual of \(X\). Then

\[
\left| \langle x^*, \int_0^T b(s)x_n(s)\Delta s - \int_0^T b(s)x(s)\Delta s \rangle \right| = \left| \int_0^T b(s) \langle x^*, x_n(s) - x(s) \rangle \Delta s \right| \leq \|b\|_C \int_0^T |\langle x^*, x_n(s) - x(s) \rangle| \Delta s.
\]
As $\|x^*,x_n(s) - x(s)\|$ tends to 0 at every point $s$ and it is bounded by $2R_0$ that is integrable it can be deduced, by applying the Lebesgue dominated convergence theorem (see Chapter 5 in [7]), that

$$\left| \left( x^*, \int_0^T b(s)x_n(s)\Delta s - \int_0^T b(s)x(s)\Delta s \right) \right|$$

also tends to 0 and so, the weak sequential continuity of $B$ is proved. The last hypothesis asserts that $B$ is a strict contraction, since $\|Bx - By\| \leq \|b\|_{C[T]}\|x - y\|_{C}$. Finally, let us check the last hypothesis of Theorem 8. Consider $x \in C(\mathbb{T}, X)$ such that, for some $y \in B_{R_0}$,

$$x(t) = \int_0^T b(s)x(s)\Delta s + (HL) \int_0^t f(s,y(s))\Delta s, \forall t \in \mathbb{T}.$$ 

Then

$$\|x\|_{C} \leq \left\| \int_0^T b(s)x(s)\Delta s \right\| + \sup_{t \in \mathbb{T}} \left\| (HL) \int_0^t f(s,y(s))\Delta s \right\|$$

$$\leq \|b\|_{C[T]}\|x\|_{C} + \frac{R_0}{2}$$

and so, $\|x\|_{C} \leq \frac{R_0}{2(1-\|b\|_{C[T]})} \leq R_0$ thanks to hypothesis $v$). This shows that $x \in B_{R_0}$. Applying Theorem 8, we obtain that the operator $A + B$ has a fixed point, therefore our problem has continuous solutions.

As it can be seen in the proof of the previous theorem, the Henstock-Lebesgue delta-integral could be replaced by the Henstock delta-integral, but in this case we only obtain a continuous solution for the integral nonlocal problem (due to the non-differentiability of the latest integral).

**Remark 12.** Theorem 8 offers a very general existence result. First of all, it is given for dynamic equations on time scales, therefore it implies, in particular, corresponding results for ordinary differential and difference equations. Also, it considers Henstock integrals, that can be defined for a much larger number of functions (comparing to Riemann or Lebesgue integrals) and it works for Banach space-valued functions on the right hand side. Moreover, the studied Cauchy problem involves a nonlocal condition, that is more natural when describing physical phenomena than the classical initial condition.
4. Existence of solutions for impulsive nonlocal differential problems on a real interval

In what follows, we apply the main result to obtain the existence of "good" (in a sense that will be made clear) solutions for an impulsive nonlocal differential problem on the real line:

\[
\begin{align*}
(1) & \quad y(t) = \tilde{f}(t, y(t)), \quad \text{a.e. } t \in [0, T_0] \setminus \{t_1, \ldots, t_m\}, \\
(2) & \quad \Delta y(t_i) = I_i(y(t_i)), \quad \forall i \in \{1, \ldots, m\}, \\
(3) & \quad y(0) = \int_0^{T_0} \tilde{b}(s)y(s)ds.
\end{align*}
\]

The interval \([0, T_0]\) is a real interval and \(0 < t_1 < \ldots < t_m < T_0\) are the a-priori known moments of impulse, while \(\Delta y(t) = y(t+) - y(t-)\) denotes the jump of the function \(y\) at \(t\) and the discontinuity at the point \(t_i\) is described by the function \(I_i : X \to X\).

Consider in the remaining of this section the following space of functions (that will contain the solutions of our problem):

**Definition 13.** \(PC([0, T_0], X)\) is the collection of functions \(y : [0, T_0] \to X\) with the following properties:

i) \(y\) is continuous at every \(t \in [0, T_0] \setminus \{t_1, \ldots, t_m\}\);

ii) at every \(t \in \{t_1, \ldots, t_m\}\), \(y\) is left continuous and there exists the right limit \(y(t+)\).

As seen in [26], \(PC([0, T_0], X)\) is a closed subspace of the space of all regulated \(X\)-valued functions which, endowed with the norm \(\|\cdot\|_C\), is complete and so, it becomes a Banach space itself.

Following the method described in [16], we "enlarge" the initial interval to obtain a time scale: take \(h > 0\) and define the set

\[
\mathbb{T} = [0, t_1] \cup [t_1 + h, t_2 + h] \cup \ldots \cup [t_m + mh, T_0 + mh].
\]

Let \(x : \mathbb{T} \to X\) be the function defined by

\[
x(t) = \begin{cases} 
y(\beta(t))+, & \text{if } t \in \{t_1 + h, t_2 + 2h, \ldots, t_m + mh\}, \\
y(\beta(t)), & \text{otherwise}.
\end{cases}
\]
where $\beta : \mathbb{T} \to [0, T_0]$ is given by

$$
\beta(t) = \begin{cases} 
  t, & \text{if } t \in [0, t_1], \\
  t - h, & \text{if } t \in [t_1 + h, t_2 + h], \\
  \vdots \\
  t - mh, & \text{if } t \in [t_m + mh, b + mh]. 
\end{cases}
$$

Note that this is a continuous functions on the time scale $\mathbb{T}$.

Denote now by $N$ the null-measure subset of $[0, T_0] \setminus \{t_1, \ldots, t_m\}$ where $y$ is not differentiable. With these notations, our impulsive differential problem can be written as

$$
x^\Delta(t) = \begin{cases} 
  \tilde{f}(\beta(t), x(t)), & \text{if } t \notin (N \cup \{t_1, t_1 + h, t_2 + h, t_2 + 2h, \ldots, t_m + mh\}), \\
  I_{t_1-h(t_1)}^T(x(t)) I^{-1}_{T_0+mh} \tilde{b}(\beta(s)) x(s) \Delta s, & \text{if } t \in \{t_1, t_2 + h, \ldots, t_m + (m-1)h\}, 
\end{cases}
$$

$$
x(0) = \int_0^{T_0+mh} \tilde{b}(\beta(s)) x(s) \Delta s.
$$

This is because, at a right-scattered point, e.g. $t_1$, 

$$
x^\Delta(t_1) = \frac{x(\sigma(t_1)) - x(t_1)}{\sigma(t_1) - t_1} = \frac{I_1(y(t_1))}{h}.
$$

Remark that the function $x$ is not differentiable on $N \cup \{t_1 + h, t_2 + 2h, \ldots, t_m + mh\}$, that is a null-measure set.

This construction enables one to study impulsive differential problems on a real interval by methods of time scales theory. Thus, applying Theorem 8, we get:

**Theorem 14.** Let $\tilde{f} : [0, T_0] \times X \to X$, $I_i : X \to X$, $i \in \{1, \ldots, m\}$ and the continuous function $\tilde{b} : [0, T_0] \to \mathbb{R}^+$ satisfy:

1) for each $y \in PC([0, T_0], X)$, the function $t \in [0, T_0] \mapsto \tilde{f}(t, y(t))$ is $HL$-integrable;

2) if $(y_n)_n \subset PC([0, T_0], X)$ weakly converges to $y$, then $(\tilde{f}(\cdot, y_n(\cdot)))_n$ pointwisely weakly converges to $\tilde{f}(\cdot, y(\cdot))$;

3) for every $R > 0$, the set 

$$
\{(HL) \int_0^T \tilde{f}(s, y(s)) ds, \|y\|_C \leq R \} \subset C([0, T_0], X)
$$

is:
i) equicontinuous and pointwisely relatively weakly compact;

ii) weakly uniformly ACG∗, i.e. \( \{(HK) \int_0^s \langle x^*, f(s, y(s)) \rangle ds, \|y\|_C \leq R \} \) is uniformly ACG∗, for all \( x^* \in X^* \);

4) \( \limsup_{R \to \infty} \frac{1}{R} \sup_{\|y\|_C \leq R} (\|f(\cdot, y(\cdot))\|_A + \sum_{i=1}^m \|I_i(y(t_i))\|) < \frac{1}{2} \);

5) \( \|\tilde{b}\|_C \leq \frac{1}{2T_0} \);

6) the functions \( I_i: X \to X \) are sequentially weakly continuous and map balls into relatively weakly compact sets.

Then the impulsive differential problem (1)-(3) possess solutions in \( PC([0, T_0], X) \).

**Proof.** We choose \( h > 0 \), transform the impulsive problem into a Cauchy problem on time scales and check the hypothesis of Theorem 11. Thus, \( f: T \times X \to X \) is given by

\[
f(t, x) = \begin{cases} \tilde{f}(\beta(t), x), & \text{if } t \notin (N \cup \{t_1, t_1 + h, t_2 + h, t_2 + 2h, \ldots, t_m + mh\}), \\ \frac{I_{t-h(t)+1}(x)}{h}, & \text{if } t \in \{t_1, t_1 + h, \ldots, t_m + (m - 1)h\}
\end{cases}
\]

and \( b(s) = \tilde{b}(\beta(s)) \).

The conditions i), iii2) and v) are easy to check. Hypothesis ii) follows from condition 2) and the sequential weak continuity of functions \( I_i \). In order to verify hypothesis iii1), it suffices to see that the equicontinuity is immediate (from 3i)), while the second part of the statement comes from 3i) and the fact that each \( I_i \) maps balls into relatively weakly compact sets.

It can be seen that

\[
\limsup_{R \to \infty} \left( \frac{1}{R} \sup_{\|x\|_C \leq R} (\|f(\cdot, x(\cdot))\|_A) \right) = \limsup_{R \to \infty} \left( \frac{1}{R} \sup_{\|x\|_C \leq R} \sup_{t \in T} \left\| (HL) \int_0^t f(s, x(s)) \Delta s \right\| \right)
\]

\[
\leq \limsup_{R \to \infty} \frac{1}{R} \sup_{\|y\|_C \leq R} \sup_{t \in T} \left\| (HL) \int_0^{\beta(t)} \tilde{f}(s, y(s))ds \right\| + \sum_{0 < t_i < \beta(t)} \|I_i(y(t_i))\|
\]

\[
\leq \limsup_{R \to \infty} \frac{1}{R} \sup_{\|y\|_C \leq R} \left( \|f(\cdot, y(\cdot))\|_A + \sum_{i=1}^m \|I_i(y(t_i))\| \right) < \frac{1}{2}
\]
and so, we get the assumption \( iv) \) of Theorem 11. In this calculus it was used the following property of integrals on time scales:

\[
\int_{t}^{\sigma(t)} g(s) \Delta s = (\sigma(t) - t)g(t).
\]

We are able to apply Theorem 11 and we obtain that the impulsive problem (1)-(3) possess solutions in \( \mathcal{P}C([0, T_0], X) \). □

**Remark 15.** As far as we know, the assumptions under which Theorem 14 asserts that the impulsive problem (1)-(3) has solutions, in particular the conditions on the jump functions, are quite different comparing to similar results in literature (e.g. [30], [4]).

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