BREZIS-BROWDER PRINCIPLE
AND DEPENDENT CHOICE

BY

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Abstract. The Brezis-Browder Ordering Principle is equivalent with the Principle of Dependent Choices; and as such, equivalent with Ekeland’s Variational Principle.

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1. Introduction

Let $M$ be a nonempty set. Take a quasi-order $(\leq)$ (i.e.: reflexive and transitive relation) over it; and a function $x \mapsto \psi(x)$ from $M$ to $R_+ := [0, \infty]$. Call the point $z \in M$, $(\leq, \psi)$-maximal when: $z \leq w \in M$ implies $\psi(z) = \psi(w)$. A basic result about such points is the 1976 Brezis-Browder ordering principle [2] (in short: BB).

Theorem 1. Suppose that

(a01) $(M, \leq)$ is sequentially inductive:

- each ascending sequence has an upper bound (modulo $(\leq)$)

(a02) $\psi$ is $(\leq)$-decreasing ($x \leq y \Rightarrow \psi(x) \geq \psi(y)$).

Then, for each $u \in M$ there exists a $(\leq, \psi)$-maximal $v \in M$ with $u \leq v$.

This statement includes (cf. Section 3) Ekeland’s Variational Principle [9] (in short: EVP) and found some useful applications to convex and non-convex analysis (cf. the above references); so, it was the subject of many
extensions. The most significant of these is contained in the 1993 paper by Cârjă and Ursescu [5] (cf. Section 2); for the remaining ones, we refer to the 1997 monograph by Hyers, Isac and Rassias [12, Ch 5]. Summing up, for each variational principle of this type (VP, say) one has VP ⇒ BB ⇒ EVP; so, it is legitimate asking of to what extent are these logical inclusions effective. As we shall see (in Section 3) the former of these is reversible; i.e., the statements VP are but logical equivalents of BB. Concerning the latter inclusion, we show (in Section 2) that BB is deductible from the Principle of Dependent Choices (in short: DC) due to Tarski [15]. So, to close the circle between these, it will suffice proving that EVP includes DC. An early result of this type was provided in 1987 by Brunner [4]. It is our aim to show (in Section 4) that a further extension of it is possible, in the sense: DC is deductible from a certain discrete Lipschitz countable version of EVP. Putting these together, it then results that (BB) and (EVP) are both equivalent with (DC); and, as such, mutually equivalent. This will remain true for all ”intermediary” statements (VP). Further aspects will be discussed elsewhere.

2. (DC) ⇒ (CU) ⇐ (BB)

Let \( M \) be a nonempty set; and \( \mathcal{R} \subseteq M \times M \) stand for a (nonempty) relation over it. For each \( x \in M \), denote \( M(x, \mathcal{R}) = \{ y \in M; x \mathcal{R} y \} \). The following ”Principle of Dependent Choices” (in short: DC) is in effect for us:

**Proposition 1.** Suppose that

\[(b01) \quad M(c, \mathcal{R}) \text{ is nonempty, for each } c \in M.\]

Then, for each \( a \in M \) there exists \( (x_n) \subseteq M \) with \( x_0 = a \) and \( x_n \mathcal{R} x_{n+1} \), for all \( n \).

This principle, due to Tarski [15], is deductible from AC (= the Axiom of Choice), but not conversely; cf. Wolk [21]. Moreover, it seems to suffice for a large part of the ”usual” mathematics; see Moore [14, Appendix 2, Table 4].

Let \( M \) be a nonempty set. Take a quasi-order (\( \leq \)) over it, as well as a function \( x \mapsto \varphi(x) \) from \( M \) to \( R \cup \{-\infty\} \cup \{\infty\} \). Define the \( (\leq, \varphi) \)-maximal property of some \( z \in M \) as in Section 1. The following ”extended” variant of (BB) due to Cârjă and Ursescu [5] (and referred to as: the Cârjă-Ursescu variational principle; in short: CU) is to be considered.
Theorem 2. Assume that (a01) and (a02) are valid (in this extended setting). Then, conclusion of Theorem 1 is attainable (with φ in place of ψ).

Proof. (i) Suppose that φ is bounded on M. Define the function β : M → R as: β(v) = \inf\{φ(M(v, ≤)), v ∈ M\}. Clearly, β is increasing; and

(2.1) \[ \varphi(v) \geq \beta(v), \text{ for all } v ∈ M. \]

Finally, (a02) gives at once a characterization like

(2.2) v is (≤, φ)-maximal iff φ(v) = β(v).

Now, assume by contradiction that the conclusion in this statement is false; i.e. [in combination with (2.2)] there must be some u ∈ M such that:

(b02) for each v ∈ Mu := M(u, ≤), one has φ(v) > β(v).

Consequently (for all such v), φ(v) > (1/2)(φ(v) + β(v)) > β(v); hence

(2.3) v ≤ w and (1/2)(φ(v) + β(v)) > φ(w),

for at least one w (belonging to Mu). The relation R over Mu introduced via (2.3) fulfills Mu(v, R) ≠ ∅, for all v ∈ Mu. So, by (DC), there must be a sequence (un) in Mu with u0 = u and

(2.4) un ≤ un+1, (1/2)(φ(un) + β(un)) > φ(un+1), for all n.

We have thus constructed an ascending sequence (un) in Mu for which the real sequence (φ(un)) is (by (b02)) strictly descending and bounded below; hence λ := limnφ(un) exists in R. By (a01), (un) is bounded from above in M; i.e., there exists v ∈ M such that un ≤ v, for all n. From (a02), φ(un) ≥ φ(v), ∀n; and (by the properties of β) φ(v) ≥ β(v) ≥ β(un), ∀n. The former of these relations gives λ ≥ φ(v) (passing to limit as n → ∞). On the other hand, the latter of these relations yields (via (2.4)) (1/2)(φ(un) + β(v)) > φ(un+1), for all n ∈ N. Passing to limit as n → ∞ gives (φ(v) ≥ β(v) ≥ λ); so, combining with the preceding one, φ(v) = β(v)(= λ), contradiction. Hence, (b02) cannot be accepted; and the conclusion follows.

(ii) Assume now that φ is unbounded. Define the auxiliary function χ : M → [0, π] as χ(x) = A(φ(x)), x ∈ M (i.e., χ = A o φ); where
$$A(t) = \pi/2 + \arctg(t) \text{ if } t \in R; \quad A(-\infty) = 0; \quad A(\infty) = \pi.$$ 

Clearly, $\chi$ is decreasing and bounded on $M$. Therefore, by the preceding step, for each $u \in M$ there exists a $(\leq, \chi)$-maximal $v \in M$ with $u \leq v$. This, along with $\chi(v) = \chi(w) \iff \phi(v) = \phi(w)$ tells us that $v$ is $(\leq, \phi)$-maximal too.

Note that, by the argument above, (DC) $\implies$ (CU) $\iff$ (BB). For a slightly different proof, we refer to CÂRJĂ, NECULA AND VRABIE [6, Ch 2, Sect 2.1]. Further metrical extensions of (CU) may be found in TURINICI [19].

3. (BB) $\implies$ (EVP)

In the following, the relationships between BB and some other maximal results in the area is discussed. And then, we show that EVP is deductible from all these.

(A) Let $(M, \leq)$ be a quasi-ordered structure; and $x \mapsto \phi(x)$ stand for a function between $M$ and $R_+ \cup \{\infty\}$.

Theorem 3. Assume that (a01) and (a02) are true, as well as

(c01) $(M, \leq)$ is almost regular (modulo $\phi$):

$$\forall x \in M, \forall \varepsilon > 0, \exists y = y(x, \varepsilon) \geq x \text{ with } \phi(y) \leq \varepsilon.$$ 

Then, for each $u \in M$ there exists $v \in M$ with $u \leq v$ and $\phi(v) = 0$ (hence $v$ is $(\leq, \phi)$-maximal).

Proof. By (c01), there must be some $z \geq u$ with $\phi(z) < \infty$. Clearly, (BB) applies to $M(z, \leq)$ and $(\leq, \phi)$. So, for $z \in M(z, \leq)$ there exists $v \in M(z, \leq)$ with i) $z \leq v$ (hence $u \leq v$) and ii) $v$ is $(\leq, \phi)$-maximal in $M(z, \leq)$. Suppose by contradiction that $\gamma := \phi(v) > 0$; and fix some $\beta$ in $]0, \gamma[$. By (c01) again, there must be $y = y(v, \beta) \geq v$ (hence $y \in M(z, \leq)$) with $\phi(y) \leq \beta < \gamma (= \phi(v)$); impossible, by the second conclusion above. Hence, $\phi(v) = 0$.

By this reasoning, Theorem 3 is deductible from (BB). The converse inclusion is also true; to verify it, we need some conventions. By a (generalized) pseudometric over $M$ we shall mean any map $d : M \times M \to R_+ \cup \{\infty\}$. Fix such an object; supposed to be reflexive $[d(x, x) = 0, \forall x \in M]$. Call $z \in M$, $(\leq, d)$-maximal, if: $u, v \in M$ and $z \leq u \leq v$ imply $d(u, v) = 0$. Note that, if $d$ is (in addition) sufficient $[d(x, y) = 0 \implies x = y]$, the $(\leq, d)$-maximal property becomes: $w \in M, z \leq w \implies z = w$ (and reads: $z$ is...
strongly \((\leq)-\)maximal). So, existence results involving such points may be viewed as "metrical" versions of the Zorn-Bourbaki maximality principle (cf. Moore [14, Ch 4, Sect 4]). A natural way of deriving them is to start from the fact that, in terms of the associated function \(\varphi_d(x) = \sup\{d(u,v); x \leq u \leq v\}, x \in M\), this property may be characterized as: \(\varphi_d(x) = 0\). So, a basic source for determining such elements is Theorem 3 above (applied to this function). To do this, note that \((a02)\) is fulfilled by \(\varphi_d\). On the other hand, the almost regularity (modulo \(\varphi_d\)) condition \((c01)\) may be written as:

\[(c02) \quad (M, \leq) \text{ is weakly regular (modulo } d): \forall x \in M, \forall \varepsilon > 0, \exists y = y(x, \varepsilon) \geq x \text{ such that } y \leq u \leq v \implies d(u,v) \leq \varepsilon.\]

Putting these together, it results (via Theorem 3) the following maximality statement involving these data (cf. Kang and Park [13]):

**Theorem 4.** Assume that \((M, \leq; d)\) is such that \((a01)\) and \((c02)\) hold. Then, for each \(u \in M\) there exists a \((\leq, d)\)-maximal \(v \in M\) with \(u \leq v\).

Clearly, Theorem 4 is a logical consequence of \((BB)\). Moreover, the reciprocal of this is also true. In fact, put \(d(x, y) = |\psi(x) - \psi(y)|, x, y \in M\); and let \(\beta : M \to R\) stand for the function of Theorem 2 (with \(\psi\) in place of \(\varphi\)). Note that, if \(\psi(x) > \beta(x)\), any point \(y \in M(x, \leq)\) with \(\beta(x) \leq \psi(y) < \beta(x) + \varepsilon\) is like in \((c02)\); so, Theorem 4 applies to these data; and, from its conclusion, all is clear. We therefore established the inclusion chain: \(BB \implies Th 3 \implies Th 4 \implies BB\). Hence, all these ordering principles are nothing but logical equivalents of \((BB)\). It is natural to ask whether the maximality principles in Altman [1] and Turinici [17] enter in this scheme. A (positive) answer to this is available with some "diagonal" version of \((DC)\); further aspects will be discussed elsewhere.

**(B)** A basic application of these facts is to "monotone" variational principles. Let \((M, \leq)\) be a quasi-ordered structure; and \(d : M \times M \to R_+\) be a metric (i.e.: sufficient semi-metric) over \(M\). Call the subset \(Z\) of \(M\), \((\leq)\)-closed when the limit of each \((\leq)\)-ascending sequence in \(Z\) belongs to \(Z\). In particular, we say that \((\leq)\) is self-closed if \(M(x, \leq)\) is \((\leq)\)-closed, for each \(x \in M\); or, equivalently: the limit of each ascending sequence is an upper bound of it (modulo \((\leq)\)). Finally, call the metric \(d, (\leq)\)-complete provided each \((\leq)\)-ascending \(d\)-Cauchy sequence converges.

We are now in position to state the announced result. Assume that

\[(c03) \quad (\leq) \text{ is self-closed and } d \text{ is } (\leq)\)-complete;\]
and take a function \( \varphi : M \to R \cup \{ \infty \} \) in accordance with

(c04) \( \varphi \) is inf-proper (\( \text{Dom}(\varphi) \neq \emptyset \) and \( \varphi_* := \inf[\varphi(M)] > -\infty \))

(c05) \( \varphi \) is \((\leq)\)-lsc: \( \{ x \in M; \varphi(x) \leq t \} \) is \((\leq)\)-closed, for each \( t \in R \).

**Theorem 5.** Let these conditions hold. Then, for each \( u \in \text{Dom}(\varphi) \) there exists \( v \in \text{Dom}(\varphi) \) with

\[
(3.1) \quad u \leq v, d(u, v) \leq \varphi(u) - \varphi(v) \quad \text{(hence} \quad \varphi(u) \geq \varphi(v))
\]

\[
(3.2) \quad d(v, x) > \varphi(v) - \varphi(x), \quad \text{for each} \quad x \in M(v, \leq) \setminus \{ v \}.
\]

**Proof.** (cf. Turinici [18]) Denote for simplicity \( M[u] = \{ x \in M; u \leq x, \varphi(u) \geq \varphi(x) \} \). Clearly, \( \emptyset \neq M[u] \subseteq \text{Dom}(\varphi) \); moreover, by (c03)+(c05),

\[
(3.3) \quad M[u] \text{ is } (\leq)\text{-closed}; \quad \text{hence} \quad d \text{ is } (\leq)\text{-complete on } M[u].
\]

Let \((\leq)\) stand for the relation (over \( M \)): \( x \leq y \) iff \( x \leq y, \ d(x, y) + \varphi(y) \leq \varphi(x) \). Clearly, \((\leq)\) acts as a *order* (antisymmetric quasi-order) on \( \text{Dom}(\varphi) \); so, it remains as such on \( M[u] \). We claim that conditions of BB are fulfilled on \( (M[u], d; \leq) \). In fact, by this very definition, \( \varphi \) is \((\leq)\)-decreasing on \( M[u] \). On the other hand, let \( (x_n) \) be a \((\leq)\)-ascending sequence in \( M[u] \):

(c06) \( x_n \leq x_m \) and \( d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m) \), if \( n \leq m \).

The sequence \( (\varphi(x_n)) \) is descending and (by (c04)) bounded from below; hence a Cauchy one. This, along with (c06), tells us that \( (x_n) \) is a \((\leq)\)-ascending \( d \)-Cauchy sequence in \( M[u] \); wherefrom (by (3.3)), there must be some \( y \in M[u] \) with \( x_n \to y \). Passing to limit as \( m \to \infty \) in (c06) one derives (via (c03)+(c05))

\[
x_n \leq y, d(x_n, y) \leq \varphi(x_n) - \varphi(y), \quad \text{(i.e.:} \quad x_n \leq y), \quad \text{for all} \quad n.
\]

In other words, \( y \in M[u] \) is an upper bound (modulo \((\leq)\)) of \((x_n)\); and this shows that \( (M[u], \leq) \) is sequentially inductive. From BB it then follows that, for the starting \( u \in M[u] \) there exists \( v \in M[u] \) with \( j \ u \leq v \) and \( j j \) \( v \leq x \in M[u] \) implies \( \varphi(v) = \varphi(x) \). The former of these is just (3.1). And the latter one gives at once (3.2). In fact, let \( y \in M \) be such that \( v \leq y \), \( d(v, y) \leq \varphi(v) - \varphi(y) \). As a consequence, \( y \in M[u] \) and \( v \leq y \); so that (by \( j j \)) above \( \varphi(v) = \varphi(y) \). Combining with the previous relation gives \( d(v, y) = 0 \) (hence \( v = y \)); and we are done.

A basic particular case of our developments corresponds to \((\leq) = M \times M \) (=the *trivial* quasi-order on \( M \)); when (c05) may be written as
(c07) \( \varphi \) is \( d \)-lsc (\( \liminf_n \varphi(x_n) \geq \varphi(x) \), provided \( x_n \to x \)).

The corresponding version of Theorem 4 under this choice of our data is the 1974 Ekeland’s variational principle [8] (in short: EVP).

**Theorem 6.** Let \((M,d)\) be complete and \( \varphi : M \to R \cup \{ \infty \} \) be as in (c04) + (c07). Then, for each \( u \in \text{Dom}(\varphi) \) there exists \( v = v(u) \in \text{Dom}(\varphi) \) with

\[
\begin{align*}
(3.4) & \quad d(u,v) \leq \varphi(u) - \varphi(v) \text{ (hence } \varphi(u) \geq \varphi(v) \text{)} \\
(3.5) & \quad d(v,x) > \varphi(v) - \varphi(x), \text{ for all } x \in M \setminus \{ v \}.
\end{align*}
\]

This principle found some basic applications to control and optimization, generalized differential calculus, critical point theory and global analysis; we refer to the 1979 paper by Ekeland [9] for a survey of these. So, it cannot be surprising that, soon after its formulation, many extensions of EVP were proposed. For example, the dimensional way of extension refers to the support space \( (R) \) of \( \text{Codom}(\varphi) \) being substituted by a (topological or not) vector space. An account of the results in this area is to be found in the 2003 monograph by Goepfert, Riahi, Tammer and Zălinescu [10, Ch 3]; see also Turinici [19]. On the other hand, the (pseudo) metrical one consists in the conditions imposed to the ambient metric over \( M \) being relaxed. The basic result in this direction was obtained by Tataru [16]; see also Hyers, Isac and Rassias [12, Ch 5].

Finally [returning to the initial framework], note that (c05) also holds under (a02) (modulo \( \varphi \)) and (c03) (the first half). For this reason, Theorem 5 will be called the monotone version of Ekeland’s Variational Principle (in short: (EVPm)). Further technical aspects may be found in Hamel [11, Ch 4].

4. (EVPdLc) \( \implies \) (DC)

By the developments above, we have the implications: (DC) \( \implies \) (CU) \( \implies \) (BB) \( \implies \) (EVPm) \( \implies \) (EVP). So, it is natural asking whether these may be reversed. Clearly, the natural setting for solving this problem is (ZF)(=the standard Zermelo-Fraenkel system) without (AC) (=the Axiom of Choice); referred to in the following as the reduced Zermelo-Fraenkel system.
Let $X$ be a nonempty set; and $(\leq)$ be an order on it. We say that $(\leq)$ has the inf-lattice property, provided: $x \land y := \inf(x, y)$ exists, for all $x, y \in X$. Further, we say that $z \in X$ is a $(\leq)$-maximal element if $X(z, \leq) = \{z\}$; the class of all these points will be denoted as $\text{max}(X, \leq)$. In this case, $(\leq)$ is called a Zorn order when $\text{max}(X, \leq)$ is nonempty and cofinal in $X$ [for each $u \in X$ there exists a $(\leq)$-maximal $v \in X$ with $u \leq v$]. Further aspects are to be described in a metric setting. Let $d : X \times X \to R_+$ be a metric over $X$; and $\varphi : X \to R_+$ be some function. Then, the natural choice for $(\leq)$ above is
\[ x \leq_{(d, \varphi)} y \text{ iff } d(x, y) \leq \varphi(x) - \varphi(y); \]
referred to as the Brøndsted order [3] attached to $(d, \varphi)$. Denote $X(x, \rho) = \{u \in X; d(x, u) < \rho\}$, $x \in X$, $\rho > 0$ [the open sphere with center $x$ and radius $\rho$]. Call the ambient metric space $(X, d)$, discrete when for each $x \in X$ there exists $\rho = \rho(x) > 0$ such that $X(x, \rho) = \{x\}$. Note that, under such an assumption, any function $\psi : X \to R$ is continuous over $X$.

Now, the statement below is a particular case of EVP:

**Theorem 7.** Let the metric space $(X, d)$ and the function $\varphi : X \to R_+$ satisfy

\begin{enumerate}
\item[(d01)] $(X, d)$ is discrete bounded and complete
\item[(d02)] $(\leq_{(d, \varphi)})$ has the inf-lattice property
\item[(d03)] $\varphi$ is $d$-nonexpansive and $\varphi(X)$ is countable.
\end{enumerate}

Then, $(\leq_{(d, \varphi)})$ is a Zorn order.

We shall refer to it as: the discrete Lipschitz countable version of EVP (in short: (EVPdLc)). Clearly, (EVP) $\implies$ (EVPdLc). The remarkable fact to be added is that this last principle yields (DC); so, it completes the circle between all these.

**Proposition 2.** We have (in the reduced Zermelo-Fraenkel system) (EVPdLc) $\implies$ (DC). So (by the above), the maximal/variational principles (CU), (BB), (EVPm) and (EVP) are all equivalent with (DC); hence, mutually equivalent.
Proof. Let \( M \) be a nonempty set; and \( R \) stand for some relation over it, with the property (b01). Fix in the following \( a \in M \); as well as some \( b \in M \) (\( a, R b \)). For each \( n \geq 2 \) in \( \mathbb{N} (= \text{the set of natural numbers}) \) let \( N(n, >) := \{0, ..., n - 1\} \) stand for the initial segment determined by \( n \); and \( X_n \) denote the class of all finite sequences \( x : N(n, >) \to M \) with: \( x(0) = a, x(1) = b \) and \( x(m) R x(m + 1) \) for \( 0 \leq m \leq n - 2 \). In this case, \( N(n, >) \) is just \( \text{Dom}(x) \) (the domain of \( x \)); and \( n = \text{card}(N(n, >)) \) will be referred to as the order of \( x \) [denoted as \( \omega(x) \)]. Put \( X = \bigcup \{X_n; n \geq 2\} \). Let \( \preceq \) stand for the partial order (on \( X \)) (d04) \( x \preceq y \) iff \( \text{Dom}(x) \subseteq \text{Dom}(y) \) and \( x = y|_{\text{Dom}(x)} \);

and \( < \) denote its associated strict order. All we have to prove is that \((X, \preceq)\) has strictly ascending infinite sequences. To this end, we need some conventions and auxiliary facts.

(A) Let \( x, y \in X \) be arbitrary fixed. Denote

\[
K(x, y) := \{ n \in \text{Dom}(x) \cap \text{Dom}(y); x(n) \neq y(n) \}.
\]

If \( x \) and \( y \) are comparable (i.e.: either \( x \preceq y \) or \( y \preceq x \); written as: \( x <> y \)) then \( K(x, y) = \emptyset \). Conversely, if \( K(x, y) = \emptyset \), then \( x \preceq y \) if \( \text{Dom}(x) \subseteq \text{Dom}(y) \) and \( y \preceq x \) if \( \text{Dom}(y) \subseteq \text{Dom}(x) \); hence \( x <> y \). Summing up, we have

\[
(x, y \in X): \ x <> y \text{ if and only if } K(x, y) = \emptyset.
\]

The negation of this property means: \( x \) and \( y \) are not comparable (denoted as: \( x\nmid y \)). By the characterization above, it is equivalent with \( K(x, y) \neq \emptyset \). Note that, in such a case, \( k(x, y) := \min(K(x, y)) \) is well defined; and \( N(k(x, y), >) \) it is the largest initial interval of \( \text{Dom}(x) \cap \text{Dom}(y) \) where \( x \) and \( y \) are identical.

Lemma 1. The partial order \( (\preceq) \) has the inf-lattice property. Moreover, \( x \mapsto \omega(x) \) is strictly increasing (\( x < y \) implies \( \omega(x) < \omega(y) \)) and

\[
(4.2) \quad 2 \leq \omega(x \wedge y) \leq \min\{\omega(x) - 1, \omega(y) - 1\}, \text{ whenever } x\nmid y.
\]

Proof of Lemma 1. i) Let \( x, y \in X \) be arbitrary fixed. The case \( x <> y \) is clear; so, without loss, one may assume that \( x\nmid y \). Note that, by the remark above, \( K(x, y) \neq \emptyset \) and \( k := k(x, y) \) exists. Let the finite
sequence $z \in X_k$ be introduced as $z = x|_{N(k, >)} = y|_{N(k, >)}$. For the moment $z \preceq x$ and $z \preceq y$. Suppose that $w \in X_k$ fulfills the same properties. Then, the restrictions of $x$ and $y$ to $N(h, >)$ are identical; wherefrom (see above) $h \preceq k$ and $w \preceq z$.

ii) Evident.

iii) By the above notations, $\omega(x \land y) = k(x, y)$. This, and $k(x, y) \in \text{Dom}(x) \cap \text{Dom}(y)$, give the desired relation.\hfill $\square$

(B) Our next objective is to introduce a metrical structure as well as an associated objective function over $X$, which should have all required properties. To this end, put $\varphi(x) = 3^{-\omega(x)}$, $x \in X$; and note that $\varphi(X) = \{3^{-n} ; n \geq 2\}$ (hence, $\varphi$ has countable many strictly positive values). Then, define

$$(d05) \quad d(x, y) = |\varphi(x) - \varphi(y)|, \text{ if } x \not\approx y; \quad d(x, y) = \varphi(x \land y), \text{ whenever } x\parallel y.$$

Lemma 2. The mapping $(x, y) \mapsto d(x, y)$ is a metric on $X$ (in the usual sense).

Proof of Lemma 2. Clearly, $d$ is reflexive and symmetric [$d(x, y) = d(y, x)$, $x, y \in X$]. On the other hand, $d$ is sufficient. In fact, assume $d(x, y) = 0$. By a previous evaluation of $\varphi(X)$, it results that $x$ and $y$ are comparable and $\omega(x) = \omega(y)$; wherefrom, $x = y$. Finally, let us verify the triangular property: $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$. Without loss, one may assume that $\omega(x) \leq \omega(z)$; for, otherwise, we simply interchange $x$ and $z$. Two alternatives are open before us.

a) The points $x$ and $z$ are comparable; that is, $x \preceq z$ (by the hypothesis above). We start from the obvious relation

$$|\varphi(s) - \varphi(t)| \leq \max\{\varphi(s), \varphi(t)\} \leq \varphi(s \land t), \quad s, t \in X.$$ 

Combining with $d(x, z) = |\varphi(x) - \varphi(z)| \leq |\varphi(x) - \varphi(y)| + |\varphi(y) - \varphi(z)|$ yields the desired fact, for all possible cases concerning $(x, y)$ and $(y, z)$.

b) The points $x$ and $z$ are not comparable ($x\parallel z$). Four sub-cases appear:

Sub-case b1): Suppose that $x \not\approx y$, $y \not\approx z$. The alternatives $[x \preceq y$, $y \preceq z]$ and $[y \preceq x$, $z \preceq y]$ give $x \not\approx z$; contradiction. So, it remains to discuss:

b11) $x \preceq y$, $z \preceq y$. Then, $x$ and $z$ are the restrictions of $y$ to $\text{Dom}(x)$ and $\text{Dom}(z)$ respectively; wherefrom $x \preceq z$, contradiction.

b12) $y \preceq x$, $y \preceq z$. We start from the direct consequence of (4.2) above

$$3\max\{\varphi(s), \varphi(t)\} \leq \varphi(s \land t) \leq 3^{-2}, \quad s, t \in X, s\parallel t.$$
The relation to be checked becomes \( \varphi(x \land z) \leq 2\varphi(y) - \varphi(x) - \varphi(z) \). By the imposed conditions, \( y \leq x \land z \); wherefrom \( \varphi(y) \geq \varphi(x \land z) \). A sufficient condition for the desired relation to be true is then \( \varphi(x \land z) \leq 2\varphi(x \land z) - \varphi(x) - \varphi(z) \); or, equivalently, \( \varphi(x) + \varphi(z) \leq \varphi(x \land z) \); evident, by the precise consequence.

Sub-case b2): Suppose that \( x || y, y < z \). Two logical possibilities occur:

b21) \( x || y, y \leq z \). We have to establish that: \( \varphi(x \land z) \leq \varphi(x \land y) + \varphi(y) - \varphi(z) \). But, evidently, \( x \land z \geq x \land y \); wherefrom \( \varphi(x \land z) \leq \varphi(x \land y) \); and then, all is clear.

b22) \( x || y, z \leq y \) (or, equivalently: \( z \leq y, y || x \)). The desired relation becomes: \( \varphi(x \land z) \leq \varphi(x \land y) + \varphi(z) - \varphi(y) \). For the moment, \( x \land z \leq x \land y \). If \( x \land z < x \land y \), we must get \( q := \omega(x \land z) < \omega(x \land y) \); so (by definition) \( x(q) = y(q) \). As \( z = y \mid \text{Dom}(z) \) and \( q \in \text{Dom}(x) \cap \text{Dom}(z) \), this yields \( y(q) = z(q) \); hence \( x(q) = z(q) \), contradiction. Consequently, \( x \land z = x \land y \); and conclusion follows.

Sub-case b3): \( x \triangleq y, y || z \). As before, two logical possibilities occur:

b31) \( x \leq y, y || z \). This is just the alternative b22), with \( (x, y, z) \) in place of \( (z, y, x) \).

b32) \( y \leq x, y || z \). The relation under consideration becomes \( \varphi(x \land z) \leq \varphi(y) - \varphi(x) + \varphi(y \land z) \). But, from hypothesis, \( x \land z \geq y \land z \); wherefrom, all is clear.

Sub-case b4): \( x || y, y || z \). We have to establish that: \( \varphi(x \land z) \leq \varphi(x \land y) + \varphi(y \land z) \). As before, the alternative \( [\omega(x \land z) \geq \omega(x \land y) \text{ or } \omega(x \land z) \geq \omega(y \land z)] \) gives the desired fact. On the other hand, the alternative \( q := \omega(x \land z) < \min\{\omega(x \land y), \omega(y \land z)\} \) yields \( x(q) = y(q), y(q) = z(q) \); hence \( x(q) = z(q) \), contradiction.

Having discussed all possible cases, the conclusion in the statement follows. \( \square \)

(C) Note that, by a previous remark involving \( \varphi(X) \), one has \( \text{diam}(X) \leq 3^{-2} \). Further properties of the triplet \( (X, d; \varphi) \) are contained in

**Lemma 3.** Under the notations above, one has (for each \( m \))

\[
(4.3) \quad x, y \in X, \quad \omega(x) \leq m, \quad d(x, y) < 2 \cdot 3^{-m-1} \implies x = y;
\]

so that, the metric space \( (X, d) \) is discrete.

**Proof of Lemma 3.** Assume that \( x \neq y \). We show that this cannot be in agreement with the accepted hypothesis. Two cases are open before us:
i) Let \( x \) and \( y \) be comparable: either \( x \prec y \) or \( y \prec x \). If \( x \prec y \), we have \( \omega(x) + 1 \leq \omega(y) \); and then \((1/3)\varphi(x) \geq \varphi(y)\); hence (by definition) \( d(x, y) = \varphi(x) - \varphi(y) \geq (2/3)\varphi(x) \geq 2 \cdot 3^{-m-1} \), contradiction. If \( y \prec x \) then (by the same way as before) \( d(x, y) \geq (2/3)\varphi(y) \geq (2/3)\varphi(x) \geq 2 \cdot 3^{-m-1} \), again a contradiction.

ii) Suppose that \( x \) and \( y \) are not comparable. Then (by definition) \( d(x, y) = \varphi(x \land y) \geq \varphi(x) \geq 3 \cdot 3^{-m-1} \), contrary to the accepted hypothesis. □

**Lemma 4.** Under the same notations above,

\[(4.4) \quad |\varphi(x) - \varphi(y)| \leq d(x, y), \ \forall x, y \in X; \]

so, \( \varphi \) is \( d \)-nonexpansive (hence, all the more \( d \)-Lipschitz).

**Proof of Lemma 4.** If \( x \) and \( y \) are comparable, then \( d(x, y) = |\varphi(x) - \varphi(y)| \); and we are done. If \( x \) and \( y \) are not comparable then, without loss, one may assume \( \omega(x) \leq \omega(y) \); hence \( \varphi(x) \geq \varphi(y) \). As \( x \land y \leq x \), we have \( d(x, y) = \varphi(x \land y) \geq \varphi(x) \geq \varphi(x) - \varphi(y) = |\varphi(x) - \varphi(y)| \); and the conclusion follows.

(D) Given the couple \((d, \varphi)\) as before, we may introduce the associated Brøndsted order \((\leq_{(d, \varphi)})\) on \( X \); also denoted as \((\leq)\), for simplicity. It is natural to ask which is the relationship between it and the initial order \((\leq)\) on \( X \).

**Lemma 5.** We necessarily have (under these conventions)

\[(4.5) \quad x \leq y \text{ if and only if } x \leq y. \]

That is: these partial orders coincide over \( X \).

**Proof of Lemma 5.** Clearly, \( x \leq y \) gives \( \omega(x) \leq \omega(y) \); wherefrom \( d(x, y) = \varphi(x) - \varphi(y) \); i.e., \( x \leq y \). Conversely, assume that \( x \leq y \). For the moment, \( x \) and \( y \) are comparable; since, otherwise, the imposed condition gives \( \varphi(x \land y) \leq \varphi(x) - \varphi(y) \leq \varphi(x) \) [hence \( \omega(x \land y) \geq \omega(x) \)]; in contradiction with \((4.2)\). The alternative \( y \leq x \) yields (by the first part) \( y \leq x \); wherefrom (as \((\leq)\) is order) \( x = y \). Hence, anyway \( x \leq y \). □

(E) We are now in position to complete the argument. As \((b01)\) holds, we necessarily have \( \max(X, \preceq) = \emptyset \); i.e.: for each \( x \in X \) there exists \( y \in X \) with \( x \prec y \). This, along with \((\text{EVPdLe})\), tells us that \((X, d)\) is not complete; i.e., there exists at least one \( d \)-Cauchy sequence \((x_n)\) in \( X \) which
is not convergent. Note that both these properties are transferable to all subsequences of \((x_n)\). This allows us to take our sequence in such a way that, for all \(m\),
\[(d06) \ d(x_p,x_q) < 3^{-m-1}, \text{ whenever } p,q \geq m.\]
In fact, the \(d\)-Cauchy property assures us (with \(\varepsilon = 3^{-m-1}\)) that
\[C(m) := \{n \in \mathbb{N}; d(x_p,x_q) < 3^{-m-1}, \text{ for all } p,q \geq n\} \neq \emptyset, \forall m \in \mathbb{N}.
\]
In addition, \(n \mapsto C(n)\) is \((\subseteq)\)-decreasing; hence \(n \mapsto g(n) := \min[C(n)]\) is \((\subseteq)\)-increasing. This finally tells us that \(n \mapsto h(n) := n + g(n)\) is strictly \((\subseteq)\)-increasing; wherefrom \((y_n = x_{h(n)}; n \in \mathbb{N})\) is a subsequence of \((x_n)\) fulfilling \((d06)\), in view of: \(p,q \geq m \Rightarrow h(p), h(q) \geq h(m) \geq g(m)\).

Now, by the divergence of \((x_n)\), we must have
\[(4.6) \ B(k,m) := \{n \in \mathbb{N}(k,\prec); \omega(x_n) > m\} \neq \emptyset, \text{ for all } k,m \in \mathbb{N}.
\]
For, otherwise, there exist \(k,m \in \mathbb{N}\) (with \(k \geq m\)) such that \(\omega(x_n) \leq m, \ \forall n \geq k\); [wherefrom \((d06)\) \(d(x_k,x_n) < 3^{-k-1} \leq 3^{-m-1}, \forall n \geq k\)] and this, by Lemma 3, leads us to \([x_n = x_k, \forall n \geq k]\); hence, \((x_n)\) is convergent, contradiction. We now consider the following sequential type algorithm:
\[
p(0) = 1, \ q(0) = \omega(x_{p(0)}) \ (\text{ hence } q(0) \geq 1) \ 
q(n+1) = \min B(p(n); q(n)), \ q(n+1) = \omega(x_{p(n+1)}), \ n \geq 0.
\]
Note that, by the precise character of such a convention, no use of \((\text{DC})\) is necessary. The rank sequence \((p(n); n \in \mathbb{N})\) is strictly ascending \([1 = p(0) < p(1) < p(2) < \ldots]\); hence, \(p(n) > n\), for all \(n\). It therefore generates a subsequence \((y_n := x_{p(n)}; n \in \mathbb{N})\) of \((x_n)\) with the supplementary property (deducible from \((d06)\))
\[(4.7) \ d(y_k,y_h) < 3^{-k-1}, \text{ whenever } k,h \in \mathbb{N} \text{ satisfy } k < h.
\]
In addition, the rank sequence \((q(n) = \omega(y_n); n \in \mathbb{N})\) is strictly ascending too \([1 \leq q(0) < q(1) < q(2) < \ldots]\); hence, \(q(n) > n\), for all \(n\). The sequence \((z_n := y_n|_{N(k,>)}; n \in \mathbb{N})\) is therefore well defined in \(X\). We claim that
\[(4.8) \ (z_n) \text{ is strictly ascending: } z_k < z_h, \text{ for } k < h.
\]
In fact, if \(y_k \prec y_h\) we must have \(y_k \prec y_k \text{ (as } \omega(y_k) < \omega(y_h))\); and, from this, \(z_k < z_h\). And, if \(y_k|y_h\) we have (by \((4.7)\)) \(\omega(y_k \land y_h) > k\); which yields
\[z_k = y_k|_{N(k,>)} = y_h|_{N(k,>)} = z_h|_{N(k,>)};
\]
wherefrom \( z_k \prec z_h \); hence the claim. But then, the sequence \( (c_n = z_{n+1}(n); n \in N) \) is well defined in \( M \); and, moreover, \( c_0 = a, c_n R c_{n+1}, \) for all \( n \). This gives us the desired conclusion. 

In particular, when the specific assumptions (d02) and (d03) are ignored in Theorem 7, Proposition 2 above reduces to the result in Brunner [4]; but, the lattice type methods used here seem to be new.

Summing up, the maximal/variational results in Section 3 are nothing but logical equivalents of (EVP). This also includes various extensions of (EVP) reducible to (BB); such as the one in Tataru [16]. So, it is natural to ask whether the remaining ones – including the asymptotic version of (BB) in Turinici [20] and Du [7] – are endowed as well with such a property. The answer to this is affirmative; further aspects will be delineated elsewhere.

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