COMPUTER VISIONS FOR THE EIGENVALUE PROBLEMS OF EMBEDDED SURFACES AND PLANE DOMAINS

BY

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Abstract. In this paper, we show the numerical computations and the computer graphics of the eigenvalues and eigenfunctions for the eigenvalue problems of the Laplacian on compact embedded surfaces. Namely, we examine the behaviors of the eigenvalues and eigenfunctions of the dumbbells connected by two spheres.

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Key words: eigenvalue problem, Laplacian, embedded surface.

1. Introduction

We consider the eigenvalue problems of the Laplacian. When $\Omega$ is a bounded domain in $\mathbb{R}^n$ with the piecewise smooth boundary $\partial\Omega$, our problem is to find the eigenfunctions $u \neq 0$ and the eigenvalues $\lambda$ which satisfy $\Delta u = \lambda u$ (on $\Omega$) and the Dirichlet or Neumann conditions (on $\partial\Omega$). We study these problems by applying the technique of functional analysis to an arbitrarily given domain. But it would be impossible in general to obtain the exact solution of the eigenvalue problem of the Laplacian except very few examples. To overcome these difficulties, we propose the finite element method. The finite element method is extensively used for solving problems described by differential equations in the structural mechanics field, such as electronic state numeration, electromagnetic-field analysis, and fluid analysis. We use here our improved version of the finite element method due to [6], [7], [8].
2. The FEM for the eigenvalue problems

Divide \( \bar{\Omega} = \Omega \cup \partial \Omega \) into a finite numbers of triangles, and set \( \Xi = \{ e_\mu \} \), the triangulation of \( \Omega \) in such a way that for any two triangles \( e_\mu \) & \( e_\nu \), \( e_\mu \cap e_\nu \neq \emptyset \) if and only if they have a common edge or a common vertex.

The bounded domain \( G \) is the interior of \( \bigcup_{\mu=1}^{l} e_\mu \) and \( \Gamma = \partial G \). Let \( P_1, \ldots, P_l \) be the vertices belonging to \( \Omega \) and let \( P_{l+1}, \ldots, P_m \) be the one belonging to \( \partial \Omega \) \( (l < m) \). Define \( \psi_i \), the basic functions associated to \( \Xi \), as follows:

1. \( \psi_i(x,y) \) is at most linear on each \( e_\mu \), i.e., \( \psi_i(x,y) = a_i^\mu x + b_i^\mu y + c_i^\mu \) if \( (x,y) \in e_\mu \).
2. \( \psi_i(P_j) = \delta_{ij} \).

Define \( \hat{u} \) a function on \( G \) by \( \hat{u}(x,y) = \sum_{i=1}^{l} u_i \psi_i(x,y) \), for every \( u = (u_1, \ldots, u_l) \in \mathbb{R}^l \). Define \( K(\Xi) = (K_{ij}(\Xi)) \) and \( M(\Xi) = (M_{ij}(\Xi)) \), called the stiffness matrix and the mass matrix, respectively, by

\[
\begin{align*}
K_{ij}(\Xi) &= \int_{G(\Xi)} \langle \nabla \psi_i, \nabla \psi_j \rangle g(\Xi) v_{g}(\Xi), \\
M_{ij}(\Xi) &= \int_{G(\Xi)} \psi_i \psi_j v_{g}(\Xi),
\end{align*}
\]

where \( \langle , \rangle_{g(\Xi)} \) is the inner product with respect to the continuous Riemannian metric \( g(\Xi) = i^*g_0 \) on \( G(\Xi) \), induced from the inclusion \( i : G(\Xi) \subset \mathbb{R}^n \).

We consider the following eigenvalue problem \( K(\Xi)u = \nu M(\Xi)u, \ u \in \mathbb{R}^l \).

Then, the following theorem is well known (cf. [1], [2], [3], [4], [5], [6]):

Theorem 1. Let \( k = 1, 2, \ldots \). Then \( \lim_{\delta(\Xi) \to 0} \nu_k(\Xi) = \lambda_k \), where \( \delta(\Xi) \to 0 \) means that the triangulation \( \Xi \) of \( \Omega \) is fine enough.

Theorem 2. \( \exists \{ \Xi_p \}_{p=1}^{\infty} \) a sequence of triangulation of \( \Omega \) s.t, for each \( k = 1, 2, \ldots \), \( \| \hat{u}_k(\Xi_p) - u_k \|_{H^1(\Omega)} \to 0 \), as \( p \to \infty \).
3. Numerical results of eigenfunction

In this section, we show the computer graphics for the eigenvalue problems of the Laplacian on compact embedded surfaces.

We show pictures of the eigenfunctions of the unit sphere, the embedded torus, and the dumbbells. We color all the figures so that the red, blue and green express positive, negative, and near zero values of the eigenfunctions, respectively. We first compare the exact eigenvalues of the standard unit sphere \( (S^2, g_0) \) and our numerical results. The exact eigenvalues are \( l(l-1), l = 1, 2, 3, \ldots \), and their multiplicities are \( 2l - 1 \). The following table shows how accurate our computation is, and that our method works correctly.

Here FEM means the result by the Finite Element Method.

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4. The eigenfunction of the dumbbell by two spheres

In this section, we show the variation of the eigenvalues by deforming either the distance between the centers or the radii. We give the numerical results on the domain given in Fig. 4. Here \( d \) is the distance between the centers, \( R \) and \( r \) are the radii of the dumbbell by two spheres. The variation of the eigenvalues can be shown in Fig. 5. Finally, \( \lambda_{1,0,1,0,0,d}^k / \lambda_{1,0,1,0,d}^k \) can be shown in Fig. 6, 7, where \( \lambda_{R,r,d}^k \) is the \( k \)-th eigenvalue when the distance between the centers is \( d \) and \( R \) and \( r \) are the radii of the dumbbell by two spheres. \( \lambda_{1,0,1,0,0,0}^k \) means the \( k \)-th eigenvalue of the unit sphere.

5. Some observations

In this work, the variations of the eigenvalues or eigenfunctions of the dumbbell by two spheres have been analyzed by FEM. The numerical results of the eigenvalues are shown in Fig. 6, 7. From these results, we can expect that

for every \( d > 0 \), there would exist a constant \( C > 0 \) such that

\[
\lim_{k \to \infty} \frac{\lambda_{1,0,1,0,0,d}^k}{\lambda_{1,0,1,0,d}^k} = C,
\]
in particular, if \( d \) is close to 2.0, then it would hold that

\[
\lim_{k \to \infty} \frac{\lambda_{1,0,1,0,0}^k}{\lambda_{1,0,1,0,d}^k} = 2.0.
\]

And the numerical results of the eigenfunctions are shown in Fig. 8, . . . , Fig. 14. Based on the next theorem, we can expect that the eigenfunction, change smoothly by both the deformations of the distance between the centers and the radii.

**Theorem** (Courant Nodal Domain Theorem).  (i) The first eigenfunction cannot have any nodes.

(ii) For \( n > 2 \), the eigenfunction corresponding to the \( n \)-th eigenvalue counting multiplicity, divides the domain \( \Omega \) into at least 2 and at most \( n \) pieces.

Figure 1: The 2nd, 6th, 10th and 15th eigenfunctions of the unit sphere

The second example is the embedded torus (see Fig. 2).

The final example is the embedded dumbbells (see Fig. 3).

Next, we show pictures of the eigenfunction, varying the radius from 0.1 to 0.7.
Figure 2: The 2nd, 6th, 9th and 14th eigenfunctions of the embedded torus

Figure 3: The 2nd, 6th, 9th and 14th eigenfunctions of the dumbbells
Figure 4: The dumbbell by two spheres

Figure 5: Variations of the eigenvalues $\lambda_{1.0.1.0,d}^k$ of the dumbbell by two spheres
Figure 6: Variations of $\lambda_{1,0,1,0,0,0}^k/\lambda_{1,0,1,0,1,0}^k$

Figure 7: Variations of $\lambda_{1,0,1,0,0,0}^k/\lambda_{1,0,1,0,1,9}^k$
Figure 8: The 2nd, 6th, 9th and 14th eigenfunctions (R=r=1.0, d=1.0) from the upper left to the lower right.

Figure 9: The 2nd, 6th, 9th and 14th eigenfunctions (R=r=1.0, d=1.3) from the upper left to the lower right.
Figure 10: The 2nd, 6th, 9th and 14th eigenfunctions (R=r=1.0, d=1.6) from the upper left to the lower right

Figure 11: The 2nd, 6th, 9th and 14th eigenfunctions (R=r=1.0, d=1.9) from the upper left to the lower right
Figure 12: The 2nd, 6th, 9th and 14th eigenfunctions (R=1.0, r=0.1, d=1.0) from the upper left to the lower right.

Figure 13: The 2nd, 6th, 9th and 14th eigenfunctions (R=1.0, r=0.5, d=1.0) from the upper left to the lower right.
Figure 14: The 2nd, 6th, 9th and 14th eigenfunction ($R=1.0$, $r=0.7$, $d=1.0$) from the upper left to the lower right
REFERENCES


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