THE SOLUTION OF THE PROBLEM OF CENTER FOR CUBIC DIFFERENTIAL SYSTEMS WITH FOUR INVARIANT STRAIGHT LINES

BY

DUMITRU COZMA* and ALEXANDRU ŢUBĂ**

Abstract. For cubic differential systems with four invariant straight lines we show that any singular point of a center or focus type is center if and only if the focal values $g_3$ and $g_5$ vanish. This is attained by constructing a first integral or an integrating factor of Darboux form.

1. Introduction. Consider the autonomous system of differential equations with polynomial right-hand sides

\[
\begin{align*}
\dot{x} &= X(x, y), \\
\dot{y} &= Y(x, y).
\end{align*}
\]

The straight line $C + Ax + By = 0$ is said to be invariant for (1), if there exists a polynomial $R(x, y)$ such that the identity holds

\[
AX(x, y) + BY(x, y) \equiv (C + Ax + By)R(x, y).
\]

The set of invariant straight lines (linear particular integrals) of the system (1) can be infinite (for example, in the case $X(x, y) \equiv 0$), finite or empty. In this paper systems of the form (1) with an infinite number of invariant straight lines will be not considered. Recently much attention has been given to the problem of investigation of the polynomial systems with invariant straight lines. There are many papers concerned with this subject. Let us distinguish some aspect of this problem.
1.1. Condition for the existence of invariant straight lines. Let $X = X_1 + X_j$, $Y = Y_1 + Y_j$, $j \geq 2$, where $X_i, Y_i$ are homogeneous polynomials in $x$ and $y$ of degree $i$. In [1,2] were found centroaffine invariant conditions for the existence of a homogeneous linear particular integral

\[(3)\quad Ax + By = 0 \quad (|A| + |B| \neq 0)\]

for (1). If $j = 2$, in [3,4] were investigated conditions for the existence of a nonhomogeneous linear particular integral

\[(4)\quad 1 + Ax + By = 0 \quad (|A| + |B| \neq 0).\]

Furthermore, in [3] is assumed that a quadratic system ($j = 2$) has a singular point of a center or focus type (a weak focus). In [5] for a cubic system ($\deg(X^2 + Y^2) = 6$) with a weak focus are given the conditions for the existence of four invariant straight lines of the form (4).

1.2. The bound of the number of invariant straight lines. Let $2n = \deg(X^2 + Y^2)$. Let us denote as in [6] by $L(n)$ the maximum number of invariant straight lines of the system (1) (we remember, that systems of the type (1) with an infinite number of invariant straight lines will be not considered). In [7] was shown, that $L(2) = 5$, in [8,9] that $L(3) = 8$, in [6] that $L(4) = 9$, in [10] that $L(5) = 14$ and $2n + 1 + \frac{1 + (-1)^n}{2} \leq L(n) \leq 3n - 1$, $n > 5$. In [10] for $n \in \{6, 7, \ldots, 20\}$ was improved the lower bound. For example, $15 \leq L(6) \leq 17$, $43 \leq L(20) \leq 59$.

1.3. The invariant straight lines and limit cycles.

A quadratic system:
- with two real invariant straight lines has no limit cycles [11];
- with two complex conjugated invariant straight lines can have no more than one limit cycle [12];
- with one invariant straight line can have only one limit cycle and no more [13–15].

A cubic system:
- with five real invariant straight lines has no limit cycles [16];
- with four real invariant straight lines or with two real and two complex conjugated invariant straight lines has at most one limit cycle [17,18];
- with four complex conjugated invariant straight lines can have at least two limit cycles [18];
the cyclicity of a center in the case of four different invariant straight lines of the form (4) (generally speaking with complex coefficients) is at most one [19].

A qualitative investigation of cubic systems with exactly eight and exactly seven invariant straight lines is done in [8,9].

In some cases, the problem of the existence of limit cycles for (1), \( \text{deg}(X^2 + Y^2) = 2n \), with \( n + 1 \) invariant straight lines was studied in [20]. In [21] was proved that (1) (\( \text{deg}(X^2 + Y^2) = 2n \)) with at least \( (n-1)(n+2)/2 \) invariant straight lines has no limit cycles (taking into account the bound \( L(n) \leq 3n - 1 \), this result works only for \( n \leq 5 \)).

1.4. The invariant straight lines and the problem of center.
We now consider the cubic systems \( \text{deg}(X^2 + Y^2) = 6 \) with which this paper is concerned. Assume that the origin is a singular point with pure imaginary eigenvalues. Coordinate changes of axis and time rescaling bring the system (1) to the form

\[
\begin{aligned}
\dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + px^2y^2 + ry^3 \equiv P(x, y), \\
\dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qxy^2 + nxy^2 + ly^3) \equiv -Q(x, y)
\end{aligned}
\]  

(5)

in which all variables and coefficients are assumed to be real. The origin is either a focus or a center for (5). The problem arises of their distinction (the problem of center or the center–focus problem). Certain sufficient center conditions for (5) with invariant straight lines were obtained in [19, 22, 23].

As follows from [5], the system (5) cannot have more than four invariant straight lines of the form (4). As homogeneous invariant straight lines (see (3)) the system (5) can have only the complex lines \( x \pm iy = 0 \), \( i^2 = -1 \) (if we consider (5) in \( \mathbb{C} \), then the origin is a saddle and the invariant straight lines are the separatrix of this saddle). Hence, the system (5) can have at most six invariant straight lines. In this paper we occur an example with six such lines.

In [19] (see also [24]) is proven

**Lemma 1.** The cubic system (5) with four different invariant straight lines (real, complex or real and complex) of the form (4) has at the origin a center if and only if the first focal value \( g_3 \) vanishes.

In coefficients of (5) \( g_3 \) looks as following

\[
g_3 = ac - bd + 2ag - 2bf + cf - dg - 3k + 3l - p + q.
\]  

(6)
The invariant straight line $C + Ax + By = 0$ is said to be $j$-invariant, if $j, j \geq 1$ is the maximum natural number such that $(C + Ax + By)^{j-1}$ divides $R(x, y)$ (see (2)). As in [25], the polynomial $R(x, y)$ is called the cofactor of the invariant straight line and number $j$ is called the degree of invariance. When we count the number of invariant straight lines of the polynomial system (1) the $j$-invariant straight line will be counted $j$ times.

By Lemma 1 to solve the problem of center completely for cubic systems with four invariant straight lines it remain to consider the cases, when among such lines may occur $j$-invariant ($j > 1$) lines or the lines $x \pm iy = 0$ (we note, that $x \pm iy = 0$ can be 1-invariant straight lines for (5)). For this purpose we shall prove the following theorem:

**Theorem 1.** Let the cubic differential system has invariant straight lines with summary degree of invariance at least four. Then any singular point with pure imaginary eigenvalues of this system is a center if and only if its first two focal values vanish ($g_3 = g_5 = 0$).

**2. The proof of Theorem 1.** The straight line (4) is an invariant straight line of (5) if and only if $A$ and $B$ are the solutions of the system

$$
\begin{align*}
F_1(A, B) &= A^2B + aA^2 - gAB - kA + sB = 0, \\
F_2(A, B) &= AB^2 - fAB + bB^2 + rA - lB = 0, \\
F_3(A, B) &= B^3 - 2A^2B + fA^2 + (c - b)AB - dB^2 - pA + nB = 0, \\
F_4(A, B) &= A^3 - 2AB^2 - cA^2 + (d - a)AB + gB^2 + mA - qB = 0.
\end{align*}
$$

The cofactor of (4) is

$$
R(x, y) = -Bx + Ay + (aA - gB + AB)x^2 + (cA - dB + B^2 - A^2)xy + (fA - bB - AB)y^2.
$$

**2.1. Two pairs of 2-invariant straight lines.** For (5) the degree of invariance of the straight lines is at most two and 2-invariant can be only the lines of the form (4).

Let each of the invariant straight lines $L_1 : 1 + A_1x + B_1y = 0$ and $L_2 : 1 + A_2x + B_2y = 0$, $L_1 \neq L_2$, is 2-invariant straight line of the system (5), i.e.

$$
A_1P - B_1Q \equiv L_1^2R_1, \ A_2P - B_2Q \equiv L_2^2R_2,
$$
where $R_{1,2} = -B_{1,2}x + A_{1,2}y$. It is obvious that $(L_1^2 R_1)/(L_2^2 R_2) \neq \text{const.}$ This means that the straight lines $L_1$ and $L_2$ are not parallel, that is $\Delta = A_1 B_2 - A_2 B_1 \neq 0$. From (9) we find that

\[(10) \quad P = (B_2 L_1^2 R_1 - B_1 L_2^2 R_2)/\Delta, \quad Q = (A_2 L_1^2 R_1 - A_1 L_2^2 R_2)/\Delta.\]

The first focal value (6) for (5), (10) looks as

\[g_3 = (A_1^2 - A_2^2 + B_1^2 - B_2^2)(A_1 A_2 + B_1 B_2)/\Delta^2.\]

Assume that $g_3 = 0$, then (5) with the right-hand sides (10) has an integrating factor of the form

\[\mu = L_1^{\alpha_1} L_2^{\alpha_2} \exp(\alpha_3 L_1^{-1} + \alpha_4 L_2^{-1}),\]

where

\[\alpha_1 = \alpha_1(A_1, B_1, A_2, B_2) = -2 + B_2^2(B_1^2 - A_2^2 - B_2^2)/\Delta^2,\]

\[\alpha_2 = \alpha_1(A_2, B_2, A_1, B_1),\]

\[\alpha_3 = \alpha_3(A_1, B_1, A_2, B_2) = [2(A_1 A_2 + B_1 B_2)(A_2^2 - A_1 A_2 + B_2^2) - (A_2^2 + B_2^2)(B_1^2 + B_2^2)]/\Delta^2,\]

\[\alpha_4 = \alpha_3(A_2, B_2, A_1, B_1).\]

We note that the existence of an integrating factor $\mu(x, y)$ for (5) ensures the origin $O(0, 0)$ to be a center (see, for example [26]).

Thus, it is proved

**Lemma 2.** The cubic system (5) with two 2-invariant straight lines has at the origin a center if and only if $g_3 = 0$.

### 2.2. One 2-invariant and two 1-invariant straight lines.
If the system (5) has only one 2-invariant straight line $L_1 = 0$, then the line must be real, i.e., must be of the form (4). Via a rotation of axis about the origin we make this line parallel to the $Oy$ axis, that is, we can assume that $L_1 = 1 + \frac{c}{2}x$ and then $P = y L_1^2$. 

2.2.1. The case of the lines \( x \pm iy = 0 \). The system (5) has the invariant straight lines \( x \pm iy = 0 \) or the particular integral \( x^2 + y^2 = 0 \), if and only if the following series of conditions
\[
\begin{align*}
  b + c - g &= a + d - f = p - q + l - k = m + n - r - s = 0
\end{align*}
\]
hold. Taking into account (11), the system (5) with particular integrals \( L_1 = 1 + \frac{c}{2}x = 0 \) (2-invariant) and \( L_2 = x^2 + y^2 = 0 \) can be written as
\[
\begin{align*}
  \dot{x} &= yL_1^2, \\
  \dot{y} &= -xL_1^2 - (b + nx + ly)L_2.
\end{align*}
\]
The vanishing of the first focal value \( g_3 \) (see (6)) gives \( l = 0 \). For \( l = 0 \), (12) has an integrating factor \( \mu = L_1^{-2}L_2 \) (a first integral \( L_1^{\alpha_1}L_2^{\alpha_2}exp(\alpha_3L_1^{-1}) = const. \), with \( \alpha_1 = -8n, \alpha_2 = -c^2, \alpha_3 = 8(bc - 2n) \)).

2.2.2. The case of the invariant straight lines \( L_1 = 1 + \frac{c}{2}x = 0 \), \( L_{2,3} = 1 + A_{2,3}x + B_{2,3}y = 0 \). The straight line \( 1 + Ax + By = 0 \) is an invariant straight line for the system of differential equations
\[
\begin{align*}
  \dot{x} &= y(2 + cx)^2/4, \\
  \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3)
\end{align*}
\]
if and only if, its coefficients are solutions of the system of algebraic equations (see (7)):
\[
\begin{align*}
  B(A^2 - gA + s) &= 0, \quad B(AB + bB - l) = 0, \\
  B[2A^2 + (b - c)A - B^2 + dB - n] &= 0, \\
  4A^3 - 8AB^2 - 4cA^2 + 4dAB + 4gB^2 + c^2A - 4qB &= 0.
\end{align*}
\]
Solving this system for \( A \) and \( B \), we find that for (13) exactly three invariant straight lines (one 2-invariant and two 1-invariant) can occur only in the following five cases:

a) \[
\begin{align*}
  g &= (c - 2b)/2, \quad d = [2l^2(c + 2b) - b(bc + 2s)^2][2l(bc + 2s)]^{-1}, \\
  n &= [4l^2 - (bc + 2s)(bc + 4s)][2(bc + 2s)]^{-1}, \\
  q &= [l^2(c^2 - 4s) + (bc + 2s)(bcs + 2s^2 - 2l^2)][2l(bc + 2s)]^{-1}.
\end{align*}
\]
The straight lines are
\[ 1 + \frac{c}{2} x = 0, \]
\[ 1 + \frac{c - 2b \pm \sqrt{(c - 2b)^2 - 16s}}{4} x + \frac{l(c + 2b \mp \sqrt{(c - 2b)^2 - 16s})}{2(bc + 2s)} y = 0. \]

The first focal value looks 
\[ g_3 = [4l^2 + (bc + 2s)^2](4l)^{-1} \neq 0 \]
and therefore \( O(0, 0) \) is a focus.

b) \( c = g = -2b, \ s = b^2, \ q = -bd, \ l = 0. \)

The straight lines are
\[ L_1 = 1 - bx, \ L_{2,3} = 1 - bx + \frac{d \pm \sqrt{d^2 - 4b^2 - 4n}}{2} y \]
and (13) has an integrating factor
\[ \mu = 16[(2 + cx)^2(c^2 x^2 + 2cdxy + (c^2 + 4n)y^2 + 4cx + 4dy + 4)]^{-1} \]
and a first integral
\[ L_1^{\alpha_1} L_2^{\alpha_2} L_3^{\alpha_3} \exp[4\alpha_4(2 + cx)^{-1}] = C, \]
with
\[ \alpha_1 = 2n\sqrt{d^2 - 4b^2 - 4n}, \]
\[ \alpha_{2,3} = b^2(\sqrt{d^2 - 4b^2 - 4n} \mp d), \]
\[ \alpha_4 = (\alpha_1 + \alpha_2 + \alpha_3)/2. \]

c) \( s = -b(b + g), \ q = d(b + g), \ n = b[4(b + g)(b + c) - c^2][4(g + 2b)]^{-1}, \ l = 0. \)

The straight lines are
\[ L_1 = 1 + \frac{c}{2} x, \ L_{2,3} = 1 - bx + y(d \pm 2\delta)[4(g + 2b)]^{-1}, \]
where \( \delta = \sqrt{(g + 2b)(b(c + 2b)^2 + d^2(g + 2b))}. \)
We can construct an integrating factor
\[ \mu = 16(g + 2b)\left[(2 + cx)^2(4(g + 2b)(1 - bx)(1 - bx + dy) - b(c + 2b)^2y^2)\right]^{-1}, \]
and a first integral of the form (14) with
\[ \alpha_1 = 2b\delta(4b^2 + 4bc + 4bg - c^2 + 4cg), \]
\[ \alpha_{2,3} = c^2(2b + g)(\delta - 2bd - dg), \]
\[ \alpha_4 = b(2b + g)(2b - c + 2g). \]

d) \[ b = g = s = q = l = 0. \]
The straight lines are
\[ L_1 = 1 + \frac{c}{2}x, \quad L_{2,3} = 1 + \frac{d \pm \sqrt{d^2 - 4n}}{2} y. \]
The system (13) admits an integrating factor \( \mu = 4[(2+cx)^2(1+dy+ny^2)]^{-1} \) and a first integral of the form (14) with
\[ \alpha_1 = 2\alpha_4 = 4n\sqrt{d^2 - 4n}, \]
\[ \alpha_{2,3} = c^2(\sqrt{d^2 - 4n} \mp d) \]
e) \[ g = -b, \quad d = (bl^2 - bs^2 - cs^2)(ls)^{-1}, \]
\[ q = (4s^2 - c^2s - 8l^2)(4l)^{-1}, \quad n = (l^2 - 2s^2)s^{-1}. \]
The straight lines are
\[ L_1 = 1 + \frac{c}{2}x, \quad L_{2,3} = 1 - \frac{b \mp \sqrt{b^2 - 4s}}{2}(x - \frac{1}{s}y). \]
The vanishing of \( g_3 \) gives \( 4l^2 + 4s^2 - sc^2 = 0 \) and therefore, the conditions for the existence of three invariant straight lines can be written as
\[ g = -b, \quad d = (bc^2 - 8bs - 4cs)(4l)^{-1}, \quad q = 3s(4s - c^2)(4l)^{-1}, \]
\[ n = (c^2 - 12s)/4, \quad 4l^2 + 4s^2 - sc^2 = 0. \]
In this conditions (13) has an integrating factor of the form
\[ \mu = L_1^{\alpha_1} L_2^{\alpha_2} L_3^{\alpha_3} \exp[4\alpha_4(2 + cx)^{-1}] \]
with
\[ \alpha_1 = -3, \]
\[ \alpha_{2,3} = -3 \pm \frac{(bc^2 + 4bs + 8cs)\sqrt{b^2 - 4s}}{2(b^2 - 4s)(c^2 - 4s)}, \]
\[ \alpha_4 = \frac{c^2 + 2bc + 4s}{2(4s - c^2)}. \]

Note that in the case e), where was necessary to equate to zero the first focal value the lines \( L_2 \) and \( L_3 \) are parallel.

**Lemma 3.** The cubic system (5) with three invariant straight lines (of which one is 2-invariant) has at \( O(0,0) \) a center if and only if \( g_3 = 0 \).

### 2.3. Four different invariant straight lines.

By Lemma 1 it is sufficient to consider the case \( L_{1,2} = x \pm iy = 0 \), \( L_{3,4} = 1 + A_{3,4}x + B_{3,4}y = 0 \). If none of these lines are parallel, then by [27] the system (5) has an integrating factor \( \mu = 1/(L_1 L_2 L_3 L_4) \) and this means, that \( O(0,0) \) is a center of (5).

Assume now \( L_1 || L_3 \).

Then we can consider that \( L_4 = L_3 \) and hence \( L_4 || L_2 \).

The straight lines \( L_{3,4} = 0 \) look respectively as \( L_3 = 1 + A(x + iy) \), \( L_4 = 1 + A(x - iy) \).

From the identity \( P \frac{\partial L_i}{\partial x} - Q \frac{\partial L_i}{\partial y} \equiv L_i R_i, \; i = 1,4, \) for \( L_{1,2} = 0 \) we find that (see (11))

\[ f = a + d, \; g = b + c, \; q = p + l - k, \; s = m + n - r, \]

\[ R_1 = -i + (a - ib - ic)x - (ia + id + b)y + (k - im - in + ir)x^2 + (r - n - il - ip)xy - (l + ir)y^2, \]

\[ R_2 = \overline{R_1}, \]
and for $L_{3,4} = 0$, taking into account (15) that (see (7))

$$F_1(A) = (2ib + ic - 2a - d)A - im - in + k - l = 0,$$

$$F_2(A) = 2(ia + id + 2b + c)A - ik + 4il + ip - m - 3r = 0,$$

(17)

$$F_3(A) = (2ib + ic - 2a - d)A + in - 3ir - 3l - p = 0,$$

$$F_4(A) = A^2 + (ia + id + b)A + il - r = 0,$$

$$R_3 = A[-ix + y + (a - bi - ci + iA)x^2 + (c - di - 2A)xy + (a - bi + d - iA)y^2],$$

$$R_4 = \overline{R_3}.$$

The system (5) has a Darboux first integral (an integrating factor) $L_1 \alpha_1 L_2 \alpha_2 L_3 \alpha_3 L_4 \alpha_4 = \text{const.}$ ($\mu = L_1 \alpha_1 L_2 \alpha_2 L_3 \alpha_3 L_4 \alpha_4$), if and only if there exist constants $\alpha_j$, $j = 1, 4$, $\sum |\alpha_j| \neq 0$, in which the identity

(19) $$\sum_{j=1}^{4} \alpha_j R_j \equiv 0$$

(20) $$\left( \sum_{j=1}^{4} \alpha_j R_j \equiv \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right)$$

is satisfied. Substituting in (19) (20) the expression for $R_j$ from (16) and (18), we get that the constant terms must verify the equality $i\alpha_1 - i\alpha_2 = 0$, yielding $\alpha_2 = \alpha_1$. Now identifying in (19) and (20) the coefficients of $x$, $y$, $x^2$, $xy$ and $y^2$, we obtain, respectively two systems of linear equations for the unknowns $\alpha_1, \alpha_3$ and $\alpha_4$:

$$\begin{align*}
(x) : & \Phi_1 = 2a\alpha_1 - iA\alpha_3 + i\bar{A}\alpha_4 = 0, \\
(y) : & \Phi_2 = -2b\alpha_1 + A\alpha_3 + \bar{A}\alpha_4 = 0, \\
(x^2) : & \Phi_3 = 2k\alpha_1 + A(a - ib - ic + iA)\alpha_3 + \bar{A}(a + ib + ic - i\bar{A})\alpha_4 = 0, \\
(xy) : & \Phi_4 = 2(r - n)\alpha_1 + A(c - id - 2A)\alpha_3 + \bar{A}(c + id + 2\bar{A})\alpha_4 = 0, \\
y^2) : & \Phi_5 = 2l\alpha_1 + A(a + d - ib - iA)\alpha_3 + \bar{A}(a + d + ib + i\bar{A})\alpha_4 = 0;
\end{align*}$$

(21)
11 CUBIC DIFFERENTIAL SYSTEMS WITH 4 INVARIANT STRAIGHT LINES 527

(22) $\Phi_1 = d - 2a$, $\Phi_2 = 2b - c$, $\Phi_3 = l + p - 4k$, $\Phi_4 = 2(n - m)$, $\Phi_5 = 3l - p$.

Under the coefficient conditions (15) the first focal value looks as $g_3 = ac - bd - k + l$. If $g_3$ vanishes then the fifth equation of the system (21) ((22)) is a consequence of the first three equations. Indeed, expressing from the first two equations of (21) ((22)) $\alpha_3$ and $\alpha_4$:

$$\alpha_3 = (b - ai)\alpha_1/A, \quad \alpha_4 = (b + ai)\alpha_1/A$$

$$(\alpha_3 = [2(b - ai)\alpha_1 + 2b - c + (d - 2a)i]/(2A),$$

$$\alpha_4 = [2(b + ai)\alpha_1 + 2b - c - (d - 2a)i]/(2A),$$

and substituting them in $\Phi_3 + \Phi_5 = 0$ ($\Phi_3 + \Phi_5 + 4k - 4l = 0$), we obtain that $2g_3\alpha_1 = 0$ ($2g_3(\alpha_1 + 2) = 0$).

Let us consider the following three determinants

$$\Delta_{123} = \begin{vmatrix} a & -i & i \\ -b & 1 & 1 \\ k & a - ib - ic + iA & a + ib + ic - i\bar{A} \end{vmatrix},$$

$$\Delta_{124} = \begin{vmatrix} a & -i & i \\ -b & 1 & 1 \\ r - n & c - id - 2A & c + id - 2\bar{A} \end{vmatrix},$$

$$\Delta = \begin{vmatrix} a & -i & i & d - 2a \\ -b & 1 & 1 & 2b - c \\ k & a - ib - ic + iA & a + ib + ic - i\bar{A} & l + p - 4k \\ r - n & c - id - 2A & c + id - 2\bar{A} & 2(n - m) \end{vmatrix}.$$

If $\Delta_{123} = \Delta_{124} = 0$, then the system (5) has a Darboux first integral. If $\Delta = 0$ and at least one of the determinants $\Delta_{123}, \Delta_{124}$ is different from zero, then (5) has an integrating factor of the Darboux form. In both cases the origin is a center for (5). To determine the cases when $\Delta = 0$ we have to solve the system (17).

The equalities $g_3 = 0$ and $F_1(A) - F_3(A) = 0$ (see (17)) give

$$l = k - ac + bd,$$

$$m = 3r - 2n,$$

$$p = 2ac - 2bd - 3k.$$
If (23) holds then (17) implies $F_1(A) \equiv F_2(A) \equiv F_3(A)$ and hence (17) is equivalent to the system

$$F_1(A) \equiv (2ib + ic - 2a - d)A - bd + ac + in - 3ir = 0,$$

(24)

$$F_4(A) \equiv A^2 + (ia + ib + b)A - iac + ibd + ik - r = 0.$$

Assume that the coefficient of $A$ in $F_1(A) = 0$ is equal to zero, i.e., $c = -2b, d = -2a$. Then $n = 3r$ and $F_4(A) = A^2 + (b - ia)A + ik - r$. If $(b - ia)^2 - 4(ik - r) \neq 0$, the equation $F_4(A) = 0$ has two solutions and the system has four different invariant straight lines of the form (4). By Lemma 1 the origin is a center of (5). The set of parameters for which the system (5) has a center at $O(0,0)$ is closed in the space of all coefficients of this system, then $O(0,0)$ is a center and by $(b - ia)^2 - 4(ik - r) = 0$.

Assume now that the coefficient of $A$ in $F_1(A) = 0$ is not equal to zero. Then from $F_1(A) = 0$ we find that $A = (3ir - in + bd - ac)/(2ib + ic - 2a - d)$ and the determinant $\Delta$ becomes

$$\Delta = 2ig_5(2ad + 2bc + c^2 + d^2 + 2n - 6r)/[(2a + d)^2 + (2b + c)^2],$$

where

$$g_5 = 4a^3c - 4a^2bd + 2a^2cd - 8a^2k - 4ab^2c - 4abc^2 -$$

$$- 2abd^2 - 8abn + 8abr - ac^3 - 4acn + 6acr - 6adk +$$

$$+ 4b^3d + 4b^2cd + 8b^2k + bc^2d + 6bck - 2bdn +$$

$$+ c^2k - cdn + cdr - d^2k$$

is the second focal value computed for (5) in conditions (15) and (23). This implies that the vanishing of the first two focal values ensure the existence of an integrating factor for (5). So, it is proved

**Lemma 4.** The cubic system (5) with four different invariant straight lines of which two are $x \pm iy = 0$ has at the origin a center if and only if $g_3 = g_5 = 0$.

Theorem 1 follows directly from the Lemmas 1–4.
REFERENCES


15. RYCHKOV, G.S. – The limit cycles of the equation $u(x+1)du=(-x+ax^2+bxu+cu+du^2)dx$, Differentsial’nye Uravneniya, 8 (1972), no. 12, 2257–2259 (in Russian).
18. KOOLJ, R.E. – Cubic systems with four line invariants, including complex conjugated lines. Preprint.

* Department of Mathematics  
State University of Tiraspol  
MD-2069 Chișinău  
REP. MOLDOVA  

** Department of Mathematics  
State University of Moldova  
MD-2069 Chișinău  
REP. MOLDOVA