NOTE ON EMBEDDED SURFACES

BY

T. J. WILLMORE
University of Liverpool, England

Dedicated to Professor Octav Mayer on his 70-th birthday

1. *Introduction.* Let S be a closed orientable surface, differentiable of class C^∞, and let f: $S \to E^3$ be a C^∞-embedding of S into euclidean space of three dimensions. The euclidean metric of E^3 induces a riemannian structure on $f(S)$. Let K be the Gaussian curvature at $P \in f(S)$. Then the theorem of Gauss-Bonnet states that

$$\frac{1}{2\pi}\int_{f(S)} K\,dS = \chi(S), \tag{1}$$

where the right-hand member of (1) is the Euler characteristic of S. Thus the left-hand member of (1) is independent of the particular embedding f.

In this Note we consider an expression analogous to the left-hand member of (1) in which the curvature K is replaced by the square of the mean curvature H of $f(S)$ considered as a hypersurface of E^3. In particular we define $\tau(f)$ by

$$\tau(f) = \frac{1}{2\pi}\int_{f(S)} H^2\,dS. \tag{2}$$

Evidently we cannot expect $\tau(f)$ to be a topological invariant of S; however, if we define $\tau(S)$ by

$$\tau(S) = \inf_{f \in \mathcal{J}} \tau(f) \tag{3}$$

where the infimum is taken over the space $\mathcal{J}$ of all C^∞-embeddings of

S in E^3, it is clear that $\tau(S)$ will be a topological invariant of S. The problem raised in this Note is to find the relation between $\tau(S)$ and $\chi(S)$. We show that for surfaces of genus 0 we have $\tau(S) = \chi(S)$, and incidentally we obtain a characterisation of the euclidean sphere. However this simple relation cannot hold for surfaces of genus $p \geq 1$ and the corresponding problem remains unsolved.

2. *Surfaces of genus* 0. We prove the following

Theorem 1. *Let S have genus* 0. *Then for all* $f \in \mathcal{J}$ *we have*

$$2 \leq \tau(f). \tag{4}$$

Moreover $\tau(f) = 2$ *if and only if* $f(S)$ *is a euclidean sphere.*

Let r_1, r_2 denote the principal curvatures at $P \in f(S)$ so that

$$K = r_1 r_2 \tag{5}$$

and

$$H = \frac{1}{2}(r_1 + r_2). \tag{6}$$

Since

$$H^2 = K + \frac{1}{4}(r_1 - r_2)^2, \tag{7}$$

we have

$$\tau(f) = \frac{1}{2\pi}\int_{f(S)} K\, dS + \frac{1}{8\pi}\int_{f(S)} (r_1 - r_2\, d)^2 S,$$

that is,

$$\tau(f) = \chi(S) + \frac{1}{8\pi}\int_{f(S)} (r_1 - r_2)^2\, dS, \tag{8}$$

where we have used (1). It follows immediately that $\tau(f) \geq \chi(S)$; since S has genus 0 we have $\chi(S) = 2$, and equation (4) follows.

Moreover, if $\tau(f) = 2$, then from (8) it follows that $r_1 = r_2$ at each point $P \in f(S)$. Thus every point of $f(S)$ is an umbilic and hence $f(S)$ is a euclidean sphere [see, for example, [3] page 128]. This completes the proof of theorem 1.

We note that $\inf \tau(f) = 2$, so that in this case we have

$$\tau(S) = \chi(S), \tag{9}$$

and there exists an embedding in which the infimum is attained.

Some information about an upper bound for $\tau(f)$ may be obtained from an early result of H. Weyl [2], also subsequently obtained by S. S. Chern in [1]. The result in question states that the square of the mean curvature H of a *convex* surface satisfies the inequality

$$H^2 \leq M \tag{10}$$

where

$$M = \sup_{P \in f(S)} \left(K - \frac{\Delta K}{K} \right). \tag{11}$$

By use of this result we have

Theorem 2. *Let $f(S)$ be a convex surface with surface area V. Then*

$$2 \leq \tau(f) \leq \frac{MV}{2\pi}. \tag{12}$$

3. *Surfaces of genus* 1. Let us consider the anchor ring $f(T)$ given by

$$x = (a + b\cos u)\cos v, \; y = (a + b\cos u)\sin v, \quad z = b\sin u. \tag{13}$$

The first fundamental coefficients are given by

$$E = b^2, \; F = 0, \; G = (a + b\cos u)^2. \tag{14}$$

The second fundamental coefficients are given by

$$L = b, \; M = 0, \; N = (a + b\cos u)\cos u. \tag{15}$$

The mean curvature is given by

$$H = \frac{a + 2b\cos u}{2b\,(a + b\cos u)}. \tag{16}$$

Then we have

$$\tau(f) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} H^2 b(a + b\cos u)\, du\, dv. \tag{17}$$

After some computation we find, on writing $b/a = c$, that

$$\tau(f) = \frac{\pi}{2c\sqrt{1 - c^2}}. \tag{18}$$

It is easy to see that $\tau(f) \to \infty$ both as $c \to 0$ and as $c \to 1$.

The minimum value of (f) occurs when $c = 1/\sqrt{2}$, when the value of $\tau(f)$ is π.

It seems reasonable to interpret $\tau(f)$ as a measure of the „niceness" of the shape of the surface $f(S)$, and to argue heuristically that a small value of $\tau(f)$ corresponds to a simple shape for $f(S)$. This suggests that (13) with $b/a = 1/\sqrt{2}$ gives the nicest shape for an embedded torus. However, whether or not $\tau(T) = \pi$ remains an open question. The problem for surfaces of genus $p \geq 2$ remains unsolved.

REFERENCES

1. Chern S. S. — Duke Math. Journ. 12, (1945), 279-290.
2. Weyl H. — Vierteljahrschrift der naturforschenden Gesellschaft in Zürich, Jahrgang 61, (1916), 40-72.
3. Willmore, T. J. — *Introduction to Differential Geometry*, Clarendon, Oxford, (1959), 128.

ASUPRA SUPRAFEŢELOR SCUFUNDATE

Rezumat

Fie S o suprafaţă închisă, orientabilă, de clasă C^∞ şi $f: S \to E^3$ o scufundare a ei în spaţiul euclidian E^3. Are loc formula (1) în care $\chi(S)$ este caracteristica lui Euler a suprafeţei S. Egalitatea (1) arată că $\frac{1}{2\pi}\int_{f(S)} K\, dS$ este un invariant topologic al suprafeţei. În § 1 autorul defineşte un nou invariant topologic folosind curbura medie, H, a acesteia. Fie $\tau(f)$ dat de (2) şi $\tau(S) = \inf_{f \in \mathcal{J}} \tau(f)$ ($\mathcal{J}$ fiind spaţiul funcţiilor C^∞ al scufundărilor lui S în E^3). $\tau(S)$ este un invariant topologic al suprafeţei. Problema stabilirii relaţiei dintre $\chi(S)$ şi $\tau(S)$ este rezolvată în § 2 numai pentru cazul cînd S este de gen zero. In § 3 sînt date cîteva exemple. Problema rămîne deschisă în cazul general.

О ПОГРУЖЕННЫХ ПОВЕРХНОСТЯХ

Краткое содержание

Пусть S замкнутая, ориентированная поверхность, класса C^∞ и пусть $f: S \to E^3$ её погружение в эвклидовом пространстве E^3. Имеет место формула (1) где $\chi(S)$ характеристика Эйлера поверхности S.

Равенство (1) показывает что $\frac{1}{2\pi}\int_{f(S)} K\, dS$ — топологический инвариант поверхности. В § 1 автор определяет новый инвариант поверхности При помощи средней кривизны её. Пусть $\tau(f)$ данный формулой (2) и $\tau(S) = \inf_{f \in \mathcal{J}} \tau(f)$ (где $\mathcal{J}$ · пространство функции C^∞ определяющих погружения S в E^3). Задача нахождения соотношений между $\chi(S)$ и $\tau(S)$ решена в § 2 только для поверхностей нулевого рода. В § 3 даны некоторые примеры. Задача нерешена в общем случае.