țări a învățămîntului matematic în școala de cultură generală menționăm o activitate rodnică a multor matematicieni ieșeni. Din inițiativa lor în cadrul filialei Iași a societății de matematică, s-au organizat primele cursuri de informare cu privire la dezvoltarea matematicii moderne pentru profesorii din învățămîntul mediu care s-au extins azi în cursul de vară organizate pe țară de societatea de matematică.

Lecțiile și conferințele prezentate la primul curs (P. Neamț 1961) au fost publicate într-o culegere Probleme actuale de matematică. *Legătura cu învățămîntul mediu se realizează și prin lecții ținute în cadrul IPCD Iași ca și prin vizite și conferințe ținute la diverse școli medii din Moldova.*

Ținînd seama de cele expuse mai sus și avînd în vedere condițiile create pentru cercetarea științifică de regimul nostru democrat-popular, avem convingerea că matematica ieșeană se va impune și în viitor atît pe plan național cît și internațional.

ON ASYMMETRIC DIOPHANTINE APPROXIMATIONS

BY

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1. *Introduction.* In a recent paper [16] and his book [17] Ivan Niven gave a new proof of the following theorem:

Every irrational θ has an infinity of rational approximations p/q such that

$$(1)\qquad -\frac{1}{q^2\xi} < \frac{p}{q} - \theta < \frac{\tau}{q^2\xi} \quad (q > 0),$$

where $\xi = (1 + 4\tau)^{1/2}$ and $\tau \geqslant 0$.

The statement holds also if $p/q - \theta$ in (1) is replaced by $\theta - p/q$.

This theorem, introduced by B. Segre [24], reduces to the fundamental result, due to Hurwitz [6] for $\tau = 1$. B. Segre gave a geometrical proof, which results from his investigations of lattice points in certain non-convex domains.

C. D. Olds [20], making use of Farey series, gave an arithmetical proof of this theorem for $\tau \geqslant 1$. His method is a generalization of that used by Khintchine [7] for the special case $\tau = 1$.

R. M. Robinson [23], making use of the convergents $p/q = [a_0, a_1, \ldots, a_n]$ of irrational $\theta = [a_0, a_1, a_2, \ldots]$, gave an interesting original proof of the same theorem for the case when $\tau \geqslant 0$.

I used in [11] a Borel's method [1] for an arithmetical proof of the theorem. I also proved there, for the first time, some properties of the convergents of irrational θ. The following two properties are more important [11, p. 13]:

(a) At least one of the convergents $\frac{p_{2n+1}}{q_{2n+1}}, \frac{p_{2n+2}}{q_{2n+2}}, \frac{p_{2n+3}}{q_{2n+3}}$ of θ satisfies (1) with $\xi = (1 + 4\tau)^{1/2}$, $\tau \geqslant 0$.

(b) At least one of the convergents $\frac{p_{2n}}{q_{2n}}, \frac{p_{2n+1}}{q_{2n+1}}, \frac{p_{2n+2}}{q_{2n+2}}$ of θ satisfies (1) with $\xi = (\tau^2 + 4\tau)^{1/2}$, $\tau \geqslant 0$.

Thus the theorem took the following equivalent form:

A. *Every irrational* θ *has an infinity of rational approximations* p/q *which satisfy* (1) *with*

$$\xi = \max\,[(1+4\tau)^{1/2},\ (\tau^2+4\tau)^{1/2}],\quad \tau \geqslant 0.$$

The statement holds also if $p/q-\theta$ in (1) is replaced by $\theta - p/q$.

In my following paper [12] on the same subject, making use of (a), (b) and the Khintchine's, Humbert's and Fujiwara's methods [7, 5, 2], I sketched three proofs of A.

From paragraphs 1 and 4 of the same paper we mention the properties

(c) At least one of the numbers $\dfrac{p_{2n+1}}{q_{2n+1}},\ \dfrac{p_{2n+2}}{q_{2n+2}},\ \dfrac{p_{2n+1}+p_{2n+2}}{q_{2n+1}+q_{2n+2}}$ satisfies (1) with $\xi=(1+4\tau)^{1/2}$, $\tau \geqslant 0$.

(d) At least one of the numbers $\dfrac{p_{2n}}{q_{2n}},\ \dfrac{p_{2n+1}}{q_{2n+1}},\ \dfrac{p_{2n}+p_{2n+1}}{q_{2n}+q_{2n+1}}$ satisfies (1) with $\xi=(\tau^2+4\tau)^{1/2}$, $\tau \geqslant 0$.

I used there the convergents of θ, instead of Farey series, for completing Olds' proof of A. In paragraph 4 also, I pointed out that R. M. Robinson and myself, independently, found for ξ the form $\xi = \max\,[(1+4\tau)^{1/2}, (\tau^2+4\tau)^{1/2}]$ [that is $\xi=(1+4\tau)^{1/2}$ for $0<\tau\leqslant 1$ and $\xi=(\tau^2+4\tau)^{1/2}$ for $\tau>1$] in order that the inequalities (1) hold for every θ and an infinity of rational numbers p/q.

W. J. Le Veque [8], using Farey series and convergents of θ, gave an interesting proof of A. He proved again the properties (a) and (b)*). From his considerations, one also deduces immediately (c) and (d).

The last proof of the theorem is that of Ivan Niven [16, 17]. It uses the Farey series and was suggested by works of Khintchine [7] and Le Veque [8].

Now there are seven complete arithmetical proofs of this Segre's theorem but none of them continues that given by C. D. Olds. The subject of this paper was suggested to me by Niven's work.

The purpose of this Note is to show that it is possible to complete Olds' proof and to make some remarks on the methods used in the proof. In paragraphs 4 and 5 I apply this Khintchine's method to prove two more general results. In the last paragraphs I give some particular cases of theorems B_1, B_2, C_1 and C_2.

2. *Proof of theorem* A. For an arbitrary positive integer n form the Farey series F_n of order n. F_n is the sequence, ordered in size, of all rational fractions a/b in lowest terms with $0<b\leqslant n$. If a/b and a'/b' are two consecutive terms in F_n which satisfy the inequalities $a/b<\theta<a'/b'$ and τ is any positive integer, then we have the following properties:

(α) the equality $a'b-ab'=1$ holds,

(β) for all sufficiently large n the two fractions a/b and a'/b' have $b>\tau$ and $b'>\tau$ [17, pp. 3–5].

To prove theorem A consider three parts.

Part I. First we consider an irrational θ and $\xi=(1+4\tau)^{1/2}$, $\tau>0$.

Let a/b, a'/b' be two consecutive terms in F_n which satisfy the inequalities $a/b<\theta<a'/b'$.

We distinguish two cases.

Case 1. (1.1). Suppose that $b/b'<(\xi-1)/2\tau$, or $b/b'>(\xi+1)/2\tau$. Then, setting $\omega=b/b'$ we see that

$$\frac{1}{\xi}\left(\tau+\frac{1}{\omega^2}\right)-\frac{1}{\omega}=\frac{\tau}{\xi\omega^2}\left(\omega-\frac{\xi-1}{2\tau}\right)\left(\omega-\frac{\xi+1}{2\tau}\right)>0; \tag{2}$$

consequently, since $a'b-ab'=1$,

$$\frac{a'}{b'}-\frac{a}{b}=\frac{1}{b'^2\omega}<\frac{1}{b'^2\xi}\left(\tau+\frac{1}{\omega^2}\right)=\frac{\tau}{b'^2\xi}+\frac{1}{b^2\xi}, \tag{3}$$

which implies that

$$\frac{a}{b}+\frac{1}{b^2\xi}>\frac{a'}{b'}-\frac{\tau}{b'^2\xi}. \tag{4}$$

Hence θ must be interior to one of the open intervals

$$\left(\frac{a}{b},\ \frac{a}{b}+\frac{1}{b^2\xi}\right)\ \text{or}\ \left(\frac{a'}{b'}-\frac{\tau}{b'^2\xi},\ \frac{a'}{b'}\right). \tag{5}$$

Then, according as θ belongs to the first or to the second interval, we have, respectively, the inequalities

$$-\frac{1}{b^2\xi}<\frac{a}{b}-\theta<0\ \text{or}\ 0<\frac{a'}{b'}-\theta<\frac{\tau}{b'^2\xi}.$$

Thus (1) is true, where for q we take either b or b'.

(1.2). If $\omega=b/b'=(\xi\pm 1)/2\tau$, $(2b\tau\pm b')^2=b'^2(1+4\tau)$ gives τ rational and hence both τ and ξ are rational. Then the inequality sign, in (4) is replaced by an equality sign, and the right and left end points of the intervals in (5) coincide. But θ, being irrational, cannot be equal to this common end point. Hence θ must be interior to one or the other of these intervals, and the proof proceeds as already explained.

Case 2. We now suppose that $\dfrac{\xi-1}{2\tau}<\dfrac{b}{b'}<\dfrac{\xi+1}{2\tau}$. We consider in turn the two open sub-intervals

*) I take the opportunity to correct a misunderstanding of my paper [12]. Among the other propositions, Le Veque asserted that I attempted to prove in [12] that of *any* three consecutive convergents of the continued fraction expansion of θ, one at least satisfies (1) for $\xi=\max\,[(1+4\tau)^{1/2}, (\tau^2+4\tau)^{1/2}]$ and arbitrary $\tau\geqslant 0$. He proved that it is false. But it is clear that this assertion does not belong to me. Probably Le Veque read it in the review of my paper [12], made by Rankin [22]. I believe that Rankin did not understand well the text of my paper because it was written in Rumanian.

(6) $$\left(\frac{a}{b}, \frac{a+a'}{b+b'}\right) \text{ and } \left(\frac{a+a'}{b+b'}, \frac{a'}{b'}\right).$$

(2.1) If θ belongs to the first interval, we write $\omega' = b/(b+b')$. Then, since $x/(1+x)$ is an increasing function, we get

(7) $$\omega' = \frac{b}{b+b'} = \frac{\omega}{1+\omega} < \frac{\xi+1}{\xi+1+2\tau}.$$

Because

(8) $$\frac{\xi+1}{\xi+1+2\tau} = \frac{\xi-1}{2\tau} \text{ for } \tau > 0,$$

we have

(9) $$\omega' < \frac{\xi-1}{2\tau},$$

and the inequality (2) is again true when we replace ω with ω'. Hence

$$\frac{a+a'}{b+b'} - \frac{a}{b} = \frac{1}{(b+b')^2\omega'} < \frac{1}{b^2\xi} + \frac{\tau}{(b+b')^2\xi},$$

and, proceeding as in case 1, we see that (1) is true where for q we take either b or $b+b'$.

(2.2) If θ belongs to the second interval (6), let the positive integer k be chosen so that

(10) $$\frac{a+ka'}{b+kb'} < \theta < \frac{a+(k+1)a'}{b+(k+1)b'} < \frac{a'}{b'}.$$

This can be done because $\frac{a+ka'}{b+kb'}$ tends to $\frac{a'}{b'}$ as k increases indefinitely.

We now consider the open sub-intervals

(11) $$\left(\frac{a+ka'}{b+kb'}, \frac{a'}{b'}\right) \text{ and } \left(\frac{a+ka'}{b+kb'}, \frac{a+(k+1)a'}{b+(k+1)b'}\right).$$

θ lies in each of these intervals.

We write $\omega' = \frac{b+kb'}{b'} = \omega + k$ and $\omega'' = \frac{b+kb'}{b+(k+1)b'} = \frac{\omega+k}{\omega+k+1} = \frac{\omega'}{\omega'+1}$.

If $\omega' < \frac{\xi-1}{2\tau}$ or $\omega' > \frac{\xi+1}{2\tau}$, proceeding as in case 1, we see that (1) is true where for q we take either b' or $b+kb'$.

If $\omega' = \frac{\xi \pm 1}{2\tau}$ the proof proceeds as in (1.2).

If $\frac{\xi-1}{2\tau} < \omega' < \frac{\xi+1}{2\tau}$, we have

(12) $$\omega'' = \frac{\omega'}{\omega'+1} < \frac{\xi+1}{\xi+1+2\tau} = \frac{\xi-1}{2\tau},$$

according to the equality (8) and, hence the inequalities (1) are again true where for q we take either $b+kb'$ or $b+(k+1)b'$. In all the cases, we get a solution p_n/q_n of (1) with q_n equal to one of $b, b', b+b', b+kb', b+(k+1)b'$ and $\xi = (1+4\tau)^{1/2}$, $\tau > 0$.

Part II. Now we consider $\xi = (\tau^2+4\tau)^{1/2}$, $\tau > 0$, and the same fractions a/b, a'/b' as in part I.

Case 1 (1.1). Suppose that $b/b' < (\xi-\tau)/2\tau$, or $b/b' > (\xi+\tau)/2\tau$. Then we have

$$\frac{a}{b} + \frac{1}{b^2\xi} > \frac{a'}{b'} - \frac{\tau}{b'^2\xi}.$$

Then, according as θ belongs to the first or to the second interval from (5), we obtain, respectively, the inequalities

$$-\frac{1}{b^2\xi} < \frac{a}{b} - \theta < 0 \text{ or } 0 < \frac{a'}{b'} - \theta < \frac{\tau}{b'^2\xi}.$$

Thus (1) is true, where for q and ξ we take either b or b' and $(\tau^2+4\tau)^{1/2}$.

(1.2). If $\omega = b/b' = (\xi \pm \tau)/2\tau$, $(2b \pm b')^2\tau^2 = b'^2(\tau^2+4\tau)$ gives τ rational and hence both τ and ξ are rational and the proof proceeds as in case (1.1).

Case 2. Now we suppose that $(\xi-\tau)/2\tau < b/b' < (\xi+\tau)/2\tau$. We consider the two sub-intervals

(6') $$\left(\frac{a}{b}, \frac{a+a'}{b+b'}\right) \text{ and } \left(\frac{a+a'}{b+b'}, \frac{a'}{b'}\right).$$

(2.1). If θ belongs to the second interval, we write $\omega'' = (b+b')/b'$. Then we get

$$\omega'' = \frac{b+b'}{b'} = 1+\omega > 1 + \frac{\xi-\tau}{2\tau} = \frac{\xi+\tau}{2\tau},$$

and the inequality (2') is again true when we replace ω with ω''. Hence (1) is true where for q we take either $b+b'$ or b'.

(2.2) If θ belongs to the first interval (6′), let the positive integer j be chosen so that

$$(10') \qquad \frac{a}{b} < \frac{(j+1)a+a'}{(j+1)b+b'} < \theta < \frac{ja+a'}{jb+b'} < \frac{a+a'}{b+b'}.$$

We now consider the intervals

$$\left(\frac{a}{b}, \frac{ja+a'}{jb+b'}\right) \quad \text{and} \quad \left(\frac{(j+1)a+a'}{(j+1)b+b'}, \frac{ja+a'}{jb+b'}\right).$$

θ lies in each of these intervals.

We write $\omega' = \dfrac{b}{jb+b'} = \dfrac{\omega}{1+j\omega}$ and $\omega'' = \dfrac{(j+1)b+b'}{jb+b'} =$

$$= \frac{1+(j+1)\omega}{1+j\omega}. \text{ We get } \omega'' = 1 + \omega'.$$

If $\omega' < \dfrac{\xi-\tau}{2\tau}$ or $\omega' > \dfrac{\xi+\tau}{2\tau}$, proceeding as in case 1, we see that (1) is true where for q we take b or $jb+b'$.

If $\omega' = \dfrac{\xi \pm \tau}{2\tau}$ the proof proceeds as in (1.2).

If $\dfrac{\xi-\tau}{2\tau} < \omega' < \dfrac{\xi+\tau}{2\tau}$, we have

$$\omega'' = 1 + \omega' > 1 + \frac{\xi-\tau}{2\tau} = \frac{\xi+\tau}{2\tau},$$

and then the inequalities (1) are again true where for q we take either $jb+b'$, or $(j+1)b+b'$. In all the cases, we get a solution p_m/q_m of (1) with q_m equal to one of b, b', $b+b'$, $jb+b'$, $(j+1)b+b'$ and $\xi = (\tau^2+4\tau)^{1/2}$, $\tau > 0$.

Thus we get that at least one of the fractions

$$(13) \qquad \frac{a}{b}, \frac{a'}{b'}, \frac{a+a'}{a+b'}, \frac{a+ka'}{b+kb'}, \frac{a+(k+1)a'}{b+(k+1)b'} \quad \text{or}$$

$$(14) \qquad \frac{a}{b}, \frac{a'}{b'}, \frac{a+a'}{b+b'}, \frac{ja+a'}{jb+b'}, \frac{(j+1)a+a'}{(j+1)b+b'}$$

satisfies (1) with $\xi = \max\{(1+4\tau)^{1/2}, (\tau^2+4\tau)^{1/2}\}$, $\tau > 0$.

Part III. It is easy to prove that all the fractions (13) or (14) are in lowest terms. For example

$$a'(b+kb') - b'(a+ka') = a'b - ab' = 1,$$

$$a(jb+b') - b(ja+a') = 1.$$

so that the greatest common divisors of $b+kb'$ and $a+ka'$ or $jb+b'$ and $ja+a'$ are 1.

We now consider F_{n_1} where n_1 is sufficiently large so that the two fractions a_1/b_1 and a_1'/b_1' adjacent to θ in F_{n_1} have denominators larger than q_n or q_m respectively. This assertion is true according to the property (β) of the Farey series.

Repeating the preceding argument, we get a new solution p_{n_1}/p_{n_1} of (1) with q_{n_1} equ l to one of b_1, b_1', b_1+b_1', $b_1+k_1 b_1'$, $b_1+(k_1+1)b_1'$ or b_1, b_1', b_1+b_1', $j_1 b_1+b_1'$, $(j_1+1)\ b_1+b_1'$.

We can prove, as above, that all the new fractions are in lowest terms. Then the new solution p_{n_1}/q_{n_1} of (1) is different from p_n/q_n or p_m/q_m.

Since this process can be repeated indefinitely, giving a new solution of (1) each time, A is proved for $\tau > 0$.

The teorem is also true in the case when $\tau = 0$. We can prove it by changing in the proceding argument of part I the numbers $\dfrac{\xi-1}{2\tau}$ and $\dfrac{\xi+1}{2\tau}$ respectively with 1 and $+\infty$.

We note that the second sentence of the statement A follows at once from the first if in the first sentence we replace τ with $\dfrac{1}{\tau}$.

3. *A few remarks on the proofs of theorem* A. 1°. Cases (1.1), (1.2) and (2.1) from part *I* follows as in Olds' proof. Part III is exactly as in Niven's paper. The essential idea in this proof of Segre's theorem is the selection of the necessary fractions which satisfy the inequalities (10) or (10′). It is due to Ivan Niven.

2°. If we want to prove directly theorem A with $\theta = p/q$ in (1) and $\xi = (1+4\tau)^{1/2}$, then in the case 1 of part I we replace the numbers

$$\frac{\xi-1}{2\tau}, \frac{\xi+1}{2\tau}, \frac{1}{\xi}\left(\tau+\frac{1}{\omega^2}\right)-\frac{1}{\omega} \text{ with } \frac{\xi+1}{2}, \frac{\xi-1}{2}, \frac{1}{\xi}\left(1+\frac{\tau}{\omega^2}\right)-\frac{1}{\omega},$$

and we have

$$-\frac{1}{b'^2\xi} < \theta - \frac{a'}{b'} < 0 \text{ or } 0 < \theta - \frac{a}{b} < \frac{\tau}{b^2\xi}.$$

If $\dfrac{b}{b'} = \dfrac{\xi \pm 1}{2}$, then both ξ and τ must be rational and the proof continues as in (1.2).

Case 2. $\frac{\xi-1}{2} < \frac{b}{b'} < \frac{\xi+1}{2}$. If θ belongs to the second interval (6'), we write $\omega'' = (b + b')/b'$. Then we get

$$\omega'' = 1 + \omega > 1 + \frac{\xi-1}{2} = \frac{\xi+1}{2},$$

and (1) is true where for q we take either $b + b'$ or b'.

(2.2) If θ belongs to the first interval (6'), let the positive integer j be chosen so that inequalities (10') are satisfied. The proof continues as in part II.

3°. If we want to prove directly theorem A with $\theta - p/q$ in (1) and $\xi = (\tau^2 + 4\tau)^{1/2}$, $\tau > 0$, then in the case 1 of part I we replace the numbers

$$\frac{\xi-1}{2\tau}, \frac{\xi+1}{2\tau}, \frac{1}{\xi}\left(\tau + \frac{1}{\omega^2}\right) - \frac{1}{\omega} \text{ with } \frac{\xi-\tau}{2}, \frac{\xi+\tau}{2}, \frac{1}{\xi}\left(1 + \frac{\tau}{\omega^2}\right) - \frac{1}{\omega}$$

and equality (12) with

$$\frac{\xi+\tau}{\xi+\tau+2} = \frac{\xi-\tau}{2}. \tag{12'}$$

The proof continues as in part I.

4°. In the course of the prof, W. J. Le Veque has obtained the following theorem: if F_n is an arbitrary Farey series and θ lies between the adjacent element a/b and a'/b' of F_n, then at least one of the numbers a/b, a'/b', and $(a + a')/(b + b')$ is a solution of the inequality

$$-\frac{1}{\beta q^2} < \frac{p}{q} - \theta < \frac{\tau}{\beta q^2},$$

where $\tau \geqslant 0$ and $\beta = \min[(1 + 4\tau)^{1/2}, (\tau^2 + 4\tau)^{1/2}]$.

It is easy to see that it is equivalent with the result proved by C. D. Olds.

5°. It is clear that if a/b and a'/b' satisfy $a/b < a'/b'$ and $a'b - ab' = 1$, then a/b and a'/b' are consecutive terms in $F_{bb'}$. We now consider, as W. J. Le Veque, the cases when a/b and a'/b' are successive convergents of θ, hence when $a/b = p_{2n}/q_{2n}$, $a'/b' = p_{2n+1}/q_{2n+1}$ or $a/b = p_{2n+2}/q_{2n+2}$, $a'/b' = p_{2n+1}/q_{2n+1}$ are consecutive terms in Farey series $F_{q_{2n}q_{2n+1}}$ or $F_{q_{2n+1}q_{2n+2}}$.

It is easy to see that the values of integers k and j, required in the proof of theorem A, are respectively a_{2n+2} and a_{2n+3}.

We obtain the following precision:

(e) If ν is any positive integer, at least one of the five numbers

$$\frac{p_{\nu-1}}{q_{\nu-1}}, \frac{p_\nu}{q_\nu}, \frac{p_{\nu+1}}{q_{\nu+1}}, \frac{p_{\nu-1}+p_\nu}{q_{\nu-1}+q_\nu}, \frac{p_\nu+q_{\nu+1}}{q_\nu+q_{\nu+1}}$$

satisfies (1) with $\xi = \max[(1 + 4\tau)^{1/2}, (\tau^2 + 4\tau)^{1/2}]$, $\tau \geqslant 0$.

This precision results also form (c) and (d).

Another precision of this kind results form (b), (c) and (a), (d). It is the following:

(f) At least one of the five numbers

$$\frac{p_{\nu-1}}{q_{\nu-1}}, \frac{p_\nu}{q_\nu}, \frac{p_{\nu-1}+p_\nu}{q_{\nu-1}+q_\nu}, \frac{p_{\nu+1}}{q_{\nu+1}}, \frac{p_{\nu+2}}{q_{\nu+2}}$$

satisfies (1) with $\xi = \max[(1 + 4\tau)^{1/2}, (\tau^2 + 4\tau)^{1/2}]$, $\tau \geqslant 0$;

We must say that the precisions (a) — (d) give other precisions which contain only four numbers. Here, we state the following two:

At least one of the four numbers

$$\frac{p_\nu}{q_\nu}, \frac{p_{\nu+1}}{q_{\nu+1}}, \frac{p_{\nu+2}}{q_{\nu+1}}, \frac{p_{\nu+3}}{q_{\nu+3}} \text{ or} \tag{g}$$

$$\frac{p_\nu}{q_\nu}, \frac{p_{\nu+1}}{q_{\nu+1}}, \frac{p_{\nu+2}}{q_{\nu+2}}, \frac{p_{\nu+1}+p_{\nu+2}}{q_{\nu+1}+q_{\nu+2}} \tag{h}$$

is a solution of (1) with $\xi = \max[(1 + 4\tau)^{1/2}, (\tau^2 + 4\tau)^{1/2}]$, $\tau \geqslant 0$.

Thus we see that the main precisions of Segre's theorem are (a) — (d), especially (a) and (b).

Now we give two generalizations of (a) and (b).

4. *First generalization.* Let $\frac{a}{b}, \frac{a'}{b'}$ denote two consecutive elements of F_n which satisfy the inequalities

$$\frac{a}{b} < \theta < \frac{a'}{b'}, \frac{a+a'}{b+b'} < \theta < \frac{a'}{b'}, \tag{15}$$

and l, m the integers so that

$$\frac{a}{b} < \frac{a+la'}{b+lb'} < \theta < \frac{a+(l+1)a'}{b+(l+1)b'} \text{ and} \tag{16}$$

$$\frac{(a+la')(m+1)+a'}{(b+lb')(m+1)+b'} < \theta < \frac{(a+la')m+a'}{(b+lb')m+b'}$$

hold.

Consider the two open intervals

$$\left(\frac{a+la'}{b+lb'}, \frac{a'}{b'}\right), \quad \left(\frac{a+la'}{b+lb'}, \frac{(a+la')m+a'}{(b+lb')m+b'}\right)$$

and set

$$\omega' = \frac{b+lb'}{b'} \text{ and } \omega'' = \frac{b+lb'}{(b+lb')m+b'} = \frac{1}{m+\frac{1}{\omega'}}$$

If $\omega' < \frac{\xi-m}{2r}$ or $\omega' > \frac{\xi+m}{2r}$ and $\xi = (m^2+4r)^{1/2}$, then at least one of the numbers $\frac{a+la'}{b+lb'}, \frac{a'}{b'}$ satisfies (1).

If $\omega' = \frac{\xi \pm m}{2r}$, then $(2r\omega'+m)^2 = m^2+4r$, hence r is rational and ξ too.

If $\frac{\xi-m}{2r} < \omega' < \frac{\xi+m}{2r}$, we have

$$\omega'' = \frac{\omega'}{m\omega'+1} < \frac{\xi+m}{m(\xi+m)+2r} = \frac{\xi-m}{2r}.$$

Then at least one of the fractions $\frac{a+la'}{b+lb'}, \frac{(a+la')m+a'}{(b+lb')m+b'}$ satisfies (1).

Thus we proved the following theorem:

B_1. *If $\frac{a}{b}, \frac{a'}{b'}$ are two elements of F_n, l, m two integers which satisfy respectively* (15) *and* (16), *then at least one of the fractions*

$$\frac{a+la'}{b+lb'}, \quad \frac{(a+la')m+a'}{(b+lb')m+b'}, \quad \frac{a'}{b'}$$

satisfies (1) *with* $\xi = (m^2+4r)^{1/2}$, $r \geqslant 0$.

A generalization of (b) can be stated in the following theorem:

B_2. *If $\frac{a}{b}, \frac{a'}{b'}$ are two elements of F_n, r, s two integers which satisfy respectively*

(15′) $$\frac{a}{b} < \theta < \frac{a'}{b'}, \quad \frac{a}{b} < \theta < \frac{a+a'}{b+b'} \text{ and}$$

(16′) $$\frac{a}{b} < \frac{(r+1)a+a'}{(r+1)b+b'} < \theta < \frac{ra+a'}{rb+b'} < \frac{a'}{b'},$$

$$\frac{(ra+a')s+a}{(rb+b')s+b} < \theta < \frac{(ra+a')(s+1)+a}{(rb+b')(s+1)+b},$$

then at least one of the fractions

$$\frac{a}{b}, \quad \frac{ra+a'}{rb+b'}, \quad \frac{(ra+a')s+a}{(rb+b')s+b}$$

satisfies (1) *with* $\xi = (s^2r^2+4r)^{1/2}$, $r \geqslant 0$.

B_2 can be proved as B_1 by considering the intervals

$$\left(\frac{a}{b}, \frac{ra+a'}{rb+b'}\right), \left(\frac{(ra+a')s+a}{(rb+b')s+b}, \frac{ra+a'}{rb+b'}\right)$$

with $\omega' = \frac{b}{rb+b'}$ and $\omega'' = \frac{(rb+b')s+b}{rb+b'} = s+\omega'$. Instead of $\frac{\xi \pm m}{2r}$ we set $\frac{\xi \pm sr}{2r}$. When $\frac{\xi - sr}{2r} < \omega' < \frac{\xi + sr}{2r}$, we use the inequalities

$$\omega'' > s + \frac{\xi - sr}{2r} = \frac{\xi + sr}{2r}.$$

5. *Second generalization.* Let $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$ and $\frac{a_3}{b_3}$ be three fractions which have the following properties: $\frac{a_2}{b_2} < \theta < \frac{a_3}{b_3} < \frac{a_1}{b_1}$. By setting

$$a_{12} = a_1b_2 - a_2b_1, \quad a_{32} = a_3b_2 - a_2b_3, \quad a_{13} = a_1b_3 - a_3b_1, \text{ we have}$$

$$\frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{a_{12}}{b_1b_2}, \quad \frac{a_3}{b_3} - \frac{a_2}{b_2} = \frac{a_{32}}{b_2b_3}, \quad \frac{a_1}{b_1} - \frac{a_3}{b_3} = \frac{a_{13}}{b_1b_3}.$$

Substracting the second from the first of these equalities and making use of the third, we get

$$\frac{a_{13}}{b_1b_3} = \frac{a_{12}}{b_1b_2} - \frac{a_{32}}{b_2b_3} \quad \text{or}$$

(17) $$\frac{b_2}{b_3} = \frac{a_{12}}{a_{13} + a_{32}\frac{b_2}{b_1}}.$$

If we solve the equation

$$\frac{a_{12}}{a_{13} + a_{32}\frac{2r}{a_{12}\xi \pm (a_{12}^2\xi^2 - 4r)^{1/2}}} = \frac{a_{32}\xi - (a_{32}^2\xi^2 - 4r)^{1/2}}{2r},$$

we obtain

(19) $$\xi = \left[\left(\frac{\alpha_{13}^2 + \tau\alpha_{12}^2 + \tau\alpha_{32}^2}{\alpha_{12}\alpha_{13}\alpha_{32}}\right)^2 - \frac{4\tau^2}{\alpha_{13}^2}\right]^{1/2}.$$

Consider the intervals

$$\left(\frac{a_2}{b_2}, \frac{a_1}{b_1}\right), \left(\frac{a_2}{b_2}, \frac{a_3}{b_3}\right)$$

and set $\omega_{12} = \frac{b_2}{b_1}$, $\omega_{32} = \frac{b_2}{b_3}$, ξ being given by (19).

We now distinguish two cases as always above.

Case 1 (1.1) Suppose that $\omega_{12} < \frac{\alpha_{12}\xi - (\alpha_{12}^2\xi^2 - 4\tau)^{1/2}}{2\tau}$ or $\omega_{12} > \frac{\alpha_{12}\xi + (\alpha_{12}^2\xi^2 - 4\tau)^{1/2}}{2\tau}$. Then we see that

(20) $$\frac{1}{\xi}\left(\tau + \frac{1}{\omega_{12}^2}\right) - \frac{\alpha_{12}}{\omega_{12}} > 0,$$

and consequently, since $a_1 b_2 - a_2 b_1 = \alpha_{12}$ and $\frac{1}{\xi}\left(\tau + \frac{1}{\omega_{12}^2}\right) > \frac{\alpha_{12}}{\omega_{12}}$ we have

(21) $$\frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{\alpha_{12}}{b_1 b_2} = \frac{\alpha_{12}}{b_1^2\omega_{12}} < \frac{1}{b_1^2\xi}\left(\tau + \frac{1}{\omega_{12}^2}\right) = \frac{\tau}{b_1^2\xi} + \frac{1}{b_2^2\xi},$$

which implies that

(22) $$\frac{a_2}{b_2} + \frac{1}{b_2^2\xi} > \frac{a_1}{b_1} - \frac{\tau}{b_1^2\xi}.$$

Hence θ must be interior to one or the other of the open intervals

(23) $$\left(\frac{a_2}{b_2}, \frac{a_2}{b_2} + \frac{1}{b_2^2\xi}\right) \quad \text{or} \quad \left(\frac{a_1}{b_1} - \frac{\tau}{b_1^2\xi}, \frac{a_1}{b_1}\right).$$

Then, according as θ belongs to the first or to the second interval, we have, respectively, the equalities

$$-\frac{1}{b_2^2\xi} < \frac{a_2}{b_2} - \theta < 0 \quad \text{or} \quad 0 < \frac{a_1}{b_1} - \theta < \frac{\tau}{b_1^2\xi}.$$

Thus (1) is true, where for q we take either b_2 or b_1.

(1.2) If $\omega_{12} = \frac{\alpha_{12}\xi \pm (\alpha_{12}^2\xi^2 - 4\tau)^{1/2}}{2\tau}$, according to (17) and (18) $\omega_{32} = \frac{\alpha_{32}\xi - (\alpha_{32}^2\xi^2 - 4\tau)^{1/2}}{2\tau}$ and we have two inequalities:

(24) $$\frac{1}{\xi}\left(\tau + \frac{1}{\omega_{12}^2}\right) = \frac{\alpha_{12}}{\omega_{12}}, \quad \frac{1}{\xi}\left(\tau + \frac{1}{\omega_{32}^2}\right) = \frac{\alpha_{32}}{\omega_{32}},$$

which give to

$$\xi = \frac{\frac{1}{\omega_{12}^2} - \frac{1}{\omega_{32}^2}}{\frac{\alpha_{12}}{\omega_{12}} - \frac{\alpha_{32}}{\omega_{32}}},$$

a rational value. One of the equalities (24) and our hypothesis (1.2) give rational values for τ and $(\alpha_{12}^2\xi^2 - 4\tau)^{1/2}$. Hence, in this case, all the numbers τ, ξ, $(\alpha_{12}^2\xi^2 - 4\tau)^{1/2}$ are rational. Then the inequality sign in (2.2) is replaced by an equality sign, and the right and left end points of the intervals in (23) coincide. But θ being irrational, cannot be equal to this common end point. Hence θ must be interior to one or the other of the open intervals (23), and the argument follows as before.

Case 2. We now suppose that

$$\frac{\alpha_{12}\xi - (\alpha_{12}^2\xi^2 - 4\tau)^{1/2}}{2\tau} < \omega_{12} < \frac{\alpha_{12}\xi + (\alpha_{12}^2\xi^2 - 4\tau)^{1/2}}{2\tau}.$$

Then, since $\frac{\alpha_{12}}{\alpha_{13} + \frac{\alpha_{32}}{x}}$ is an increasing function, we get

(25) $$\omega_{32} = \frac{\alpha_{12}}{\alpha_{13} + \frac{\alpha_{32}}{\omega_{12}}} < \frac{\alpha_{12}}{\alpha_{13} + \alpha_{32}\frac{2\tau}{\alpha_{12}\xi + (\alpha_{12}^2\xi^2 - 4\tau)^{1/2}}}.$$

From (18), (19) and (25) we have

$$\omega_{32} < \frac{\alpha_{32}\xi - (\alpha_{32}^2\xi^2 - 4\tau)^{1/2}}{2\tau},$$

which gives

$$\frac{1}{\xi}\left(\tau + \frac{1}{\omega_{32}^2}\right) > \frac{\alpha_{32}}{\omega_{32}}$$

and, exactly as in case (1.1), we have that (1) is true, where for q we take either b_2 or b_3.

Thus we proved the following theorem:

C$_1$. *If* θ *is an irrational and* $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ *are some fractions satisfying the inequalities*

$$\frac{a_2}{b_2} < \theta < \frac{a_3}{b_3} < \frac{a_1}{b_1},$$

then at least one of these fractions satisfies (1) *with* ξ *given by* (19) *and* $\tau \geqslant 0$.

We can also state the theorem:

C_2. *If θ is an irrational and $\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}$ are some fractions satisfying the inequalities*

$$\frac{a_0}{b_0} < \frac{a_2}{b_2} < \theta < \frac{a_1}{b_1},$$

then at least one of these fractions satisfies (1) *with*

$$(26) \qquad \xi = \left[\left(\frac{\tau a_{20}^2 + a_{10}^2 + a_{12}^2}{a_{20}a_{10}a_{12}}\right)^2 - \frac{4}{a_{20}^2}\right]^{1/2}, \quad \tau \geqslant 0.$$

The proof of this theorem follows from an argument analogous to that used in the proof of C_1.

6. *Particular cases of theorems* B_1 *and* B_2. If we set in B_1

$$\frac{a}{b} = \frac{p_{2n}}{q_{2n}}, \quad \frac{a'}{b'} = \frac{p_{2n+1}}{q_{2n+1}}, \quad l = a_{2n+2}, \quad m = a_{2n+3},$$

we have

$$\frac{a+la'}{b+lb'} = \frac{p_{2n+2}}{q_{2n+2}}, \quad \frac{(a+la')m+a'}{(b+lb')m+b'} = \frac{p_{2n+3}}{q_{2n+3}}$$

and (15), (16) are satisfied.

Thus we obtain the following theorem:

B_3. At least one of the convergents $\frac{p_{2n+1}}{q_{2n+1}}, \frac{p_{2n+2}}{q_{2n+2}}, \frac{p_{2n+3}}{q_{2n+3}}$ of θ satisfies (1) with $\xi = (a_{2n+3}^2 + 4\tau)^{1/2}$, $\tau \geqslant 0$.

If we set in B_2

$$\frac{a}{b} = \frac{p_{2n}}{q_{2n}}, \quad \frac{a'}{b'} = \frac{p_{2n-1}}{q_{2n-1}}, \quad r = a_{2n+1}, \quad s = a_{2n+2},$$

we get the theorem:

B_4. At least one of the convergents $\frac{p_{2n}}{q_{2n}}, \frac{p_{2n+1}}{q_{2n+1}}, \frac{p_{2n+2}}{q_{2n+2}}$ of θ satisfies (1) with $\xi = (a_{2n+2}^2\tau^2 + 4\tau)^{1/2}$, $\tau \geqslant 0$.

Form B_3 and B_4 follow the theorems:

B_5. At least one of the convergents $\frac{p_{2n+1}}{q_{2n+1}}, \frac{p_{2n+2}}{q_{2n+2}}, \frac{p_{2n+3}}{q_{2n+3}}, \frac{p_{2n+4}}{q_{2n+4}}$ of θ satisfies (1) with $\xi = \max[(a_{2n+3}^2 + 4\tau)^{1/2}, (a_{2n+4}^2\tau^2 + 4\tau)^{1/2}]$, $\tau \geqslant 0$.

B_6. At least one of the convergents $\frac{p_{2n}}{q_{2n}}, \frac{p_{2n+1}}{q_{2n+1}}, \frac{p_{2n+2}}{q_{2n+2}}, \frac{p_{2n+3}}{q_{2n+3}}$ of θ satisfies (1) with $\xi = \max[(a_{2n+3}^2 + 4\tau)^{1/2}, (a_{2n+2}^2\tau^2 + 4\tau)^{1/2}]$, $\tau \geqslant 0$.

Because all the integers a_n are greater than 1, we obtain (g).

If we make $\tau = 1$ in B_3 and B_4, we have:

B_7. At least one of any three consecutive convergents satisfies (1) with $\xi = (a_{n+2}^2 + 4)^{1/2}$.

Theorem B_7 is a result of Morimoto [9] in 1926, found again by Obrejkov [18, 19] and M. Müller [10] in 1949 and 1955 respectively. These results are cited in the book of Sudan [25].

As a particular case of theorem B_7 we can cite the following Perron's theorem [21]:

B_8. At least one of three consecutive convergents satisfies (1) with $\xi = (m^2 + 4)^{1/2}$, where $a_{n+2} \geqslant m$.

If $m = 2$, then we get Humbert's theorem [5]:

B_9 At least one of three consecutive convergents satisfies (1) with $\xi = 8^{1/2}$, if $a_{n+2} \geqslant 2$.

We can also state the above theorems in the following form:

B_{10}. If an infinity of a_{2n+1} and an infinity of a_{2n} satisfy

$$a_{2n+1} \geqslant m_1, \quad a_{2n} \geqslant m_2,$$

then θ has an infinity of convergents which satisfy (1) with

$$\xi = \max[(m_1^2 + 4\tau)^{1/2}, (m_2^2\tau^2 + 4\tau)^{1/2}], \quad \tau \geqslant 0.$$

This theorem contains, as a particular case, theorem A, because for any irrational θ we have $a_{2n+1} \geqslant 1$ and $a_{2n} \geqslant 1$.

7. *Particular cases of theorems* C_1 *and* C_2. If we set in C_1

$\frac{a_1}{b_1} = \frac{p_n}{q_n}, \frac{a_2}{b_2} = \frac{p_m}{q_m}, \frac{a_3}{b_3} = \frac{p_l}{q_l}$, $a_{12} = p_nq_m - p_mq_n = q_{n,m}$ etc. where $\frac{p_n}{q_n}, \frac{p_m}{q_m}, \frac{p_l}{q_l}$ are three convergents of θ with $n < m < l$, $m - n$ and $l - m$ odd integers $\geqslant 1$ and n odd, $q_{n,m}$ is the denominator of convergent $[a_{n+1}, \ldots, a_m]$ of irrational $\theta_n = [a_{n+1}, a_{n+2}, \ldots]$, then we get the theorem:

C_3. At least one of the three convergents $\frac{p_n}{q_n}, \frac{p_m}{q_m}, \frac{p_l}{q_l}$ of θ satisfies (1) with

$$\xi = \left[\left(\frac{\tau q_{nm}^2 + \tau q_{ml}^2 + q_{nl}^2}{q_{nm}q_{ml}q_{nl}}\right)^2 - \frac{4\tau^2}{q_{nl}^2}\right]^{1/2}, \quad \tau \geqslant 0.$$

If we take $\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}$ as the three convergents $\frac{p_n}{q_n}, \frac{p_m}{q_m}, \frac{p_l}{q_l}$ of θ in C_2 with $n < m < l$, $m - n$ and $l - m$ odd integers $\geqslant 1$ and n even integer we have the theorem:

C_4. At least one of the three convergents $\frac{p_n}{q_n}, \frac{p_m}{q_m}, \frac{p_l}{q_l}$ of θ satisfies (1) with

$$\xi = \left[\left(\frac{q_{nm}^2 + q_{ml}^2 + \tau q_{nl}^2}{q_{nm}\, q_{ml}\, q_{nl}}\right)^2 - \frac{4}{q_{nl}^2}\right]^{1/2}, \quad \tau \geqslant 0,$$

where $q_{n,m}$ has the same meaning as in C_3.

In [13, 14] I proved and stated these theorems in the same form:

C_5. If $n < m < l$, $m - n$ and $l - m$ are odd integers > 1, at least one of the convergents $\frac{p_n}{q_n}, \frac{p_m}{q_m}, \frac{p_l}{q_l}$ of θ satisfies (1) with

$$\xi = \left[\left(\frac{\tau^{\varepsilon_l} q_{n,m}^2 + \tau^{\varepsilon_n} q_{ml}^2 + \tau^{\varepsilon_m} q_{nl}^2}{q_{nm}\, q_{ml}\, q_{nl}}\right)^2 - 4\frac{\tau^{\varepsilon_n + \varepsilon_l}}{q_{nl}^2}\right]^{1/2}, \quad \tau \geqslant 0,$$

where $\varepsilon_n = \frac{1-(-1)^n}{2}$.

If we take $\tau = 1$, we get a theorem, due to Fujiwara [3].

Setting $a_{2n+4} = 1$, $a_{2n+5} = 1$ and considering $\frac{p_{2n+1}}{q_{2n+1}}, \frac{p_{2n+2}}{q_{2n+2}}, \frac{p_{2n+5}}{q_{2n+5}}$ or $\frac{p_{2n}}{q_{2n}}, \frac{p_{2n+1}}{q_{2n+1}}, \frac{p_{2n+4}}{q_{2n+4}}$, we obtain the theorems:

C_6. At least one of the convergents $\frac{p_{2n+1}}{q_{2n+1}}, \frac{p_{2n+2}}{q_{2n+2}}, \frac{p_{2n+5}}{q_{2n+5}}$ of θ satisfies (1) with

$$\xi = \left[\left(\frac{1 + 4\tau + (2a_{2n+3} + 1)^2 \tau}{2(2a_{2n+3} + 1)}\right)^2 - \frac{4\tau^2}{(2a_{2n+3} + 1)^2}\right]^{1/2}, \quad \tau \geqslant 0.$$

C_7. At least one of the convergents $\frac{p_{2n}}{q_{2n}}, \frac{p_{2n+1}}{q_{2n+1}}, \frac{p_{2n+4}}{q_{2n+4}}$ satisfies (1) with

$$\xi = \left[\left(\frac{\tau + 4 + (2a_{2n+2} + 1)^2}{2(2a_{2n+2} + 1)}\right)^2 - \frac{4}{(2a_{2n+2} + 1)^2}\right]^{1/2}, \quad \tau \geqslant 0.$$

When we take $a_{2n+3} = 2$ in C_6 or $a_{2n+2} = 2$ in C_7 and $\tau = 1$, we get:

C_8. At least one of the convergents $\frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}}, \frac{p_{n+4}}{q_{n+4}}$ of θ satisfies (1) with $\xi = \frac{\sqrt{221}}{4}$, where $a_{n+2} = 2$, $a_{n+3} = 1$, $a_{n+4} = 1$. This theorem is also due to Fujiwara [4].

If we take $m = n + 1$, $l = n + 2$, then $q_{n,n+1} = 1$, $q_{n+1,n+2} = 1$, $q_{n,n+2} = a_{n+2}$. According as n is an odd or even integer, we have respectively

$$\xi = \left[\left(\frac{2\tau + a_{n+2}^2}{a_{n+2}}\right)^2 - 4\frac{\tau^2}{a_{n+2}^2}\right]^{1/2} = (a_{n+2}^2 + 4\tau)^{1/2}, \quad \xi = (a_{n+2}^2 \tau^2 + 4\tau)^{1/2}.$$

Thus, in the case when n, m, l are three consecutive integers, we get theorems B_3 and B_4.

Then all the particular cases of theorems B_3 and B_4 are also particular for $C_3 - C_5$, that is theorems C_1 and C_2 are the best general ones among all those given here.

8. *Other particular cases of theorems* C_1 *and* C_2. Now we consider the numbers

$$\frac{p_{2n+2}}{q_{2n+2}}, \quad \frac{jp_{2n+2} + p_{2n+1}}{jq_{2n+2} + q_{2n+1}}, \quad \frac{p_{2n+3} - kp_{2n+2}}{q_{2n+3} - kq_{2n+2}}. \tag{27}$$

It is easy to see that we have the following inequalities:

$$\frac{p_{2n+2}}{q_{2n+2}} < \theta < \frac{p_{2n+3} - kp_{2n+2}}{q_{2n+3} - kq_{2n+2}} < \frac{jp_{2n+2} + p_{2n+1}}{jq_{2n+2} + q_{2n+1}},$$

$$a_{12} = (jp_{2n+2} + p_{2n+1})\, q_{2n+2} - (jq_{2n+2} + q_{2n+1})\, p_{2n+2} = 1,$$

$$a_{32} = (p_{2n+3} - kp_{2n+2})\, q_{2n+2} - (q_{2n+3} - kq_{2n+2})\, p_{2n+2} = 1.$$

$$a_{13} = (jp_{2n+2} + p_{2n+1})(q_{2n+3} - kq_{2n+2}) - (jq_{2n+2} + q_{2n+1})(p_{2n+3} - kp_{2n+2}) =$$
$$= a_{2n+3} - (j + k).$$

If we replace these numbers in C_1, we obtain the following theorem:

D_1. If $a_{2n+3} \geqslant j + k$, then at least one of the fractions (27) satisfies (1) with $\xi = [(a_{2n+3} - (j+k))^2 + 4\tau]^{1/2}$, $\tau \geqslant 0$.

In a similar way, we can state the theorem:

D_2. If $a_{2n+2} \geqslant j + k$, then at least one of the fractions

$$\frac{p_{2n+1}}{q_{2n+1}}, \quad \frac{jp_{2n+1} + p_{2n}}{jq_{2n+1} + q_{2n}}, \quad \frac{p_{2n+2} - kp_{2n+1}}{q_{2n+2} - kp_{2n+1}}$$

satisfies (1) with $\xi = [(a_{2n+2} - (j+k))^2 \tau^2 + 4\tau]^{1/2}$, $\tau \geqslant 0$.

I proved D_1 and D_2 for $\tau = 1$ in [15].

Theorems D_1 and D_2 are more general than B_3 and B_4, because we get these by setting $j = k = 0$ in D_1 and D_2.

9. *General remarks* 1°. It is clear that we could state a great deal of theorems as particular cases of theorems C_1 and C_2. But we gave only some theorems which generalize the well known ones.

2°. In the proofs of C_1 and C_2 the fractions $\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ are supposed as arbitrarly satisfying only some inequalities. They do not depend on the convergents of θ or Farey series.

3°. If in paragraph 5 case 1 (1.1) we set $\xi = \frac{2}{\alpha_{12}}\sqrt{r}$, then

$$\frac{1}{\xi}\left(r + \frac{1}{\omega_{12}^2}\right) - \frac{\alpha_{12}}{\omega_{12}} > 0 \text{ and hence:}$$

E_1. At least one of the fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}$ satisfies (1) with

$$\xi = \frac{2}{\alpha_{12}}\sqrt{r}, \quad r > 0.$$

Setting

$$\frac{a_1}{b_1} = \frac{p_{2n-1}}{q_{2n-1}}, \ \frac{a_2}{b_2} = \frac{p_{2n}}{q_{2n}} \text{ or } \frac{a_1}{b_1} = \frac{p_{2n+1}}{q_{2n+1}}, \ \frac{a_2}{b_2} = \frac{p_{2n}}{q_{2n}}, \text{ we have:}$$

E_2. At least one of the fractions $\frac{p_\nu}{q_\nu}, \frac{p_{\nu+1}}{q_{\nu+1}}$ satisfies (1) with $\xi = 2\sqrt{r}$, $r > 0$. If we set $r = 1$ in E_2, we get Vahlen's theorem [26]:

E_3 At least one of the fractions $\frac{p_\nu}{q_\nu}, \frac{p_{\nu+1}}{q_{\nu+1}}$ satisfies (1) with $\xi = 2$.

If we take

$$\frac{a_1}{b_1} = \frac{jp_n + p_{n-1}}{jq_n + q_{n-1}}, \quad \frac{a_2}{b_2} = \frac{p_n - kp_{n-1}}{q_n - kq_{n-1}},$$

we obtain the following theorem:

F. If $a_n \geqslant \max(1,k)$ and $a_{n+1} \geqslant \max(1,j)$, then at least one of fractions $\frac{jp_n + p_{n-1}}{jq_n + q_{n-1}}, \frac{p_n - kp_{n-1}}{q_n - kq_{n-1}}$ satisfies (1) with $\xi = \frac{2}{1+kj}\sqrt{r}$, $r > 0$

I proved directly this theorem in [15].

4°. In my paper [14, p. 200], it is easy to see that the equalities (24), written for the convergents of θ, prove better and faster that ξ is rational.

5°. I must say that the problem of the best approximation of irrational numbers by rational numbers cannot be solved with the methods given here. Robinson's works and my papers, written in 1947—1948 and which are not cited here, show it immediately. On this occasion R. Robinson solved the important problem of critical numbers for asymmetrical approximation and I gave a geometrical interpretation for this kind of approximation and generalized a Perron's and Shibata's theorem.

6°. I must also add that, in many occasions, I formulated the theorems and their proofs using Olds', Niven's and Le Veque's words.

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ASUPRA APROXIMĂRILOR DIOFANTICE ASIMETRICE

Rezumat

În primele trei paragrafe se dă o demonstraţie completă a teoremei lui B. Segre [24]: Orice iraţional θ are o infinitate de aproximări raţionale p/q aşa că (1) să fie satisfăcută cu $\xi = \max[(1+4\tau)^{1/2}, (\tau^2+4\tau)^{1/2}]$ şi $\tau \geqslant 0$.

Folosind o idee dată de I. Niven [16, 17], autorul arată cum trebuie continuată demonstraţia aritmetică dată de C. D. Olds [20] numai pentru $\tau \geqslant 1$.

Tot aici se fac unele observaţii asupra precizărilor teoremei date de demonstraţie.

În paragrafele următoare se dau două generalizări ale teoremei lui B. Segre. Generalizarea a doua conţine drept cazuri particulare multe din teoremele clasice asupra aproximărilor diofantice simetrice.

ОБ АССИММЕТРИЧЕСКИХ ДИОФАНТОВЫХ ПРИБЛИЖЕНИЯХ

Краткое содержание

В первых трех параграфах дается полное доказательство теоремы Б. Сегре [24]. Каждый иррационал θ имеет безконечное множество рациональных приближений p/q так что (1) удовлетворяется $\xi = \max[(1+4\tau)^{1/2}, (\tau^2+4\tau)^{1/2}]$ и $\tau \geqslant 0$.

Используя идею И. Нивена [16, 17], автор указывает как должно продолжеться арифметическое доказательство которое С. Д. Олдс [20] дает только для $\tau \geqslant 1$.

Здесь же делаются некоторые замечания на счет уточнений доказаной теоремы.

В последующих параграфах даются обобщения теоремы Б. Сегре. Второе обобщение содержит как частные случаи многие классические теоремы о симметрических диофантовых приближениях.

ASUPRA ŞIRURILOR RECURENTE

DE

CORINA REISCHER-HAIMOVICI şi V. TAMAŞ

Comunicare prezentată în sesiunea ştiinţifică a Universităţii „Al. I. Cuza" din 26–27 octombrie 1963

1. Fie

$$u_m = a_1 u_{m-1} + a_2 u_{m-2} + \ldots + a_n u_{m-n} \tag{1}$$

o relaţie de recurenţă de ordin n, unde a_j $(j = 1, 2, \ldots, n)$ sînt numere reale şi fie

$$\lambda^n = a_1 \lambda^{n-1} + \ldots + a_n \tag{2}$$

ecuaţia generatoare a relaţiei (1).

Dacă termenii şirului $u_1, u_2, \ldots, u_n, \ldots$ satisfac relaţia (1) şirul se numeşte recurent.

O problemă importantă care se pune în legătură cu şirurile recurente este aceea de găsire a unei formule care să dea termenul general u_m în funcţie de coeficienţii relaţiei (1) sau de rădăcinile ecuaţiei (2) şi de primii n termeni ai şirului.

Astfel I. J. Schwatt [5], folosind inducţia matematică, ajunge, după calcule laborioase, la formula

$$u_m = u_0 a_2 \sum_{k=0}^{\left[\frac{m-2}{2}\right]} C^k_{m-2-k} a_1^{m-2-2k} a_2^k + u_1 \sum_{k=0}^{\left[\frac{m-1}{2}\right]} C^k_{m-1-k} a_1^{m-1-2k} a_2^k \tag{3}$$

pentru termenul general al unui şir oarecare ce satisface relaţia de recurenţă de ordin 2:

$$u_m = a_1 u_{m-1} + a_2 m_{m-2}.$$

L. Toscano [7], plecînd de la ecuaţia generatoare $x^2 - px + q = 0$, găseşte pentru termenul general al unui şir ce satisface relaţia de recurenţă corespunzătoare: