TENSIONS THERMIQUES QUASISTATIONNAIRES DANS UNE PLAQUE CIRCULAIRE MINCE DANS L'HYPOTHÈSE DE LA VARIATION LINÉAIRE DE LA TEMPÉRATURE D'APRÈS SON ÉPAISSEUR

Résumé

On montre d'abord que la détermination du champ de températures en régime non-stationnaire et axial symétrique dans une plaque circulaire mince se réduit, dans les cas d'échanges convectifs de la chaleur sur sa frontière et (17), à l'intégration du système d'équations (18) avec les conditions initiales et aux limites (19). Le champ de température (39) est donné par les formules (36) et (38).

Les tensions thérmiques correspondantes à ce champ de températures sont données par les formules (59) — (60).

ON CERTAIN MONGE-AMPÈRE EQUATIONS AND THE ONE-DIMENSIONAL UNSTEADY POTENTIAL EQUATION

BY

E. GOLDHAGEN

Introduction. In this paper we deal with the hyperbolic equations of the form:

$$A(p,q)\frac{\partial^2\Phi}{\partial t^2}+B(p,q)\frac{\partial^2\Phi}{\partial t\,\partial x}+C(p,q)\frac{\partial^2\Phi}{\partial x^2}=0, \tag{I.1}$$

where t and x are independent variables, Φ is an unknown function, $p=\partial\Phi/\partial t$, $q=\partial\Phi/\partial x$, and A, B, C are real functions of class C^2.

We denote by $\lambda_1(p,q)$ and $\lambda_2(p,q)$ the solutions of algebraic equation

$$A\lambda^2-B\lambda+C=0$$

and with

$$u(p,q)=\alpha,\quad v(p,q)=\beta, \tag{I.2}$$

the solutions of the differential equations

$$dp+\lambda_2\,dq=0,\quad dp+\lambda_1\,dq=0, \tag{I.3}$$

α and β being constants.

We can easily show that the integration of the nonlinear equation (I.1) leads to the integration of the linear equation

$$(\mu_1-\mu_2)\frac{\partial^2 t}{\partial\alpha\partial\beta}+\frac{\partial\mu_1}{\partial\alpha}\frac{\partial t}{\partial\beta}-\frac{\partial\mu_2}{\partial\beta}\frac{\partial t}{\partial\alpha}=0, \tag{I. 4}$$

where the coefficients

$$\mu_i(\alpha,\beta)\equiv\lambda_i[p(\alpha,\beta),\ q(\alpha,\beta)] \tag{I.5}$$

are obtained with the aid of the relations (I.2).

By applying this integration method we get the general solution for the potential equation of an isentropic fluid in an onedimensional unsteady flow.

By using the general solution of (I.4) ([5] and [8]), we may get analogous results for the polytropic flow, the isothermal flow and for some other barotropic flows.

In order to solve the boundary-value problems, we may use the methods of [3], [7], [8], [5].

1. *The setting of the problem.* 1°. The first order characteristics of (I.1) are noted down as follows:

$$\mu = \alpha, \quad \frac{\partial \Phi}{\partial \beta} - p\frac{\partial t}{\partial \beta} - q\frac{\partial x}{\partial \beta} = 0, \quad \frac{\partial x}{\partial \beta} - \mu_1 \frac{\partial t}{\partial \beta} = 0 \tag{1.1}$$

and

$$v = \beta, \quad \frac{\partial \Phi}{\partial \beta} - p\frac{\partial t}{\partial \alpha} - q\frac{\partial x}{\partial \alpha} = 0, \quad \frac{\partial x}{\partial \alpha} - \mu_2 \frac{\partial t}{\partial \alpha} = 0. \tag{1.2}$$

Out of $(1.1)_3$ and $(1.2)_3$ we get

$$(\mu_1 - \mu_2)\frac{\partial^2 t}{\partial \alpha \partial \beta} + \frac{\partial \mu_1}{\partial \alpha}\frac{\partial t}{\partial \beta} - \frac{\partial \mu_2}{\partial \beta}\frac{\partial t}{\partial \alpha} = 0. \tag{1.3}$$

As soon as we know t, we can further determine x and Φ.

2. *One-dimensional unsteady potential-equation.* 1°. The equation of the one-dimensional unsteady flow of an inviscid, elastic fluid has the form

$$\frac{\partial^2 \Phi}{\partial t^2} + 2q\frac{\partial^2 \Phi}{\partial t \partial x} + (q^2 - a^2)\frac{\partial^2 \Phi}{\partial x^2} = 0. \tag{2.1}$$

Here a ist the sound velocity. For (2.1) we have

$$\lambda_1 = q + a, \qquad \lambda_2 = q - a. \tag{2.2}$$

For the polytropic flow $p = c\rho^k\,(k > 1)$, and the sound velocity is given by the formula

$$a^2 = (1-k)(p + 1/2\, q^2). \tag{2.3}$$

By using (2.3) and integrating (I.3), we get

$$\begin{aligned} 2\sqrt{p + q^2/2} - \sqrt{1-k}\,q &= \alpha, \\ 2\sqrt{p + q^2/2} + \sqrt{1-k}\,q &= \beta. \end{aligned} \tag{2.4}$$

When the expressions (2.3) and (2.4) are substituted in (2.2), we find that

$$\begin{aligned} \mu_1 &= \frac{1}{2\sqrt{1-k}}(\beta - \alpha) + \frac{\sqrt{1-k}}{4}(\beta + \alpha), \\ \mu_2 &= \frac{1}{2\sqrt{1-k}}(\beta - \alpha) - \frac{\sqrt{1-k}}{4}(\beta + \alpha). \end{aligned} \tag{2.5}$$

The equation (1.3) corresponding to the polytropic fluid may be written in the form

$$\frac{\partial^2 t}{\partial \alpha \partial \beta} + \frac{1}{\alpha + \beta}\frac{k+1}{2(k-1)}\left(\frac{\partial t}{\partial \alpha} + \frac{\partial t}{\partial \beta}\right) = 0. \tag{2.6}$$

The equation (2.6) was studied by several authors [1], [4], [5].

For the isentropic flow ($k = 1,4$) we obtain from (2.6) and (2.5)

$$\frac{\partial^2 t}{\partial \alpha \partial \beta} + \frac{3}{\alpha + \beta}\left(\frac{\partial t}{\partial \alpha} + \frac{\partial t}{\partial \beta}\right) = 0 \tag{2.7}$$

and

$$\mu_1 = \frac{3\alpha - 2\beta}{\sqrt{10}}\,i, \quad \mu_2 = \frac{2\alpha - 3\beta}{\sqrt{10}}\,i, \tag{2.8}$$

The general solution of (2.7) is

$$t = \frac{2^5}{(\alpha+\beta)^5}\left|\varphi(\alpha) + \psi(\beta) - \frac{\alpha+\beta}{2}(\psi' + \psi') + \frac{(\alpha+\beta)^2}{12}(\varphi'' + \psi'')\right|, \tag{2.9}$$

where φ and ψ are arbitrary functions of class C^4 of α respectively β.

2°. From $(1.1)_3$, we get

$$x = \frac{3\alpha - 2\beta}{\sqrt{10}}\,it + \frac{2i}{\sqrt{10}}\int t\,d\beta + H(\alpha, \varphi, \varphi', \varphi''), \tag{2.10}$$

where

$$\int t\,d\beta = -\frac{2^3}{(\alpha+\beta)^4}(\psi + \psi) + \frac{2^3}{3(\alpha+\beta)^3}(\psi' + 2\varphi') - \frac{2^2}{3(\alpha+\beta)^2}\varphi''. \tag{2.11}$$

By using the above mentioned formulas we get from $(1.2)_3$

$$\frac{\partial H}{\partial \alpha} + \frac{\partial H}{\partial \varphi}\varphi' + \frac{\partial H}{\partial \varphi'}\varphi'' + \frac{\partial H}{\partial \varphi''}\varphi''' = 0.$$

Therefore H is constant.

The relation $(1.2)_3$ shows that

$$x = \frac{2\alpha - 3\beta}{\sqrt{10}}\,it - \frac{2i}{\sqrt{10}}\int t\,d\alpha + K(\beta, \psi, \psi', \psi''), \tag{2.12}$$

where

$$(2.13)\qquad \int t\,d\alpha = -\frac{2^3}{(\alpha+\beta)^4}(\varphi+\psi) + \frac{2^3}{3(\alpha+\beta)^3}(\varphi'+2\psi') - \frac{2^2}{3(\alpha+\beta)^2}\psi''.$$

It results from (1.1) that K is also constant. By adding (2.10) and (2.12) we get

$$(2.14)\qquad x = \frac{5(\alpha-\beta)}{2\sqrt{10}}\,it + \frac{2^3 i}{3\sqrt{10}(\alpha+\beta)^3}\left[\varphi'-\psi'+\frac{\alpha+\beta}{2}(\varphi''-\psi'')\right] + L,$$

where L is a constant.

3°. And now let us try to determine the expression of Φ. First we rewrite $(1.2)_2$, by introducing $\partial x/\partial\alpha$ from the $(1.2)_3$. Integrating we obtain

$$\Phi = (p+q\mu_2)\,t - \int\frac{\partial}{\partial\alpha}(p+q\mu_2)\,t\,d\alpha + L(\beta,\psi,\psi',\psi'').$$

Then, if we use (2.4) and (2.8), we get

$$(2.15)\qquad \begin{gathered} p = \frac{3(\alpha^2+\beta^2)-4\alpha\beta}{8}, \quad q = \frac{\alpha-\beta}{\sqrt{1,6}}, \\ q\mu_1 = \frac{1}{4}(\beta-\alpha)(3\alpha-2\beta), \quad q\mu_2 = \frac{1}{4}(\beta-\alpha)(2\alpha-3\beta). \end{gathered}$$

These latter relations allow to write

$$(2.16)\qquad \Phi = \frac{1}{8}(6\alpha\beta-3\beta^2-\alpha^2)\,t - \beta\int t\,d\alpha + \frac{1}{4}\int(\alpha+\beta)\,t\,d\alpha + M(\beta,\psi,\psi',\psi'').$$

Similarly, from $(1.1)_2$ and $(1.1)_3$, we obtain

$$(2.17)\qquad \Phi = \frac{1}{8}(6\alpha\beta-3\alpha^2-\beta^2)\,t - \alpha\int t\,d\beta + \frac{1}{4}\int(\alpha+\beta)\,t\,d\beta + N(\alpha,\varphi,\varphi',\varphi'').$$

In the same way as before we can prove that M and N are also constants. By adding (2.16) and (2.17) we get

$$(2.18)\qquad \begin{aligned} \Phi = {} & \frac{1}{4}(3\alpha\beta-\alpha^2-\beta^2)\,t - \frac{1}{2}\left[\alpha\int t\,d\beta + \beta\int t\,d\alpha\right] + \\ & + \frac{1}{8}\left[\int(\alpha+\beta)\,t\,d\alpha + \int(\alpha+\beta)\,t\,d\beta\right] + S, \end{aligned}$$

where S is a constant, while

$$\int(\alpha+\beta)\,t\,d\alpha = \frac{-2^5}{3(\alpha+\beta)^3}(\varphi+\psi) + \frac{2^3}{3(\alpha+\beta)^2}(\varphi'+3\psi') - \frac{2^3}{3(\alpha+\beta)}\psi'',$$

$$\int(\alpha+\beta)\,t\,d\beta = \frac{-2^5}{3(\alpha+\beta)^3}(\varphi+\psi) + \frac{2^3}{3(\alpha+\beta)^2}(\psi'+3\varphi') - \frac{2^3}{3(\alpha+\beta)}\psi''.$$

The formulas (2.18), (2.14), (2.9) represent the general solution of (2.1) in the isentropic case.

3. *The using of the above method.* In order to use with the same good results the above method to the general equation (I.1), it is quite enough that (I.4) be a Moutard-Cosserat equation [2]. Indeed, in this case we have

$$(3.1)\qquad t = f[\alpha,\beta,\varphi(\alpha),\varphi',\dots,\varphi^{(n)};\ \psi(\beta),\psi',\dots,\psi^{(m)}],$$

where f is a given function of class C^2, φ and ψ are arbitrary functions of class C^{n+2}, respectively C^{m+2} of α, respectively β. The integrability condition (1.3) being satisfied, we may obtain, by using $(1.1)_3$ and $(1.2)_3$, the relation

$$(3.2)\qquad dx = \mu_2\frac{df}{d\alpha}\,d\alpha + \mu_1\frac{df}{d\beta}\,d\beta,$$

where

$$\frac{df}{d\alpha} = \frac{\partial f}{\partial\alpha} + \frac{\partial f}{\partial\varphi}\varphi' + \dots + \frac{\partial f}{\partial\varphi^{(n)}}\varphi^{(n+1)}, \quad \frac{df}{d\beta} = \frac{\partial f}{\partial\beta} + \frac{\partial f}{\partial\psi}\psi' + \dots + \frac{\partial f}{\partial\psi^{(m)}}\psi^{(m+1)}.$$

By integrating (3.2) we obtain

$$(3.3)\qquad x = g[\alpha,\beta,\varphi,\varphi',\dots,\varphi^{(n)};\ \psi,\psi',\dots,\psi^{(m)}].$$

Φ is given by the equation

$$d\Phi = p\,dt + q\,dx.$$

By using (3.1) and (3.2), we get

$$d\Phi = (p+q\mu_2)\frac{df}{d\alpha}\,d\alpha + (p+q\mu_1)\frac{df}{d\beta}\,d\beta.$$

In order that the second member could be an exact total differential, the following identity must occur:

$$(p+q\mu_2)\frac{d^2f}{d\alpha d\beta} + \frac{\partial}{\partial\beta}(p+q\mu_2)\frac{df}{d\alpha} = (p+q\mu_1)\frac{d^2f}{d\alpha d\beta} + \frac{\partial}{\partial\alpha}(p+q\mu_1)\frac{df}{d\beta}.$$

By using (I.4) and grouping the terms, we get

$$(3.4)\qquad \left(\frac{\partial p}{\partial\alpha} + \mu_2\frac{\partial q}{\partial\beta}\right)\frac{df}{d\alpha} = \left(\frac{\partial p}{\partial\alpha} + \mu_1\frac{\partial q}{\partial\alpha}\right)\frac{df}{d\beta}.$$

To satisfy this relation it is necessary and sufficient that

$$(3.5)\qquad \frac{\partial p}{\partial\alpha} + \mu_1\frac{\partial q}{\partial\alpha} = 0, \quad \frac{\partial p}{\partial\beta} + \mu_2\frac{\partial q}{\partial\beta} = 0.$$

These conditions are realized owing to the fact that

$$dp + \mu_1\,dq = d\beta, \quad dp + \mu_2\,dq = d\alpha$$

results from (I.3).

4 *Supplementary remarks.* 1°. We have obtained the general solution of the potential-equation for the isentropic flow, after knowing the general solution of equation (2.7). Moreover, for the polytropic fluid $p = c\rho^k$ $(k > 1)$. In that case we have

$$\frac{\partial^2 t}{\partial\alpha\partial\beta} + \frac{c}{\alpha+\beta}\left(\frac{\partial t}{\partial\alpha} + \frac{\partial t}{\partial\beta}\right) = 0 \quad \left[c = \frac{k+1}{2(k-1)}\right]. \tag{4.1}$$

This equation was studied in [5] an [7] when c is an integer. By applying these results, we are able to integrate the potential-equation in the isoterm flow and in the polytropic flow if $(k+1)/[2(k-1)]$ is an integer. The integration of (1.3), according to Cosserats method may lead to more general results.

2°. In order to solve some boundary-value problems with the aid of general solutions, we may use [3], as well as Mises works [6], [7].

REFERENCES

1. Copson, N. T. — *On sound waves of finite amplitude.* Proc. Roy. Soc. A — 216 (1953), pp. 539—547.
2. Cosserat, E. — *Sur la théorie des équations aux dérivées partielles du second ordre.* Théorie générale des surfaces de Darboux. t. IV, p. 405—422
3. Goldhagen, E. — *Rezolvarea problemelor la limită pentru unele ecuații hiperbolice și parabolice.* Studii și Cerc. St. Acad RPR, Fil. Iași, An. XIV, 1963, p. 29—54.
4. Goursat, E. — *Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes.* Paris, 1886, p. 113—115.
5. Love, A. E., Pidduck F. B. — *Lagrange's ballistic problem.* Phil. Trans. Roy. Soc. London, A 222 (1922) pp. 167—226.
6. Mackie, A. G. — *Contour integral solutions of a class of differential equations.* J. Rational Mech. Anal. 4 (1955), pp. 733—750.
7. Mises, R. v. — *One-dimensional adiabatic flow of an inviscid fluid.* NAVORD Rept. 1719 (1951).
8. Mises, R. v. — *Mathematical theory of compressible fluid flow.* New York, 1958, pp. 165—179.
9. Taub, A. H. — *Interaction of progressive rarefaction waves.* Ann. Math. 47 (1946), p. 811—828

ASUPRA UNOR ECUAȚII MONGE-AMPÈRE ȘI ECUAȚIA POTENȚIALĂ UNIDIMENSIONALĂ, NESTAȚIONARĂ

Rezumat

Se știe că integrarea ecuației cu derivate parțiale neliniare (1.1) conduce la integrarea ecuației liniare (1.4). Demonstrăm aici că, dacă (1.4) este o ecuație integrabilă explicit, putem obține soluția generală a ecuației neliniare corespunzătoare. Studiem mai departe amănunțit ecuația potențială unidimensională pentru cazul isentropic. Pentru rezolvarea problemelor la limită cu ajutorul soluției generale putem folosi metodele din [3, 6, 7].

PLASTIC PLANE DEFORMATIONS OF ORTHOTROPIC BODIES

BY

C. I. BORȘ and V. DIACONIȚA

The problem of plastic plane deformations of orthotropic bodies has already been studied by a great number of researchers who, in their studies, have made use of various plasticity conditions.

We confine ourselves only to quote the following scientific works [1] — [7].

In the present paper we shall study this problem when the plastic potential is given by [8]:

$$\Psi(\sigma_{ij}) = K_{12}\left(\sigma_{11} - \frac{\Delta_{11}}{\Delta_{22}}\sigma_{22}\right)^2 + K_{23}\left(\sigma_{22} - \frac{\Delta_{22}}{\Delta_{33}}\sigma_{31}\right)^2 + K_{31}\left(\sigma_{33} - \frac{\Delta_{33}}{\Delta_{11}}\sigma_{11}\right)^2 + 2K_{44}\sigma_{12}^2 + 2K_{55}\sigma_{23}^2 + 2K_{66}\sigma_{31}^2 \tag{1}$$

and the yield condition by

$$\Psi(\sigma_{ij}) = 1\,, \tag{1'}$$

where K_{ij} are constants or not, according to whether the body is homogeneous or not, which characterise the plastic properties of the material.

The coefficients Δ_{ii}/Δ_{jj} $(i, j = 1, 2, 3)$ are being expressed by the elastic constants referring to the body under study.

The form of the potential was obtained from the supposition that the volume deformations are always elastic. Both, the potential (1) and the yield condition (1') are natural generalizations of the well-known results in the case of the isotropic body.

In the present case the relationships between strain rates on the one hand and stresses on the other hand are taken thus: