4 *Supplementary remarks.* 1°. We have obtained the general solution of the potential-equation for the isentropic flow, after knowing the general solution of equation (2.7). Moreover, for the polytropic fluid $p = c\rho^k$ $(k > 1)$. In that case we have

$$\frac{\partial^2 t}{\partial \alpha \partial \beta} + \frac{c}{\alpha + \beta}\left(\frac{\partial t}{\partial \alpha} + \frac{\partial t}{\partial \beta}\right) = 0 \quad \left[c = \frac{k+1}{2(k-1)}\right]. \tag{4.1}$$

This equation was studied in [5] an [7] when c is an integer. By applying these results, we are able to integrate the potential-equation in the isoterm flow and in the polytropic flow if $(k+1)/[2(k-1)]$ is an integer. The integration of (1.3), according to Cosserats method may lead to more general results.

2°. In order to solve some boundary-value problems with the aid of general solutions, we may use [3], as well as Mises works [6], [7].

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ASUPRA UNOR ECUAŢII MONGE-AMPÈRE ŞI ECUAŢIA POTENŢIALĂ UNIDIMENSIONALĂ, NESTAŢIONARĂ

Rezumat

Se ştie că integrarea ecuaţiei cu derivate parţiale neliniare (1.1) conduce la integrarea ecuaţiei liniare (1.4). Demonstrăm aici că, dacă (1.4) este o ecuaţie integrabilă explicit, putem obţine soluţia generală a ecuaţiei neliniare corespunzătoare. Studiem mai departe amănunţit ecuaţia potenţială unidimensională pentru cazul isentropic. Pentru rezolvarea problemelor la limită cu ajutorul soluţiei generale putem folosi metodele din [3, 6, 7].

PLASTIC PLANE DEFORMATIONS OF ORTHOTROPIC BODIES

BY

C. I. BORŞ and V. DIACONIŢA

The problem of plastic plane deformations of orthotropic bodies has already been studied by a great number of researchers who, in their studies, have made use of various plasticity conditions.

We confine ourselves only to quote the following scientific works [1] — [7].

In the present paper we shall study this problem when the plastic potential is given by [8]:

$$\Psi(\sigma_{ij}) = K_{12}\left(\sigma_{11} - \frac{\Delta_{11}}{\Delta_{22}}\sigma_{22}\right)^2 + K_{23}\left(\sigma_{22} - \frac{\Delta_{22}}{\Delta_{33}}\sigma_{33}\right)^2 + K_{31}\left(\sigma_{33} - \frac{\Delta_{33}}{\Delta_{11}}\sigma_{11}\right)^2 + 2K_{44}\sigma_{12}^2 + 2K_{55}\sigma_{23}^2 + 2K_{66}\sigma_{31}^2 \tag{1}$$

and the yield condition by

$$\Psi(\sigma_{ij}) = 1, \tag{1'}$$

where K_{ij} are constants or not, according to whether the body is homogeneous or not, which characterise the plastic properties of the material.

The coefficients Δ_{ii}/Δ_{jj} $(i, j = 1, 2, 3)$ are being expressed by the elastic constants referring to the body under study.

The form of the potential was obtained from the supposition that the volume deformations are always elastic. Both, the potential (1) and the yield condition (1') are natural generalizations of the well-known results in the case of the isotropic body.

In the present case the relationships between strain rates on the one hand and stresses on the other hand are taken thus:

$$
(2)\quad \begin{cases}
\dot{\varepsilon}_{11} = \lambda_1 \left| K_{12}\left(\sigma_{11} - \dfrac{\Delta_{11}}{\Delta_{22}}\sigma_{22}\right) - \dfrac{\Delta_{33}}{\Delta_{11}} K_{31}\left(\sigma_{33} - \dfrac{\Delta_{33}}{\Delta_{11}}\sigma_{11}\right)\right|, \\
\dot{\varepsilon}_{22} = \lambda_2 \left| K_{23}\left(\sigma_{22} - \dfrac{\Delta_{22}}{\Delta_{33}}\sigma_{33}\right) - \dfrac{\Delta_{11}}{\Delta_{22}} K_{12}\left(\sigma_{11} - \dfrac{\Delta_{11}}{\Delta_{22}}\sigma_{22}\right)\right|, \\
\dot{\varepsilon}_{33} = \lambda_3 \left| K_{31}\left(\sigma_{33} - \dfrac{\Delta_{33}}{\Delta_{11}}\sigma_{11}\right) - \dfrac{\Delta_{22}}{\Delta_{33}} K_{23}\left(\sigma_{22} - \dfrac{\Delta_{22}}{\Delta_{33}}\sigma_{33}\right)\right|, \\
\dot{\varepsilon}_{12} = 2\lambda K_{44}\sigma_{12}; \quad \dot{\varepsilon}_{23} = 2\lambda K_{55}\sigma_{23}; \quad \dot{\varepsilon}_{31} = 2\lambda K_{66}\sigma_{31}
\end{cases}
$$

where λ_1, λ_2, λ_3, λ are connected by

$$
(3)\quad \begin{cases}
\lambda_1 = \dfrac{\Delta_{11}}{\Delta_{22}}\lambda, \\
\lambda_2 = \lambda, \\
\lambda_3 = \dfrac{\Delta_{33}}{\Delta_{22}}\lambda .
\end{cases}
$$

On account of (3) the incompressibility condition $\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} + \dot{\varepsilon}_{33} = 0$ is indentically satisfied.

In the case of plane deformation we have

$$
(4)\quad \dot{\varepsilon}_{33} = 0, \quad \sigma_{23} = \sigma_{31} = 0 .
$$

On account of the first condition we obtain

$$
(5)\quad \sigma_{33} = \frac{\dfrac{\Delta_{33}}{\Delta_{11}} K_{31}\sigma_{11} + \dfrac{\Delta_{22}}{\Delta_{33}} K_{23}\sigma_{22}}{K_{31} + \left(\dfrac{\Delta_{22}}{\Delta_{33}}\right)^2 K_{23}} .
$$

According to (4) and (5) the yield condition (1′) becomes

$$
(6)\quad (\sigma_x - a\sigma_y)^2 + M^2\tau_{xy}^2 = N^2,
$$

where

$$
(7)\quad \begin{cases}
M_2 = \dfrac{2K_{44}\left(\dfrac{\Delta_{11}}{\Delta_{22}}\right)^2 \left| K_{31} + \left(\dfrac{\Delta_{22}}{\Delta_{33}}\right)^2 K_{23}\right|}{\left(\dfrac{\Delta_{11}}{\Delta_{22}}\right)^2 K_{12}K_{31} + \left(\dfrac{\Delta_{11}}{\Delta_{22}}\right)^2\left(\dfrac{\Delta_{22}}{\Delta_{33}}\right)^2 K_{12}K_{23} + K_{23}K_{31}}, \\
N^2 = \dfrac{\left(\dfrac{\Delta_{11}}{\Delta_{22}}\right)^2 \left| K_{31} + \left(\dfrac{\Delta_{22}}{\Delta_{33}}\right)^2 K_{23}\right|}{\left(\dfrac{\Delta_{11}}{\Delta_{22}}\right)^2 K_{12}K_{13} + \left(\dfrac{\Delta_{11}}{\Delta_{22}}\right)^2\left(\dfrac{\Delta_{22}}{\Delta_{33}}\right)^2 K_{12}K_{23} + K_{23}K_{31}}, \quad a = \dfrac{\Delta_{12}}{\Delta_{22}} .
\end{cases}
$$

The relatioships between the strain rates and stresses in the case of plane deformtion become

$$
(8)\quad \begin{cases}
\dot{\varepsilon}_x = \dfrac{a\lambda}{N^2}(\sigma_x - a\sigma_y), \\
\dot{\varepsilon}_y = -\dfrac{a\lambda}{N^2}(\sigma_x - a\sigma_y), \\
\dot{\gamma}_{xy} = \dfrac{\lambda M^2}{N^2}\tau_{xy}; \quad \dot{\gamma}_{yz} = \dot{\gamma}_{zx} = 0
\end{cases}
$$

or

$$
(9)\quad \begin{cases}
\dfrac{\dfrac{\partial v_x}{\partial x}}{\sigma_x - a\sigma_y} = \dfrac{\dfrac{\partial v_y}{\partial y}}{-(\sigma_x - a\sigma_y)} = \dfrac{\dfrac{\partial v_x}{\partial y} + \dfrac{\partial v_y}{\partial x}}{M^2\tau_{xy}} \\
\dot{\varepsilon}_x = \dfrac{\partial v_x}{\partial x}, \ldots
\end{cases}
$$

where v_x and v_y are the velocity components.

The differential equations of equilibrium in our case are:

$$
(10)\quad \begin{cases}
\dfrac{\partial\sigma_x}{\partial x} + \dfrac{\partial\tau_{xy}}{\partial y} = 0, \\
\dfrac{\partial\tau_{xy}}{\partial x} + \dfrac{\partial\sigma_y}{\partial y} = 0 .
\end{cases}
$$

If the boundary conditions are given in stresses, then we have to do with a statically determinated problem; we can therefore determine from (10) and the stresses σ_x, σ_y, τ_{xy}, and then from (9) we can determine the velocity components.

If we take

$$
(11)\quad \begin{cases}
\sigma_x = p + \dfrac{N}{2}\cos 2\theta, \\
\sigma_y = \dfrac{p}{a} - \dfrac{N}{2a}\cos 2\theta, \\
\tau_{xy} = \dfrac{N}{M}\sin 2\theta,
\end{cases}
$$

then the yield condition will be identically satisfied.

In the relations (11) p and θ are unknown functions of x and y, which should be determined, in the next.

Introducing the stresses given by (11) into the differential equations of equilibrium (10) we get the following system for the new unknown functions p and θ:

$$\left\{\begin{aligned} &\frac{\partial p}{\partial x} - N \sin 2\theta \frac{\partial \theta}{\partial x} + \frac{2N}{M} \cos 2\theta \frac{\partial \theta}{\partial y} = 0, \\ &\frac{\partial p}{\partial y} + \frac{2aN}{M} \cos 2\theta \frac{\partial \theta}{\partial x} + N \sin 2\theta \frac{\partial \theta}{\partial y} = 0. \end{aligned}\right. \tag{12}$$

Adding to the equations (12) the relations

$$\left\{\begin{aligned} dp &= \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy, \\ d\theta &= \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy \end{aligned}\right. \tag{13}$$

and using a known method from the theory of the systems of partial differential equations, we find that the system has two families of characteristics whose differential equations are

$$\left(\frac{dy}{dx}\right)_{1,2} = \frac{M \sin 2\theta \pm \sqrt{M^2 \sin^2 2\theta + 4a \cos^2 2\theta}}{2a \cos 2\theta}, \tag{14}$$

and along the characteristics we have still the relations

$$p \pm F(\theta) = \text{const.}, \tag{15}$$

where

$$F(\theta) = \int \frac{N}{M} \sqrt{M^2 \sin^2 2\theta + 4a \cos^2 2\theta}\, d\theta. \tag{16}$$

As for the orthotropic bodies we always have $a > 0$, it results the system is of an hyperbolic type.

It can also be immediately seen that:

$$\left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2 = -\frac{1}{a}$$

and therefore the characteristics are no longer orthogonal as in the isotropic case.

If we take into account (11), then the equations (9) for the determination of velocity components become

$$\frac{\dfrac{\partial v_x}{\partial x}}{\cos 2\theta} = \frac{\dfrac{\partial v_y}{\partial y}}{-\cos 2\theta} = \frac{\dfrac{\partial v_x}{\partial y} + \dfrac{\partial v_y}{\partial x}}{M \sin 2\theta} \tag{17}$$

or

$$\left\{\begin{aligned} &\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \\ &\cos 2\theta \frac{\partial v_x}{\partial y} + \cos 2\theta \frac{\partial v_y}{\partial x} + M \sin 2\theta \frac{\partial v_y}{\partial y} = 0. \end{aligned}\right. \tag{18}$$

If now we add to equations (18) the relations

$$\left\{\begin{aligned} dv_x &= \frac{\partial v_x}{\partial x} dx + \frac{\partial v_x}{\partial y} dy, \\ dv_y &= \frac{\partial v_y}{\partial x} dx + \frac{\partial v_y}{\partial y} dy \end{aligned}\right. \tag{19}$$

we find that the differential equations of the characteristics are

$$\left(\frac{dy}{dx}\right)_{1,2} = \frac{M \sin 2\theta \pm \sqrt{M^2 \sin^2 2\theta + 4 \cos^2 2\theta}}{2 \cos 2\theta}, \tag{20}$$

which shows that the system is of a hyperbolic type, the characteristic lines being orthogonal.

It can also be easily proved that along the characteristics we have

$$dv_\alpha - v_\beta\, d\theta = 0 \quad \text{on the lines } \alpha,$$

$$dv_\beta - v_\alpha\, d\theta = 0 \quad \text{on the lines } \beta;$$

these are the H i l d a G e i r i n g e r relations.

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DEFORMAREA PLANĂ A CORPURILOR ORTOTROPE ÎN DOMENIUL PLASTIC

Rezumat

Pentru studiul deformării plastice plane se utilizează potenţialul plastic de forma (1). Legăturile dintre vitezele de deformare şi tensiuni sînt date de (8). Se arată că ecuaţiile pentru definirea stării de tensiune (12) sînt de tip hiperbolic, caracteristicile nefiind ortogonale,

Starea de deformare se determină din ecuaţiile (18) care sînt de tip hiperbolic.

TEOREME VARIAŢIONALE ÎN PLASTICITATEA CORPURILOR ORTOTROPE. CAZUL DEFORMĂRILOR PLANE

DE

C. I. BORŞ

1. Postulînd faptul că deformările de volum sînt totdeauna elastice, s-a ajuns la concluzia că potenţialul plastic, din care se obţine condiţia de plasticitate şi legea de curgere, trebuie să fie de forma [1]:

$$2\Psi = K_{12}\left(\sigma_{11} - \frac{\Delta_{11}}{\Delta_{22}}\sigma_{22}\right)^2 + K_{23}\left(\sigma_{22} - \frac{\Delta_{22}}{\Delta_{33}}\sigma_{33}\right)^2 + K_{31}\left(\sigma_{33} - \frac{\Delta_{33}}{\Delta_{11}}\sigma_{11}\right)^2 + K_{44}\sigma_{23}^2 + K_{55}\sigma_{31}^2 + K_{66}\sigma_{12}^2. \tag{1}$$

Condiţia de plasticitate este dată de

$$\Psi = K^2, \tag{2}$$

iar legea de curgere de

$$\left\{\begin{aligned} \dot{\varepsilon}_{11} &= \lambda\Delta_{11}\left[K_{12}\left(\sigma_{11} - \frac{\Delta_{11}}{\Delta_{22}}\sigma_{22}\right) - K_{31}\frac{\Delta_{33}}{\Delta_{11}}\left(\sigma_{33} + \frac{\Delta_{33}}{\Delta_{11}}\sigma_{11}\right)\right], \\ \dot{\varepsilon}_{22} &= \lambda\Delta_{22}\left[K_{23}\left(\sigma_{22} - \frac{\Delta_{22}}{\Delta_{33}}\sigma_{33}\right) - K_{12}\frac{\Delta_{11}}{\Delta_{22}}\left(\sigma_{11} - \frac{\Delta_{11}}{\Delta_{22}}\sigma_{22}\right)\right], \\ \dot{\varepsilon}_{33} &= \lambda\Delta_{33}\left[K_{31}\left(\sigma_{33} - \frac{\Delta_{33}}{\Delta_{11}}\sigma_{11}\right) - K_{23}\frac{\Delta_{22}}{\Delta_{33}}\left(\sigma_{22} - \frac{\Delta_{22}}{\Delta_{33}}\sigma_{33}\right)\right]; \end{aligned}\right. \tag{3}$$

$$\dot{\varepsilon}_{23} = \lambda K_{44}\sigma_{23}, \quad \dot{\varepsilon}_{31} = \lambda K_{55}\sigma_{31}, \quad \dot{\varepsilon}_{12} = \lambda K_{66}\sigma_{12}. \tag{3'}$$

Se poate constata uşor că legea de curgere verifică în mod identic condiţia de incompresibilitate.