équations indépendantes. Dans ce cas le système obtenu en différentiant extérieurement (1.1) peut être mis sous la forme (2.6).

Si $\rho = n - s = 4$ il n'y a pas d'élément caractéristique et la formule bien connue:

$$n = s + s_1 + \ldots + s_p + p$$

montre que l'on a $p = 1, s_1 = 3$.

Conformément au deuxième théorème d'existence de Cartan l'intégrale générale du système (1.1) est unidimensionnelle et dépend de trois fonctions arbitraires d'une seule variable indépendante. Le système peut être regardé comme un système de s équations différentielles ordinaires à $s + 3$ fonctions inconnues et on peut prendre arbitrairement trois de ces fonctions inconnues.

Si $\rho > 4$, on a des caractéristiques à $\rho - 4$ dimensions et l'on a

$$p = \rho - 3, \quad s_1 = 3, \quad s_2 = s_3 = \ldots = s_p = 0.$$

En effectuant alors des opérations d'ordre $s + 4$, $s + 3, \ldots, 2, 1$ nous sommes ramenés au cas où $\rho = 4$, c'est-à-dire à des équations différentielles ordinaires.

L'efficacité et la facilité de maniement des méthodes purement algébriques que nous avons utilisées, nous a encore permis de généraliser les résultats obtenus ici dans le cas des systèmes de Pfaff de s équations indépendantes dont le premier caractère s_1 est quelconque, mais dont le système dérivé a moins de $s - s_1$ équations indépendantes.

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ASUPRA SISTEMELOR PFAFF DE CARACTER TREI CU SISTEM DERIVAT AVÎND MAI PUŢIN DECÎT $s-3$ ECUAŢII

Rezumat

Autorul se ocupă de problema sistemelor Pfaff de caracter trei pentru care sistemul derivat are efectiv mai puţin de $s - 3$ ecuaţii independente. Se arată care este forma sistemului obţinut prin diferenţierea exterioară a sistemului Pfaff dat în acest caz şi se stabileşte o teoremă care arată cînd poate sistemul derivat avea mai puţin de $s - 3$ ecuaţii. Se integrează sistemul în această eventualitate, folosindu-se teoremele de existenţă ale lui Cartan.

Rezultatele obţinute se extind simplu la cazul sistemelor Pfaff cu s_1 arbitrar avînd sistem derivat format cu mai puţin ca $s - s_1$ ecuaţii independente.

AN EXTREMAL PROBLEM ASSOCIATED WITH A CERTAIN NON-LINEAR INTEGRAL EQUATION[1]

BY

R. H. ROLWING and I. A. BARNETT

Introduction. In a previous paper entitled *On a System of Quadratic Equations and its Integral Analogue* (An. St. Univ. „Al. I. Cuza", Iaşi. Sect. I, 1965, p. 58) the authors have discussed the integral equation

$$y^2(x) = [y(x) - K(x)] \int_0^1 y(s)\, ds.$$

Let us denote by $I(c)$ the totality of values c for which

$$1 - cK(x) \geqslant 0 \text{ for every } x \text{ on } 0 \leqslant x \leqslant 1.$$

Here, $K(x)$ denotes a real valued function defined on the interval $0 \leqslant x \leqslant 1$ having the following properties:

(i) $K(x)$ is bounded on $0 \leqslant x \leqslant 1$,

(ii) $K(x)$ is continuous almost everywhere on $0 \leqslant x \leqslant 1$,

(iii) $\int_0^1 K(x)\, dx \neq 0$,

(iv) $K(x)$ is positive on some subinterval of $0 \leqslant x \leqslant 1$.

Such a function will be called *admissible*. Furthermore, let $\varepsilon(x)$ be a step function assuming values ± 1 on $0 \leqslant x \leqslant 1$; otherwise it is continuous a.e. with the assumption that there is at least one jump discontinuity. Then it has been shown that there exists a c on $I(c)$ for which

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$$\int_0^1 \varepsilon(x)\sqrt{1-cK(x)}\,dx = 1,$$

and hence a solution

$$y(x) = \frac{2}{c}[1+\varepsilon(x)\sqrt{1-cK(x)}]$$

of the above integral equation with $\varepsilon(x)$ of the same type ($\varepsilon(x)$ is of type r if it has exactly r points of discontinuity and type ∞ if the number of points of discontinuity is infinite).

We observe that corresponding to the function $\overline{K}(x) = \rho K(x)$ where ρ is an arbitrary constant not zero and corresponding to the same step function $\varepsilon(x)$ of the last equation, the constant $\bar{c} = c/\rho$ leads to the solution $\overline{y(x)} = \rho y(x)$ of the integral equation $y^2(x) = [y(x) - \rho K(x)]\int_0^1 y(x)\,dx$.

Thus we see that if $K(x)$ is an arbitrary admissible function, the associated constant c which leads to a solution of type $\geqslant 1$ may assume any value other than zero. If, however, the class of functions $K(x)$ is restricted by some additional condition as for example $\int_0^1 K^r(x)\,dx = 1$, it turns out that the constants c which lead to solution are also restricted. The subject of this paper is to determine the interval on which c must fall as the functions $K(x)$ vary over a certain class for a prescribed $\varepsilon(x)$ function of type $\geqslant 1$.

Section 1. Analytic statement of the problem. Let $\varepsilon(x)$ be an arbitrary but prescribed step function of type $\geqslant 1$. Consider the constants c which satisfy the following:

(1) $$1 - cK(x) \geqslant 0, \quad K(x) \ \textit{admissible},$$

(2) $$\int_0^1 \varepsilon(x)\sqrt{1-cK(x)}\,dx = 1,$$

(3) $$\int_0^1 K^r(x)\,dx = 1, \quad r \ \textit{fixed integer} \geqslant 1.$$

The problem is to find the range of values c for which conditions (1), (2), (3) are valid for the prescribed step function $\varepsilon(x)$ and for the fixed positive integer r. If we set $\varphi(x) = -cK(x)$, these conditions become

(4) $$\varphi(x) \geqslant -1, \quad \int_0^1 \varepsilon(x)\sqrt{1+\varphi(x)}\,dx = 1,$$

(5) $$\int_0^1 \varphi^r(x)\,dx = (-c)^r.$$

Thus the problem is that of finding the extreme values of $\int_0^1 \varphi^r(x)\,dx$, r fixed, for functions satisfying (4). We first exhibit a function such that $\int_0^1 \varphi^r(x)\,dx$ may take on arbitrarily large positive values and such that condition (4) is satisfied. Let p be the measure of the set for which $\varepsilon(x) = +1$ so that $p = \int_0^1 [(1+\varepsilon(x))/2]\,dx$ and consider the function φ defined as follows.

$$\varphi(x) = \begin{cases} -1 \text{ for all } x \text{ for which } \varepsilon(x) = -1, \\ -1 \text{ for all } x \text{ for which } \varepsilon(x) = +1 \text{ except for a set, of points of measure } \lambda p \text{ where } \lambda \text{ is a fixed but arbitrary number between 0 and 1}, \\ \varphi_0 \text{ on the set of measure } \lambda p. \end{cases}$$

We see then that $\int_0^1 \varepsilon(x)\sqrt{1+\varphi(x)}\,dx = \lambda p\sqrt{1+\varphi_0}$ and since this last expression must satisfy (4), we find that $\varphi_0 = \dfrac{1-\lambda^2p^2}{\lambda^2p^2}$. Hence,

$$\int_0^1 \varphi^r(x)\,dx = \int_{\lambda p} \varphi^r(x)\,dx + \int_{1-\lambda p} \varphi^r(x)\,dx = \lambda p\left(\frac{1-\lambda^2p^2}{\lambda^2p^2}\right)^r + (1-\lambda p)(-1)^r,$$

a quantity which can be made arbitrarily large by letting $\lambda \to 0$. It should be observed that in this argument p is fixed since $\varepsilon(x)$ is prescribed.

The fact that $\int_0^1 \varphi^r(x)\,dx$ has a finite lower bound is clear since $\varphi(x) \geqslant -1$ for all x on [0, 1] so that this integral could never be smaller than -1. However, the function $\varphi(x)$ cannot be negative on the whole interval $0 \leqslant x \leqslant 1$ since $\int_0^1 \varepsilon(x)\sqrt{1+\varphi(x)}\,dx$ must equal unity for an ε, function of type $\geqslant 1$ by condition (4). Let us note that if, for example, $\varepsilon(x)$ is prescribed to be positive on a very small range of $[0, 1]$, the value of $\varphi(x)$ would have to be large for that positive range of $\varepsilon(x)$, and thus the infimum may be an extremely large positive number.

Section 2. A restatement of the problem. Let P be the set of points for which $\varepsilon(x) = +1$ and N the set of points for which $\varepsilon(x) = -1$. Further let

$$\eta(x) = \varphi(x) \quad x \text{ on } P, \quad \eta(x) = 0 \quad x \text{ on } N,$$
$$\zeta(x) = \varphi(x) \quad x \text{ on } N, \quad \zeta(x) = 0 \quad x \text{ on } P.$$

Thus we can write

$$\varphi(x) = \frac{1+\varepsilon(x)}{2}\eta(x) + \frac{1-\varepsilon(x)}{2}\zeta(x),$$

$$1+\varphi(x) = \frac{1+\varepsilon(x)}{2}(1+\eta(x)) + \frac{1-\varepsilon(x)}{2}(1+\zeta(x)),$$

$$\varepsilon(x)\sqrt{1+\varphi(x)} = \frac{1+\varepsilon(x)}{2}\sqrt{1+\eta(x)} - \frac{1-\varepsilon(x)}{2}\sqrt{1+\zeta(x)},$$

Our problem now becomes that of minimizing the expression

$$\int_P \eta^r(x)\,dx + \int_N \zeta^r(x)\,dx \tag{6}$$

subject to the conditions

$$\int_P \sqrt{1+\eta(x)}\,dx - \int_N \sqrt{1+\zeta(x)}\,dx = 1, \quad \eta(x) \geqslant -1, \quad \zeta(x) \geqslant -1. \tag{7}$$

Section 3. The case $r = 1$. Let η_0 and ζ_0 be a pair of admissible functions such that the condition (7) is satisfied. We show first that there exists a pair of functions which are constant on P and N, respectively, such that:

$$\int_P \sqrt{1+\eta_1}\,dx - \int_N \sqrt{1+\zeta_1}\,dx = \int_P \sqrt{1+\eta_0(x)}\,dx - \int_N \sqrt{1+\zeta_0(x)}\,dx = 1$$

and for which

$$\int_P \eta_1\,dx + \int_N \zeta_1\,dx \leqslant \int_P \eta_0(x)\,dx + \int_N \zeta_0(x)\,dx.$$

To prove this, let η_1 and ζ_1 be constants defined by the relations

$$\sqrt{1+\eta_1} = \frac{1}{p}\int_P \sqrt{1+\eta_0(x)}\,dx,$$
$$\sqrt{1+\zeta_1} = \frac{1}{n}\int_N \sqrt{1+\zeta_0(x)}\,dx, \tag{8}$$

where p and n are the measures of the sets P and N respectively, so that $p + n = 1$. It then follows that η_1 and ζ_1 satisfy (7), namely,

$$\int_P \sqrt{1+\eta_1}\,dx - \int_N \sqrt{1+\zeta_1}\,dx = 1.$$

By the Schwarz inequality, we may write

$$p^2(1+\eta_1) = \left(\int_P \sqrt{1+\eta_1}\,dx\right)^2 = \left(\int_P \sqrt{1+\eta_0(x)}\,dx\right)^2 \leqslant p\int_P (1+\eta_0(x))\,dx \tag{9}$$

so that $\int_P \eta_1\,dx \leqslant \int_P \eta_0(x)\,dx$. Similarly, $\int_N \zeta_1\,dx \leqslant \int_N \zeta_0(x)\,dx$.

Thus, we have shown the existence of two constant functions for which:

$$\int_P \eta_1\,dx + \int_N \zeta_1\,dx \leqslant \int_P \eta_0(x)\,dx + \int_N \zeta_0(x)\,dx.$$

In other words if there is a pair of functions which give a minimum value to

$$\int_P \eta(x)\,dx + \int_N \zeta(x)\,dx$$

subject to

$$\int_P \sqrt{1+\eta(x)}\,dx - \int_N \sqrt{1+\zeta(x)}\,dx = 1,$$

this pair of functions must be constant over this domain.

We next define the constant $k \geqslant 0$ by

$$k = \int_N \sqrt{1+\zeta_1}\,dx = n\sqrt{1+\zeta_1}\,. \tag{10}$$

Using this and (7), we see that $\int_P \sqrt{1+\eta_1}\,dx - \int_N \sqrt{1+\zeta_1}\,dx = 1$ leads to

$$p\sqrt{1+\eta_1} = k+1\,. \tag{11}$$

Solving for η_1 and ζ_1 in (10) and (11), we obtain

$$\eta_1 = \frac{(k+1)^2 - p^2}{p^2}, \quad \zeta_1 = \frac{k^2-n^2}{n^2}. \tag{12}$$

Using these in (6) for $r=1$, we see that

$$\int_P \eta_1\,dx + \int_N \zeta_1\,dx = \frac{(k+1)^2}{p} + \frac{k^2-n^2}{n} = \frac{(n+k)^2}{pn}\,. \tag{13}$$

Thus (13) is minimum at $k=0$, and the minimum of $\int_0^1 \varphi(x)\,dx$ subject to $\int_0^1 \varepsilon(x)\sqrt{1+\varphi(x)}\,dx = 1$ is attained for

$$\varphi = \begin{cases} -1 \text{ on } N, \\ \dfrac{1}{p^2} - 1 \text{ on } P, \end{cases}$$

and the value of the minimum is $\frac{n}{p}$. Thus we have proved that the minimum value of the integral $\int_0^1 \varphi(x)\,dx$ subject to the condition $\int_0^1 \varepsilon(x)\sqrt{1+\varphi(x)}\,dx = 1$, where $\varepsilon(x)$ is a prescribed step function of type $\geqslant 1$ is given by

$$\frac{n}{p} = \frac{1-m}{1+m} \text{ where } m = \int_0^1 \varepsilon(x)\,dx = p - n. \text{ Since } \int_0^1 \varphi(x)\,dx = (-c),$$

we see that the largest c is given by

$$c = -\frac{1-m}{1+m}.$$

Section 4. Fundamental lemma. The following analogue of the Euler-Lagrange multiplier rule will be found useful in the sequel. As defined in Section 1, the problem is to minimize

$$\int_0^1 \varphi^r(x)\,dx, \quad r \text{ fixed positive integer}, \tag{14}$$

for all admissible functions $\varphi(x)$ defined on $0 \leqslant x \leqslant 1$ satisfying the condition

$$\int_0^1 \varepsilon(x)\sqrt{1+\varphi(x)}\,dx = 1, \tag{15}$$

where $\varepsilon(x)$ is a prescribed step function assuming values ± 1.

We now show that the minimizing function $\varphi(x)$ must be a step function. Let us define the functions $F(x,\varphi)$ and $G(x,\varphi)$ respectively, by

$$F(x,\varphi) = \varphi^r(x)\,,$$

$$G(x,\varphi) = \varepsilon(x)\sqrt{1+\varphi(x)}\,.$$

Our problem then is to minimize

$$I \equiv \int_0^1 F(x,\varphi)\,dx\,,$$

subject to the condition

$$\mathcal{J} \equiv \int_0^1 G(x,\varphi)\,dx = 1.$$

We express the variation of φ in terms of two parameters t_1 and t_2

$$\delta\varphi(x) = t_1\theta_1(x) + t_2\theta_2(x),$$

where $\theta_1(x)$ and $\theta_2(x)$ are admissible functions which each vanish at both $x=0$ and $x=1$. Then, with the abbreviations

$$I(t_1,t_2) = \int_0^1 F(x, \varphi + t_1\theta_1 + t_2\theta_2)\,dx = \int_0^1 [\varphi(x) + t_1\theta_1(x) + t_2\theta_2(x)]^r\,dx\,,$$

$$\mathcal{J}(t_1,t_2) = \int_0^1 G(x, \varphi + t_1\theta_1 + t_2\theta_2)\,dx = \int_0^1 \varepsilon(x)\sqrt{1+\varphi(x) + t_1\theta_1(x) + t_2\theta_2(x)}\,dx,$$

it must be true that $I(t_1, t_2)$ takes on its relative minimum subject to the condition $\mathcal{J}(t_1, t_2) = 1$, when $t_1 = t_2 = 0$. If we now introduce the Lagrange multiplier λ, we obtain

$$I(t_1, t_2) + \lambda \mathcal{J}(t_1, t_2) = \int_0^1 \{F(x, \varphi + t_1\theta_1 + t_2\theta_2) + \lambda G(x, \varphi + t_1\theta_1 + t_2\theta_2)\}\, dx =$$

$$= \int_0^1 \{[\varphi(x) + t_1\theta_1(x) + t_2\theta_2(x)]^r + \lambda\varepsilon(x)\sqrt{1 + \varphi(x)\, t_1\theta_1(x) + t_2\theta_2(x)}\}\, dx.$$

The relevant necessary conditions are

$$\frac{\partial}{\partial t_1}[I(t_1, t_2) + \lambda\mathcal{J}(t_1, t_2)] = \frac{\partial}{\partial t_2}[I(t_1, t_2) + \lambda\mathcal{J}(t_1, t_2)] = 0$$

when $t_1 = t_2 = 0$. When the indicated operations are performed, we obtain

$$\int_0^1 \left[r\varphi^{r-1}(x) + \frac{\lambda\varepsilon(x)}{2\sqrt{1+\varphi(x)}}\right]\theta_1(x)\, dx = 0, \text{ except when } \varphi = -1,$$

and $$\int_0^1 \left[r\varphi^{r-1}(x) + \frac{\lambda\varepsilon(x)}{2\sqrt{1+\varphi(x)}}\right]\theta_2(x)\, dx = 0, \text{ except when } \varphi = -1.$$

Here, we note that if we let $H(x, \varphi) = F + \lambda G$, $\widetilde{F} = F(x, \varphi + t_1\theta_1 + t_2\theta_2)$, and $\widetilde{G} = G(x, \varphi + t_1\theta_1 + t_2\theta_2)$, then $\widetilde{H} = \widetilde{F} + \lambda\widetilde{G}$ so that $I(t_1, t_2) + \lambda\mathcal{J}(t_1, t_2) = \int_0^1 \widetilde{H}\, dx$.

We also see that

$$\frac{\partial\widetilde{H}}{\partial t_1} = \frac{\partial\widetilde{H}}{\partial\varphi}\theta_1$$

and

$$\frac{\partial\widetilde{H}}{\partial t_2} = \frac{\partial\widetilde{H}}{\partial\varphi}\theta_2.$$

Thus, we can write

$$\frac{\partial}{\partial t_1}[I(t_1, t_2) + \lambda\mathcal{J}(t_1, t_2)] = \int_0^1 \frac{\partial\widetilde{H}}{\partial\varphi}\theta_1\, dx$$

and

$$\frac{\partial}{\partial t_2}[I(t_1, t_2) + \lambda\mathcal{J}(t_1, t_2)] = \int_0^1 \frac{\partial\widetilde{H}}{\partial\varphi}\theta_2\, dx,$$

by differentiating under the integral sign. This differentiation is permissible except at $\varphi = -1$, since $\widetilde{H}$ as a function of φ is a continuously differentiable function. Finally, since $\widetilde{H} \to H$ when t_1 and $t_2 \to 0$ and the same is true of the partial derivatives, the necessary conditions take the form

$$\int_0^1 \frac{\partial H}{\partial\varphi}\theta_1\, dx = \int_0^1 \left[r\varphi^{r-1}(x) + \frac{\lambda\varepsilon(x)}{2\sqrt{1+\varphi(x)}}\theta_1(x)\right] dx = 0, \quad \text{except when } \varphi = -1,$$

and $$\int_0^1 \frac{\partial H}{\partial\varphi}\theta_2\, dx = \int_0^1 \left[r\varphi^{r-1}(x) + \frac{\lambda\varepsilon(x)}{2\sqrt{1+\varphi(x)}}\right]\theta_2(x)\, dx = 0 \text{ except when } \varphi = -1.$$

The function θ_2 can be chosen so that $\int_0^1 \frac{\varepsilon(x)\theta_2(x)}{2\sqrt{1+\varphi(x)}}\, dx \neq 0$, so that λ can be so determined that the second of these necessary conditions is satisfied. Since θ_1 is arbitrary on $(0, 1)$, its coefficient in the integrand of the first of these necessary conditions must vanish giving the Euler equation:

$$r\varphi^{r-1}(x) + \frac{\lambda\varepsilon(x)}{2\sqrt{1+\varphi(x)}} = 0 \text{ except when } \varphi = -1.$$

Thus, we must have $2r\varphi^{r-1}\sqrt{1+\varphi} = \varepsilon\lambda$ for each $x \in [0, 1]$ except when $\varphi = -1$.
Hence, the minimizing function φ must be a step function on $[0, 1]$.

Section 5. The case r odd, where r is greater than 1. The problem is to minimize

(16) $$\int_0^1 \varphi^r(x)\, dx, \; r \text{ fixed odd integer greater than } 1,$$

subject to the condition

$$\int_0^1 \varepsilon(x)\sqrt{1+\varphi(x)}\,dx = 1, \tag{17}$$

where $\varepsilon(x)$ is a prescribed step function assuming values ± 1. In Section 4, we found that for each x in $[0, 1]$, we must have

$$2r\varphi^{r-1}\sqrt{1+\varphi} = \varepsilon\lambda \quad \text{except when } \varphi = -1. \tag{18}$$

Thus, we have

$$\begin{aligned} &2r\varphi^{r-1}\sqrt{1+\varphi} = \lambda \quad &&\text{for } x \in P, \\ &2r\varphi^{r-1}\sqrt{1+\varphi} = -\lambda \quad &&\text{for } x \in N, \text{ except when } \varphi = -1. \end{aligned} \tag{19}$$

If $\lambda = 0$, then $\varphi = 0$ or $\varphi = -1$ on all of P but this is impossible since equation (17) could not be satisfied. Hence, since r is odd, we must have $\lambda > 0$. We also see that φ cannot be negative on all of P because again equation (17) could not be satisfied. Squaring equation (19) we obtain

$$\varphi^{2r-1} + \varphi^{2r-2} - \lambda_1^2 = 0, \quad \lambda_1 = \frac{\lambda}{2r}.$$

This equation has a single positive root and depending on λ possibly two negative roots between -1 and 0. Here, since r is odd, we need not be concerned with these negative roots since $\varphi = -1$ would give a smaller value to equation (16) than these two negative roots between -1 and 0.

Thus, we find from equation (19) that

$$\begin{cases} \varphi = -1 \text{ on } N, \\ \varphi = -1 \text{ on } P_1 \text{ where } P_1 \text{ is a subset of } P \text{ with measure } p_1, \\ \varphi > 0,\ \varphi^{r-1}\sqrt{1+\varphi} = \lambda > 0 \text{ on } P_2 = P - P_1. \end{cases} \tag{20}$$

Thus $\varphi = -1$ on $N + P_1$ with measure $n + p_1$ and $\varphi = x_1$ on P_2 with measure p_2, so that $\int_0^1 \varepsilon(x)\sqrt{1+\varphi(x)}\,dx = p_2\sqrt{1+\psi_1} = 1$ and hence $x_1 = 1/p_2^2 - 1$. Thus, we find that $p_2 = 1/\sqrt{1+x_1}$ and

$$\int_0^1 \varphi^r(x)\,dx = -p_1 - n + p_2 x_1^r = -1 + \frac{x_1^r + 4}{\sqrt{1+x_1}}, \tag{21}$$

where $x_1 = \dfrac{1}{p_2^2} - 1 \geqslant \dfrac{1}{p^2} - 1$, since $p_2 \leqslant p$.

Our problem has now been reduced to that of finding a value of x wich minimizes

$$f(x) = \frac{x^r + 1}{\sqrt{1+x}} \quad \text{for } x \geqslant \frac{1}{p^2} - 1. \tag{22}$$

Differentiating this last expression, we find

$$f'(x) = \frac{(2r-1)x^r + 2rx^{r-1} - 1}{2\sqrt{1+x}\,(1+x)}. \tag{23}$$

Since $f'(0) = -1/2$ and $f'(1) = (2r-1)/2\sqrt{2} > 0$, we see that the equation $\varphi^{2r-1} + \varphi^{2r-2} - \lambda_1^2 = 0$ has a unique zero X and that since $f'(x)$ becomes positive, it will remain positive. Thus for $x > X$, the function $f(x)$ will be increasing. Hence if $X \leqslant 1/p^2 - 1$, *the minimum of* $f(x)$ *falls at* $x = 1/p^2 - 1$, the end point; and if $X > 1/p^2 - 1$, *the minimum falls at* $x = X$. Thus, the minimum for the case r odd, $r > 1$, is

$$\begin{cases} \text{If } p^2 \leqslant \dfrac{1}{1+X}, & \text{minimum of } \displaystyle\int_0^1 \varphi^r(x)\,dx = p\left(\frac{1}{p^2} - 1\right)^r - n \\ \text{If } p^2 > \dfrac{1}{1+X}, & \text{minimum of } \displaystyle\int_0^1 \varphi^r(x)\,dx = -1 + \frac{X^r + 1}{\sqrt{1+X}}. \end{cases} \tag{24}$$

The last expressions for the minimum are obtained by substituting $x = 1/p^2 - 1$ in equation (21) and simplifying the result. The first expression above is also true for the case $r = 1$, although the method of deriving the result is not valid since $f'(x)$ in (23) becomes $(x+1)/[2\sqrt{1+x}\,(1+x)]$ which does not have a positive root.

Section 6. The case r even. For the case r even, we shall find it more convenient to return to the notation of Section 2. Our problem is to minimize

$$\int_P \eta^r(x)\,dx + \int_N \zeta^r(x)\,dx, \tag{25}$$

subject to the condition

$$\int_P \sqrt{1+\eta(x)}\,dx - \int_N \sqrt{1+\zeta(x)}\,dx = 1. \tag{26}$$

(a) *The Function* $\eta(x)$. Let k be an arbitrary but fixed constant greater than or equal to one; and let us first seek the function $\eta(x)$ which minimizes

$$\int_P \eta^r(x)\,dx, \tag{27}$$

subject to the condition

$$\int_P \sqrt{1+\eta(x)}\,dx = k \geqslant 1. \tag{28}$$

Applying the same lemma of Section 4, we find that the minimizing function $\eta(x)$ must be a step function and that

$$2r\,\eta^{r-1}(x)\sqrt{1+\eta(x)} = +\lambda \quad \text{except when } \eta(x) = -1, \tag{29}$$

where λ is a constant. At this point in the case r even we redefine the sets P and N. We now let P be the set where the first of these equations is satisfied and we still call its measure p. The set of points where $\eta = -1$ will be joined to the set N and from this point in the sequel we use the function ζ as the function defined on this set which we still name N with measure n so that $p + n = 1$. We note from equations (25) and (26) that the values of these two equations will not be changed by this change in the sets P and N. From equations (29), we see that $\eta(x)$ cannot change sign on the interval $0 \leqslant x \leqslant 1$, since λ is constant and r s even. Further, $\eta(x) \geqslant 0$ on P, for if $\eta(x) \leqslant 0$ on P then $\sqrt{1+\eta(x)} \leqslant 1$ and hence $\int_P \sqrt{1+\eta(x)}\,dx \leqslant p < 1$ and this contradicts (28). Squaring equation (29) we find that

$$\eta^{2r-1} + \eta^{2r-2} - \frac{\lambda^2}{4r^2} = 0. \tag{30}$$

This equation has only one positive or zero root. The case $\eta = 0$ is possible only when $\lambda = 0$ but $\eta = 0$ contradicts equation (28). Hence the unique root η of (30) is positive throughout the set P. Let η_1 be the positive root of (30) so that $\eta(x) = \eta_1$ on the set P with measure p. From equation (28)

$$\int_P \sqrt{1+\eta(x)}\,dx = p\sqrt{1+\eta_1} = k. \tag{31}$$

Solving for η_1 we obtain

$$\eta_1 = \frac{k^2}{p^2} - 1. \tag{32}$$

Thus the *minimum value* of $\int_P \eta^r(x)\,dx$ is given by

$$\int_P \eta^r(x)\,dx = p\left(\frac{k^2}{p^2} - 1\right)^r. \tag{33}$$

(b) *The function* $\zeta(x)$. We now seek the function $\zeta(x)$ which minimizes

$$\int_N \zeta^r(x)\,dx \tag{34}$$

subject to the condition

$$\int_N \sqrt{1+\zeta(x)}\,dx = k - 1. \tag{35}$$

Again, applying the lemma of Section 4, we find that the minimizing function $\zeta(x)$ is a step function and that

$$2r\zeta^{r-1}(x)\sqrt{1+\zeta(x)} = -\lambda, \quad \text{except when } \zeta = -1, \tag{36}$$

where λ is a constant. It is clear here also that $\zeta(x)$ cannot change sign on N. Squaring (32) we find

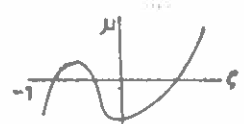
$$\zeta^{2r-1} + \zeta^{2r-2} - \mu^2 = 0, \tag{37}$$

where $\mu^2 = \lambda^2/4r^2$. Equation (37) has two negative roots if and only if

$$0 < \mu^2 < \frac{(2r-2)^{2r-2}}{(2r-1)^{2r-1}} \tag{38}$$

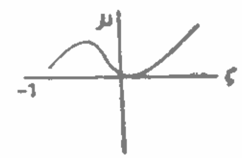
as will now be shown. The left member of (37) is less than or equal to zero at $\zeta = 0$ and $\zeta = -1$. Further, its derivates is given by

$$\zeta^{2r-3}[(2r-1)\zeta + 2r - 2] = 0. \tag{39}$$



$0 < \mu^2 < \dfrac{(2r-2)^{2r-2}}{(2r-1)^{2r-1}}$

Fig. 1

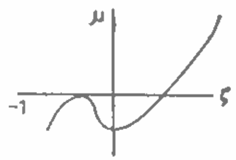


$\mu = 0$

Fig. 2

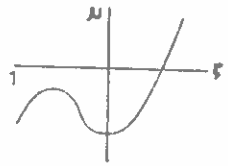
Hence at a certain critical point, we have $\zeta = 0$ or $\zeta = -(2r-2)/(2r-1)$. If at the critical point $-(2r-2)/(2r-1)$ equation (37) is positive and $\mu \neq 0$, then equation (37) will have two negative roots between 0 and -1.

Thus (37) has two negative roots if and only if the inequality (38) holds. The graph of this equation is given by fig. 1, 2 or 3. The case when there are no negative roots is illustrated by fig. 4.



$$\mu^2 = \frac{(2r-2)^{2r-2}}{(2r-1)^{2r-1}}$$

Fig. 3



$$\mu^2 > \frac{(2r-2)^{2r-2}}{(2r-1)^{2r-1}}$$

Fig. 4

If μ^2 is positive, then the equation (37) has a unique positive root. Thus all the possibilities may be divided into the following cases:

Case A. ζ is a positive, negative, or a zero constant throughout N.

Case B. There exist two negative constants ζ_1 and ζ_2 between 0 and -1 or one of the ζ_i is 0 while the other is -1 such that by (36) we have

$$\zeta_1^{r-1}\sqrt{1+\zeta_1} = \zeta_2^{r-1}\sqrt{1+\zeta_2},$$

and at each point of N, ζ is equal to one or the other of these constants.

Case A. ζ constant on N. Equation (35) now gives us

(40) $$\int_N \sqrt{1+\zeta(x)}\,dx = n\sqrt{1+\zeta} = k-1,$$

so that

(41) $$\zeta = -1+\theta^2, \text{ where } k = 1+\theta n, \quad 0 \leqslant \theta.$$

For the given fixed constant k, this ζ gives us the following minimum for our integral

(42) $$\int_N \zeta^r(x)\,dx = n(1-\theta^2)^r.$$

Collecting our results for the function η from (33) and ζ from (42) we have

(43) $$\int_0^1 \varphi^r(x)\,dx = p\left[\frac{(1+n\theta)^2}{p^2}-1\right]^r + n(1-\theta^2)^r.$$

Since r is even and $n>0$, it is readily seen that the value of this integral for $\theta>1$ is greater than the value for $=1$. Hence in Case A, we need consider only $0 \leqslant \theta \leqslant 1$. But in view of equation (41), we then know that ζ will be less than or equal to zero. Thus Case A may be regarded as a special instance of Case B, i.e. when $\zeta_1=\zeta_2$ and we shall so consider it in the future.

Case B. Let ζ_1 and ζ_2 be two constants such that $-1 \leqslant \zeta_1 < \zeta_2 \leqslant 0$ and

(44) $$\zeta_1^{r-1}\sqrt{1+\zeta_1} = \zeta_2^{r-1}\sqrt{1+\zeta_2}.$$

Let $\zeta_2/\zeta_1 = \sigma$ so that $0 \leqslant \sigma < 1$. Then from equation (44) we have

(45) $$1+\zeta_1 = \sigma^{2r-2}(1+\sigma\zeta_1).$$

Hence in terms of σ which may be regarded as a parameter, we have

(46) $$\begin{cases} \zeta_1 = -\dfrac{1-\sigma^{2(r-1)}}{1-\sigma^{2r-1}} & \text{on } N_1 \text{ with measure } n_1 \\ \zeta_2 = \sigma\zeta_1 = -\sigma\dfrac{1-\sigma^{2(r-1)}}{1-\sigma^{2r-1}} & \text{on } N_2 \text{ with measure } n_2 \text{ where } n_1+n_2=n. \end{cases}$$

We note that as $\sigma \to 1$, $\zeta_1 \to \zeta_2$ and they both have a fixed limit $-(2r-2)/(2r-1)$. Also, we have

(47) $$\begin{cases} \sqrt{1+\zeta_1} = \sigma^{r-1}\sqrt{\dfrac{1-\sigma}{1-\sigma^{2r-1}}}, \\ \sqrt{1+\zeta_2} = \sqrt{\dfrac{1-\sigma}{1-\sigma^{2r-1}}}. \end{cases}$$

To obtain a more useful form of $\int_0^1 \varphi^r(x)\,dx$, we let

(48) $$\begin{cases} N_1 \text{ be the subset of } N \text{ for which } \zeta=\zeta_1;\ n_1 = \text{measure of } N_1, \\ N_2 \text{ be the subset of } N \text{ for which } \zeta=\zeta_2;\ n_2 = \text{measure of } N_2 \\ \qquad \text{with } n_1+n_2=n. \end{cases}$$

Equation (35) now becomes

$$(49)\qquad k-1=\int_N \sqrt{1+\zeta}\,dx=\int_{N_1}\sqrt{1+\zeta_1}\,dx+\int_{N_2}\sqrt{1+\zeta_2}\,dx=$$
$$=n_1\sqrt{1+\zeta_1}+n_2\sqrt{1+\zeta_2}=\sqrt{1+\zeta_2}\,(n_1\sigma^{r-1}+n_2).$$

Using (47), (48) and (49) we see that

$$(50)\qquad \sigma^{r-1}n_1+n_2=(k-1)\sqrt{\frac{1-\sigma^{2r-1}}{1-\sigma}},\quad n_1+n_2=n.$$

From these equations we obtain

$$(51)\qquad \begin{cases} n_1=\dfrac{n-(k-1)\sqrt{\dfrac{1-\sigma^{2r-1}}{1-\sigma}}}{1-\sigma^{r-1}}, \\[2ex] n_2=\dfrac{-n\sigma^{r-1}+(k-1)\sqrt{\dfrac{1-\sigma^{2r-1}}{1-\sigma}}}{1-\sigma^{r-1}}. \end{cases}$$

Using the fact that $0\leqslant n_1\leqslant n$, we find from (51)

$$(52)\qquad \begin{cases} \sigma^{r-1}\sqrt{\dfrac{1-\sigma}{1-\sigma^{2r-1}}}\leqslant\theta\leqslant\sqrt{\dfrac{1-\sigma}{1-\sigma^{2r-1}}}, \\ \text{or equivalently} \\ \sqrt{1+\zeta_1}\leqslant\theta\leqslant\sqrt{1+\zeta_2}. \end{cases}$$

From equation (46) we see that

$$(53)\qquad \int_N \zeta^r(x)\,dx=n_1\zeta_1{}^r+n_2\zeta_2{}^r=\zeta_1{}^r(n_1+\sigma^r n_2)=$$
$$=\left(\frac{1-\sigma^{2(r-1)}}{1-\sigma^{2r-1}}\right)^r\left[\frac{n(1-\sigma^{2r-1})-(k-1)\sqrt{\dfrac{1-\sigma^{2r-1}}{1-\sigma}}\,(1-\sigma^r)}{1-\sigma^{r-1}}\right].$$

Let us rewrite equation (53) using $k=1+n$; we obtain

$$(54)\qquad \int_N \zeta^r(x)\,dx\equiv\psi(\sigma,\theta)=n\left(\frac{1-\sigma^{2(r-1)}}{1-\sigma^{2r-1}}\right)^r\left(\frac{1-\sigma^{2r-1}-\theta\sqrt{\dfrac{1-\sigma^{2r-1}}{1-\sigma}}\,(1-\sigma^r)}{1-\sigma^{r-1}}\right)^r.$$

From the inequality (52) we see that

$$(55)\qquad \sigma^{r-1}\leqslant\theta\sqrt{\frac{1-\sigma^{2r-1}}{1-\sigma}}\leqslant 1.$$

Let us define

$$(56)\qquad \begin{cases} \theta_1(\sigma)=\sigma^{r-1}\sqrt{\dfrac{1-\sigma}{1-\sigma^{2r-1}}}, \\[2ex] \theta_2(\sigma)=\sqrt{\dfrac{1-\sigma}{1-\sigma^{2r-1}}}, \end{cases}$$

so that from (55) we have

$$\theta_1(\sigma)\leqslant\theta\leqslant\theta_2(\sigma).$$

If we now substitute $\theta_1(\sigma)$ and $\theta_2(\sigma)$ from (56) into (54) we obtain

$$(57)\qquad \begin{cases} \psi(\sigma,\theta_1(\sigma))=n\left(\dfrac{1-\sigma^{2r-2}}{1-\sigma^{2r-1}}\right)=n\,(1-\theta_1^2(\sigma))^r, \\[2ex] \psi(\sigma,\theta_2(\sigma))=n\sigma^r\left(\dfrac{1-\sigma^{2r-2}}{1-\sigma^{2r-1}}\right)=n\,(1-\theta_2^2(\sigma))^r. \end{cases}$$

We now define s, t such that

$$(58)\qquad s\psi(\sigma,\theta_1)+t(\sigma,\theta_2)=\psi(\sigma,\theta),\quad s+t=1.$$

Solving for s and t, we obtain

$$(59)\qquad \begin{cases} s=\left[1-\theta\sqrt{\dfrac{1-\sigma^{2r-1}}{1-\sigma}}\right]:(1-\sigma^{r-1}), \\[2ex] t=\left[\theta\sqrt{\dfrac{1-\sigma^{2r-1}}{1-\sigma}}-\sigma^{r-1}\right]:(1-\sigma^{r-1}). \end{cases}$$

From $\theta_1(\sigma)\leqslant\theta\leqslant\theta_2(\sigma)$, it follows that $s\geqslant 0$ and $t\geqslant 0$ and since $s+t=1$, we see that s and t are each less than or equal to one. Hence $\psi(\sigma,\theta)$ lies between $\psi(\sigma,\theta_1(\sigma))$ and $\psi(\sigma,\theta_2(\sigma))$ for each value of σ in $0\leqslant\sigma\leqslant 1$. Thus in examining $\psi(\sigma,\theta)$ for a minimum, we need only examine the values of (σ,θ) on the boundary of the region of definition, which consists of the three parts

(a) $(\sigma,\theta_1(\sigma))$ (b) $(\sigma,\theta_2(\sigma))$ (c) $\sigma=0 \quad 0\leqslant\theta\leqslant 1$

shown in the figure 5.

Confining ourselves first to the boundaries (a) and (c) with $0\leqslant\theta\leqslant 1/\sqrt{2r-1}$, we see from (56) that we must consider

$$
(60)\qquad \begin{cases} \sigma = \sigma_1 \quad \theta = \sigma_1^{r-1}\sqrt{\dfrac{1-\sigma_1}{1-\sigma_1^{2r-1}}} & \text{boundary (a)} \\[2ex] \sigma = 0, \quad 0 \leqslant \theta \leqslant \dfrac{1}{\sqrt{2r-1}} & \text{boundary (c)} \end{cases}
$$

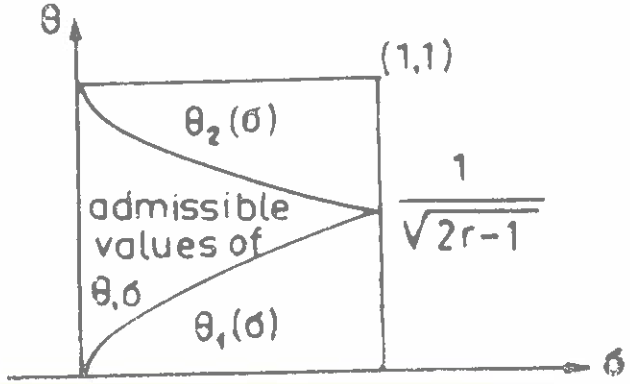


Fig. 5

For the value of $\sigma = \sigma_1$ we have

$$
(61)\qquad \frac{1-\sigma_1^{2r-2}}{1-\sigma_1^{2r-1}} = 1-\theta^2.
$$

Thus it follows that for θ in $0 \leqslant \theta \leqslant 1$, in examining $\psi(\sigma,\theta)$ for a minimum, we must compare

$$
(62)\qquad \psi(0,\theta) = n(1-\theta) \text{ with } \psi(\sigma_1,\theta) = n(1-\theta^2)^r.
$$

Finally proceeding to the other boundaries (b) and (c) with $1/\sqrt{2r-1} \leqslant \theta \leqslant 1$, we must consider

$$
(63)\qquad \begin{cases} \sigma = \sigma_2, \quad \theta = \sqrt{\dfrac{1-\sigma_2}{1-\sigma_2^{2r-1}}} & \text{boundary (b)} \\[2ex] \sigma = 0, \quad \dfrac{1}{\sqrt{2r-1}} \leqslant \theta \leqslant 1, & \text{boundary (c)} \end{cases}
$$

For $\sigma = \sigma_2$, we have

$$
(64)\qquad \sigma_2\frac{1-\sigma_2^{2r-2}}{1-\sigma_2^{2r-1}} = 1-\theta^2.
$$

Thus it follows that for θ in $1/\sqrt{2r-1} \leqslant \theta \leqslant 1$ in examining the minimum of $\psi(\sigma,\theta)$ we must compare

$$
(65)\qquad \psi(0,\theta) = n(1-\theta) \text{ with } \psi(\sigma_2,\theta) = n(1-\theta^2)^r.
$$

Combining the results of (62) and (65), we see that for each value of θ in $0 \leqslant \theta \leqslant 1$, we must compare

$$
(66)\qquad n(1-\theta) \text{ with } n(1-\theta^2)^r, \quad 0 \leqslant \theta \leqslant 1.
$$

If $\sigma = 0$, we see from (46) that $\xi_1 = -1$ on N_1 and $\zeta_2 = 0$ on N_2. From (51), we see that $n_1 = n(1-\theta)$ and $n_2 = n\theta$ using the fact that $k-1 = n\theta$. Thus from (53) we see that if $\sigma = 0$ we have

$$
(67)\qquad \begin{gathered} \int_N \zeta^r(x)\,dx = n_1\zeta_1^r + n_2\zeta_2^r = n_1 = n(1-\theta) \\ \text{and} \\ \int_N \sqrt{1+\zeta(x)}\,dx = \int_{N_2} dx = n_2 = n\theta = k-1. \end{gathered}
$$

If, however, $\sigma = \sigma_1$ or $\sigma = \sigma_2$ then we obtain

First, let $\sigma = \sigma_2$ then we have from (60) that

$$
\theta = \sigma_1^{r-1}\sqrt{\frac{1-\sigma_1}{1-\sigma_1^{2r-1}}}.
$$

Using this value of θ in (51) we see that $n_1 = n$ and $n_2 = 0$. From equation (46) we see that $\zeta_1 = -1+\theta^2$ on N_1. We need not consider ζ_2 since N_2 is the empty set.

Secondly, let $\sigma = \sigma_2$, then we have from (63) that

$$
\theta = \frac{1-\sigma_2}{1-\sigma_2^{2r-1}}.
$$

Using this value of θ in (51), we see that $n_1 = 0$ and $n_2 = n$. From (46), we see that $\zeta_2 = -1+\theta^2$ on N_2. We need not consider ξ_1 since N_1 is empty. Again using (53), we see that if $\sigma = \sigma_1$ or $\sigma = \sigma_2$ we obtain:

$$
(68)\qquad \begin{gathered} \int_N \zeta^r(x)\,dx = n(1-\theta^2)^r, \\ \text{and} \\ \int_N \sqrt{1+\zeta(x)}\,dx = n\theta = k-1, \end{gathered}
$$

respectively.

Section 7. The integral $\int_0^1 \varphi^r(x)\,dx$. Combining equation (33) with equation (67) and calling it Case I, we get:

Case I

$$\int_0^1 \varphi^r(x)\,dx = p\left|\frac{(1+n\theta)^2}{p^2}-1\right|^r + n(1-\theta) \quad 0\leqslant\theta\leqslant 1, \tag{69}$$

and combining equation (33) with (68) and calling it Case II, we get:

Case II

$$\int_0^1 \varphi^r(x)\,dx = p\left|\frac{(1+n\theta)^2}{p^2}-1\right|^r + n(1-\theta^2)^r \quad 0\leqslant\theta\leqslant 1. \tag{70}$$

For a prescribed p, our problem reduces to determining the minima of Case I and Case II as functions of θ where $0\leqslant\theta\leqslant 1$ and then taking as the minimum of $\int_0^1 \varphi^r(x)\,dx$ subject to $\int_0^1 \varepsilon(x)\sqrt{1+\varphi(x)}\,dx = 1$, r even, whichever of the two gives the smaller value. We note here, also, that equation (70) is the same as equation (43) so that in discussing Case B we are also discussing Case A at the same time.

Section 8. Discussion of Case I. We define τ by

$$\tau = \frac{k}{p} = \frac{1+n\theta}{p}. \tag{71}$$

Since θ lies in $0\leqslant\theta\leqslant 1$, we see that $1<\frac{1}{p}\leqslant\tau\leqslant\frac{1+n}{p}$. Rewriting (69) in terms of τ, we find

$$\int_0^1 \varphi^r(x)\,dx = 1+n+p[(\tau^2-1)^r-\tau]. \tag{72}$$

We now differentiate with respect to θ and obtain

$$\frac{d\left(\int_0^1 \varphi^r(x)\,dx\right)}{d\theta} = n[2r\tau(\tau^2-1)^{r-1}-1], \tag{73}$$

where we have used the fact that $d\tau/d\theta = n/p$. We define function $f(\tau)$ by

$$f(\tau) = 2r\tau(\tau^2-1)^{r-1}-1, \tag{74}$$

so that $d\left(\int_0^1 \varphi^r(x)\,dx\right)/d\theta = nf(\tau)$. We now show that $f(\tau)$ is an increasing function of τ for τ in $1/p\leqslant\tau\leqslant(1+n)/p$ and that $f(\tau)$ has a unique zero between $\tau=1$ and $\tau=\sqrt{2}$. We find that

$$f'(\tau) = 2r[(2r-1)\tau^2-1](\tau^2-1)^{r-2}, \tag{75}$$

which is clearly positive for $\tau>1$. We see that $f(1)=-1$ and $f(\sqrt{2}) = 2r\sqrt{2}-1>0$; hence $f(\tau)$ has a unique zero between $\tau=1$ and $\tau=\sqrt{2}$. Let X be this root of $f(\tau)=0$ so that we have

$$\begin{aligned} &f(X) = 2rX(X^2-1)^{r-1}-1=0, \quad 1<X<\sqrt{2},\\ &f(\tau)>0 \text{ for } \tau>X,\\ &f(\tau)<0 \text{ for } \tau<X. \end{aligned} \tag{76}$$

We now consider the three possibilities for X, viz.

$$\begin{aligned} &\text{Case I}_a \quad X<\frac{1}{p} \text{ or equivalently } p<\frac{1}{X},\\ &\text{Case I}_b \quad \frac{1}{p}\leqslant X\leqslant\frac{1+n}{p} \text{ or equivalently } \frac{1}{X}\leqslant p\leqslant\frac{2}{1+X},\\ &\text{Case I}_c \quad X\geqslant\frac{1+n}{p} \text{ or equivalently } p>\frac{2}{1+X}. \end{aligned} \tag{77}$$

Case I_a. In this instance we show that the minimum of (69) occurs at $\theta=0$. By assumption we have $X<1/p\leqslant\tau$ which from (76) gives $f(\tau)>0$. From (75), we see that $f(\tau)$ is increasing and since $d\left(\int_0^1 \varphi^r(x)\,dx\right)/d\theta = nf(\tau)$, we have $d\left(\int_0^1 \varphi^r(x)\,dx\right)/d\theta>0$. Thus the minimum must occur at $\tau=1/p$ which corresponds to $\theta=0$ and the value of the minimum of (69) is

$$\int_0^1 \varphi^r(x)\,dx\,\Big|_{\theta=0} = f_1 = p\left(\frac{1-p^2}{p^2}\right)^r + n, \tag{78}$$

where we have used f_1 to denote this value.

Case I_b. Here the minimum occurs at $\tau = X$, since we have $1/p \leqslant X \leqslant (1+n)p$ and at X, $d\left(\int_0^1 \varphi^r(x)\,dx\right)/d\theta = 0$. At $\tau = X$, we have $\theta = (pX-1)/n$.

The value of the minimum for this case is by (69)

$$f_2 = 2 - \alpha p, \tag{79}$$

where

$$\alpha = \frac{(X+1)\,[(2r-1)\,X+1]}{2rX} \tag{80}$$

and X is the unique zero of $f(\tau) = 2r\tau(\tau^2-1)^{r-1} - 1 = 0$ for $1/p \leqslant \tau \leqslant (1+n)/p$.

Case I_c. Here we show that the minimum occurs at $\theta = 1$. By assumption, we have $\tau \leqslant (1+n)/p < X$. Since τ is less than X, we have $f(\tau)$ negative. Using this fact and also that $f(\tau)$ is increasing, we see that the minimum must occur at $\tau = (1+n)/p$, which corresponds to $\theta = 1$. The value of the minimum is in this case given by

$$f_3 = \frac{4^r n^r}{p^{2r-1}}. \tag{81}$$

Section 9. Discussion of Case II. We define w by

$$w = \frac{n(1+p+n\theta)}{p^2(1-\theta)} \tag{82}$$

to simplify the discussion. Rewriting (70), we obtain

$$\int_0^1 \varphi^r(x)\,dx = (1-\theta^2)^r\,(pw^r + n). \tag{83}$$

We now differentiate with respect to θ and obtain

$$\frac{d}{d\theta}\left(\int_0^1 \varphi^r(x)\,dx\right) = 2rn\,(1+n\theta)\,(1-\theta^2)^{r-1}\left[w^{r-1} - \frac{p\theta}{1+n\theta}\right], \tag{84}$$

where we have used the fact that $dw/d\theta = 2n/[p^2(1-\theta)^2]$. To find the zeros of the right member of (84) we note that

$$\left.\frac{d\left(\int_0^1 \varphi^r(x)\,dx\right)}{d\theta}\right|_{\theta=0} = \frac{2rn}{p}\left(\frac{1-p^2}{p^2}\right)^{r-1} \quad \text{is positive}$$

and at $\theta = 1$ we get $2^r rn\,[(1+n)\,(2n/p^2)^{r-1}]/p$ which is also positive. Therefore, if $d\left(\int_0^1 \varphi^r(x)\,dx\right)/d\theta$ is to vanish, we must have

$$w^{r-1} = \frac{p\theta}{1+n\theta}, \tag{85}$$

where

$$\theta = \frac{wp^2 - n - np}{wp^2 + n^2} \tag{86}$$

is the solution of (82). Writing (85) in terms of w, we obtain

$$(1-n^2)\,w^r - nw^{r-1} - p^2 w + 1 - p^2 = 0. \tag{87}$$

If there are no real roots for this equation, the minimum of (70) Case II occurs at $\theta = 0$ since the derivative would be positive and the function would be increasing. Thus, the minimum must occur at the left endpoint which is $\theta = 0$. It would follow from (70) and (78) that this value would be

$$f_1 = p\left(\frac{1-p^2}{p^2}\right)^r + n.$$

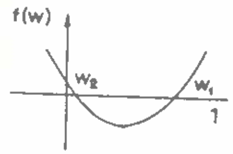
If the equation (87) has real roots, we will show that there are exactly two positive roots and no negative roots. If the two roots are distinct, then the larger of the two roots is the value where the relative minimum of Case II occurs. The minimum of (70), Case II would be then the smaller of the two values: the value at $\theta = 0$ and the value of the relative minimum. If the two roots are not distinct, then the minimum of Case II occurs at $\theta = 0$ since the derivative is positive everywhere except at the one root of multiplicity two.

By Descartes' Rule of Signs, we see that the equation (87) will have exactly two positive roots if it has any at all and no negative roots. If we use equation (87) to define $f(w)$ by

$$f(w) = (1-n^2)\,w^r - nw^{r-1} - p^2 w + 1 - p^2, \tag{88}$$

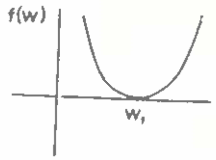
we have $f(0) = 1 - p^2 > 0$ and $f(1) = 4np > 0$. We will show later in the sequel, page 70, that we need consider only $f(w)$ for w between 0 and 1. Suppose that $f(w)$ has two roots on the interval $0 < w < 1$: The graph of $f(w)$ becomes fig. 6 and fig. 7.

Let the zeros of $f(w)$ be w_1 and w_2 with $w_1 > w_2$. The fact that w_1 gives the value of w where the relative minimum occurs is seen by observing that for $0 < w \leq w_2$, the integral in equation (83) is increasing and, for $w_2 \leq w \leq w_1$ this integral decreases and for $w_1 \leq w \leq 1$, this integral is again increasing so that at $w = w_1$ we have the point of re-



$w_2 < w_1$

Fig. 6



$w_2 = w_1$

Fig. 7

lative minimum. If $w_1 = w_2$ then w_1 is not a point of relative minimum since the derivative is positive on $0 \leq w \leq 1$ except at $w = w_1$. Hence the minimum in this case would occur at $0 = 0$ which is f_1. We now compute the relative minimum when $w_1 > w_2$. The two roots w_1, w_2 correspond to θ_1 and θ_2. The value of the relative minimum of (70) (Case II) is given by

(89) $$f_4 = \frac{n(1-\theta_1^2)^{r-1}(1+\theta_1)^2}{1+n\theta_1}.$$

where $\theta_1 = [p^2 w_1^2 - n - np]/(p^2 w_1 + n^2)$.

Thus if $f(w)$ has two roots w_1 and w_2 with $w_1 > w_2$, the minimum of (70) (Case II) is given by the smaller of the two values f_1 and f_4 as given in equation (87) and (89).

Our problem then for Case II reduces to determining for fixed r the range of p for which the function $f(w)$ as given by equation (88) has real zeros.

We can rewrite (87) in the form

(90) $$w^{r-1} = \frac{wp^2 + p^2 - 1}{wp(2-p) - (1-p)^2}.$$

We now examine this equation by considering the graphs of the two functions of w for $w \geq 0$ given by

(91) $$y = w^{r-1} \text{ and } y = \frac{wp^2 + p^2 - 1}{wp(2-p) - (1-p)^2}$$

to determine their intersection points. The sketch of the graph of $y = w^{r-1}$ is given by

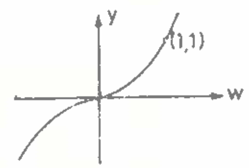


Fig. 8

The sketch of the graph of $y = (wp^2 + p^2 - 1)/[wp(2-p) - (1-p)^2]$ is shown by

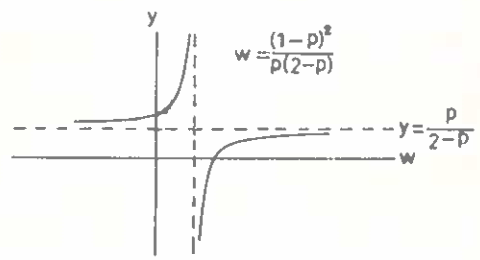


Fig. 9

The graph of $y = (wp^2 + p^2 - 1)/[wp(2-p) - (1-p)^2]$ is a rectangular hyperbola with horizontal asymptote given by

(92) $$y = \frac{p}{2-p}$$

and vertical asymptote given by

(93) $$w = \frac{(1-p)^2}{p(2-p)}$$

Since p lies in the range $0 < p < 1$, we see that the horizontal asymptote is between the graphs of $y = 0$ and $y = 1$, and the vertical asymptote is between $w = 0$ and $w = \infty$.

If we sketch both graphs on the same set of coordinate axes, we obtain

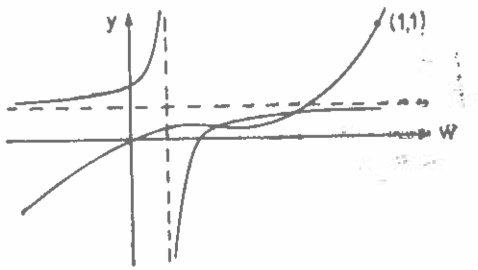


Fig. 10

which gives us a clearer view of the possibility of having two roots of equation (88).

We now show that we need not consider the values of w for which $y = w^{r-1}$ intersects the upper part of the hyperbola. Since θ must lie in the range $0 \leqslant \theta \leqslant 1$, we see from equation (86) that

$$w \geqslant \frac{1-p^2}{p^2}. \tag{94}$$

If $y = w^{r-1}$ is to intersect the upper part of the hyperbola, we see that the value of w at the point of intersection must lie to the left of the vertical asymptote, i.e.

$$w < \frac{(1-p)^2}{p(2-p)}. \tag{95}$$

Thus if θ is to be on the range $0 \leqslant \theta \leqslant 1$ and if $y = w^{r-1}$ is to intersect the upper part of the hyperbola, we must have

$$\frac{1-p^2}{p^2} < \frac{(1-p)^2}{p(2-p)} \tag{96}$$

equivalent to $p(1-p) < 0$, but this leads to a contradiction since $0 < p < 1$. Hence, we need consider only points of intersection of $y = w^{r-1}$ with the lower part of the hyperbola. We first note that if the

curve of $y = w^{r-1}$ intersects the lower part of the hyperbola, the value of w at the point of intersection must be less than one. This is seen by observing that the lower part of the hyperbola is bounded above by $y = 1$, since the horizontal asymptote $y = p/(2-p)$ is never above the line $y = 1$ and by the fact that the curve $y = w^{r-1}$ passes through the point (1, 1) and lies to the left of $w = 1$ for $w < 1$. We see thus that w at a point of intersection must lie on the interval $0 < w < 1$.

We can be assured that the lower part of the hyperbola intersects $y = w^{r-1}$ for r fixed if and only if the ordinate of the hyperbola is greater than or equal to the value of $y = w^{r-1}$ at the abscissa of the vertex. The coordinates of the lower part of the right hand vertex of the hyperbola are

$$\left(\frac{(1-p)^2 + \sqrt{2p(1-p)}}{p(2-p)},\ \frac{p^2 - \sqrt{2p(1-p)}}{p(2-p)}\right). \tag{97}$$

Analytically, we have a necessary and sufficient condition for the intersection of the two functions in (91)

$$\frac{p^2 - \sqrt{2p(1-p)}}{p(2-p)} \geqslant \left|\frac{(1-p)^2 + \sqrt{2p(1-p)}}{p(2-p)}\right|^{r-1}. \tag{98}$$

The abscissa of the vertex must be less than one, i.e.

$$\frac{(1-p)^2 + \sqrt{2p(1-p)}}{p(2-p)} < 1. \tag{99}$$

This is equivalent to $4p^4 - 16p^3 + 22p^2 - 10p + 1 > 0$ which is valid for $61/100 < p < 1$ approximately. We also must have the ordinate of the vertex greater than zero, i.e.

$$\frac{p^2 - \sqrt{2p(1-p)}}{p(2-p)} > 0. \tag{100}$$

This is equivalent to

$$p^3 + 2p - 2 > 0, \tag{101}$$

which is valid for $77/100 < p < 1$ approximately. Hence we see that (98) and (99) will simultaneously be true for $p^3 + 2p - 2 > 0$.

For a given p, we note that, as a function of r, the expression $\{[(1-p)^2+\sqrt{2p(1-p)}]/[p(2-p)]\}^{r-1}$ is decreasing. Hence if for a given p the inequality (97) is satisfied for $r=r_0$, it will be satisfied for all $r>r_0$.

In summary, we find that the range of p, for a given r, for which the function $f(w)$ given by (88) has zeros is the range of p for which both the inequalities (97) and (100) simultaneously are satisfied. The minimum of $\int_0^1 \varphi^r(x)\,dx$ for the *case r even* is given by the smaller of the minima obtained in Cases I and II.

Section 10. Summary. A summary of the preceeding discussion is very effectively brought out by the diagrams that follow: Let us consider the minimum of $\int_0^1 \varphi^r(x)\,dx$ in Case I, viz.

$$\int_0^1 \varphi^r(x)\,dx = p\left[\frac{(1+n\theta)^2}{p^2}-1\right]^r + n(1-\theta). \quad 0\leqslant\theta\leqslant 1.$$

Here we found that the minimum is given by f_1, f_2, f_3 according as p lies on different parts of $0<p<1$ which are mutually exclusive. We found that

$$f_1 = n + \left[\frac{1-p^2}{p^2}\right]^r \text{ for } p<\frac{1}{X_r},$$

where X_r is the unique zero of $f(\tau)=2r\tau(\tau^2-1)^{r-1}-1$.

$$f_2 = 2-\alpha p \text{ for } \frac{1}{X_r}\leqslant p\leqslant\frac{2}{1+X_r},$$

where $\alpha=\{(X_r+1)[(2r-1)X_r+1]\}/2rX_r)$

$$f_3=\frac{4^r n^r}{p^{2r-1}} \text{ for } p>\frac{2}{1+X_r}.$$

These are illustrated by fig. 11.

Let us now consider the minimum of $\int_0^1 \varphi^r(x)\,dx$ in Case II, viz.

$$\int_0^1 \varphi^r(x)\,dx = p\left[\frac{(1+n\theta)^2}{p^2}\right]^r + n(1-\theta^2)^r \quad 0\leqslant\theta\leqslant 1.$$

In Case II, we found that we must compare f_1 with f_4, given by

$$f_4=\frac{n(1-\theta_1^2)^{r-1}(1+\theta_1)^2}{1+n\theta},$$

where

$$\theta_1=\frac{p^2w_1-n(1+p)}{p^2w_1+n^2},$$

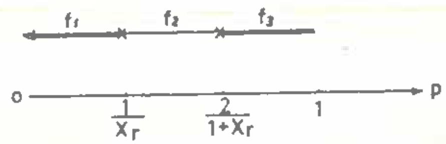


Fig. 11

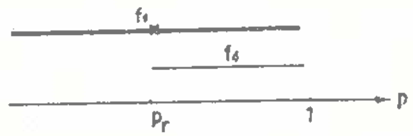


Fig. 12

and where w_1 is the larger of the two positive roots of the equation $(1-n^2)w^r-nw^{r-1}-p^2w+1-p^2=0$. To insure that there will be two positive roots w_1 and w_2 with $w_1>w_2$ for which $0\leqslant\theta_1\leqslant 1$, the prescribed p must satisfy the following two inequalities

$$p^3+2p-2>0$$

and

$$\frac{p^2-\sqrt{2p(1-p)}}{p(2-p)} > \left[\frac{(1-p)^2+\sqrt{2p(1-p)}}{p(2-p)}\right]^{r-1}$$

The range of p for which the above inequalities are valid increases as r increases; for example the range of p for which the inequalities are valid for $r=12$ includes the range for which the inequalities are valid for $r=10$.

Diagramatically, we may represent the functions f_1 and f_4 as in fig. 12. where p_r is the left endpoint of the interval for the range for which the above inequalities are valid for some fixed r.

Finally the absolute minimum is found by comparing the minimum found in Diagram 1 (Case I) with the minimum found in Diagram 2 (Case II).

Section 11. The case $r=2$. We now consider the special case $r=2$. Cases I and II for $r=2$ reduce to

$$\text{Case I}_2 \quad \int_0^1 \varphi^2(x)\,dx = p^2\left[\frac{(1+n\theta)^2}{p^2}-1\right]^2 + n(1-\theta) \quad 0\leqslant\theta\leqslant1,$$

$$\text{Case II}_2 \quad \int_0^1 \varphi^2(x)\,dx = p^2\left[\frac{(1+n\theta)^2}{p^2}-1\right]^2 + n(1-\theta^2)^2 \quad 0\leqslant\theta\leqslant1.$$

(a) *Discussion of Case* I_2. The zero of the function $f(\tau)=4\tau^3-4\tau-1$ is $X_2=1.10716$ approximately. By our discussion of Case I for all r, the minimum of Case I_2 is given by

$$f_1 = n + p\left(\frac{1-p^2}{p^2}\right)^2 \quad \text{for } p<\frac{1}{X_2}\approx .90321182$$

($\approx$ stands for approximately)

$$f_2 = 2-\alpha p \quad \text{for } \frac{1}{X_2}\leqslant p\leqslant\frac{2}{1+X_2}\approx .94914482$$

where $\alpha = \dfrac{(X_2+1)(3X_2+1)}{4}\approx 2.05612$

$$f_3 = \frac{16(1-p)^2}{p^3} \quad \text{for } p>\frac{2}{1+X_2}.$$

(b) *Discussion of Case* II_2. The inequality (97) for $r=2$ reduces to

$$1-12p+12p^2\geqslant0,$$

which is valid for $p\leqslant\dfrac{3-\sqrt{6}}{2}$ and for $p\geqslant\dfrac{3+\sqrt{6}}{2}$. However the inequality (100), viz.

$$p^3+2p-2>0$$

is not satisfied for $p\leqslant(3-\sqrt{6})/2$, while both of these inequalities are valid for $p\geqslant(3+\sqrt{6})/2$. Hence by our discussion of Case II for all r, we see that the minimum for Case II_2 is given by

$$f_1 = n+p\left(\frac{1-p^2}{p^2}\right)^2 \quad \text{for } p\leqslant\frac{3+\sqrt{6}}{2}\approx .90825.$$

If, however, $p>(3+\sqrt{6})/2$ we compare the value of f_1 with that of f_4 given by

$$f_4 = \frac{n(1-\theta_1^2)(1+\theta_1)^2}{1+n\theta_1},$$

where $\theta_1=[p^2w_1-n(1+p)]/(p^2w_1+n^2)$ and where w_1 is the larger of the two positive roots of the equation

$$(1-n^2)w^2-nw+p^2w+1-p^2=0.$$

The absolute minimum of $\int_0^1\varphi^2(x)\,dx$ subject to the condition $\int_0^1\varepsilon(x)\sqrt{1+\varphi(x)}\,dx=1$, where $\varepsilon(x)$ is a step function assuming the value $+1$ on the set P with measure p and -1 on the set N with measure n where $n+p=1$, is the smaller of the minima obtained in the Case I_2 and Case II_2.

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APPENDIX

TABLE I (*r* odd)

r	X	$\frac{1}{1+X}$
3	.35825800	.73623715
5	.51154400	.66157518

TABLE II (*r* odd)

r	p	$p\left(\frac{1}{p^2}-1\right)^r-n$	$-1+\frac{X^r+1}{\sqrt{1+X}}$
3	.1	.97028946 E + 05	−.10250290
3	.2	.27639996 E + 04	
3	.3	.30941147 E + 03	
3	.4	.57281236 E + 02	
3	.5	.12999996 E + 02	
3	.6	.29711929 E + 01	
3	.7	.48926030 E + 00	
3	.8	.57617200 E − 01	
3	.9	.88384300 E − 01	
5	.1	.95098980 E + 09	−.13813660
5	.2	.15925237 E + 07	
5	.3	.31703396 E + 05	
5	.4	.15947513 E + 04	
5	.5	.12099994 E + 03	
5	.6	.10254633 E + 02	
5	.7	.55500450 E + 00	
5	.8	−.15494920 E + 00	
5	.9	−.993360900 E − 01	

TABLE III (*r* even)

r	X	$\frac{1}{X}$	$\frac{4}{1+X}$
2	1.107160	.90321182	.94914482
4	1.212006	.82507842	.90415667

TABLE IV (*r* even)

r	p	w	θ_1	$n+p\left(\frac{1-p^2}{p^2}\right)^r$	$2-\alpha p$	$\frac{4^n n^r}{p^2 r-1}$	$\frac{n(1-\theta_1^2)^{r-1}(1+\theta_1)^2}{1+n\theta_1}$
2	.91	.48762400	.56300217	.12921280 E + 00	.12888270	.17198115	.14293353
2	.92	.60112700	.68943713	.11029840 E + 00	.10832090	.13150325	.11354013
2	.93	.67207300	.76116380	.92691500 E − 01	.08775920	.09746917	.08670677
2	.94	.73048300	.81511564	.76312600 E − 01	.06719750	.06934880	.06324516
2	.95	.78232800	.85886687	.61087700 E − 01	.04663570	.04665403	.04345933
2	.96	.83007900	.89564320	.46947400 E − 01	.02607400	.02893518	.02745136
2	.97	.87504800	.92720494	.33827100 E − 01	.00551230	.01577783	.01520867
2	.98	.91803100	.95465240	.21666200 E − 01	.01504940	.00679988	.00664628
2	.99	.95955700	.97873610	.10408200 E − 01	.03561120	.00164897	.00163146
4	.84	.61145000	.33083384	.18545640 E + 00	.18254250	.56854501	.19008163
4	.85	.70214600	.43374894	.16849280 E + 00	.16090600	.40427092	.15492050
4	.86	.74104500	.50676152	.15321530 E + 00	.13926970	.28265907	.12182985
4	.87	.77181200	.56744850	.13925770 E + 00	.11763320	.19381108	.09270702
4	.88	.79800700	.62047929	.12633830 E + 00	.09599690	.12989315	.06822066
4	.89	.82120100	.66796125	.11422370 E + 00	.07436040	.08473847	.04842800
4	.90	.84225200	.71107620	.10272470 E + 00	.05272400	.05352323	.03302659
4	.91	.86168600	.75057571	.91689700 E − 01	.03108770	.03250276	.02150673
4	.92	.87985700	.78698212	.80997900 E − 01	.00945130	.01879685	.01325632
4	.93	.89701400	.82068004	.70553700 E − 01	.01218510	.01021531	.00763648
4	.94	.91334100	.85196664	.60283100 E − 01	.03382150	.00511623	.00403408
4	.95	.92897800	.88108012	.50129500 E − 01	.05545790	.00229115	.00189691
4	.96	.94403400	.90821703	.40050300 E − 01	.07709420	.00087213	.00075507
4	.97	.95859900	.93354336	.30015100 E − 01	.09873070	.00025663	.00023147
4	.98	.97274300	.95720189	.20002900 E − 01	.12036700	.00004718	.00004418
4	.99	.98652600	.97931732	.10000200 E − 01	.14200350	.00000274	.00000266

O PROBLEMĂ EXTREMALĂ ASOCIATĂ UNEI ANUMITE ECUAŢII INTEGRALE NELINIARE

(Rezumat)

În [1] este studiată următoarea ecuaţie integrală

$$y^2(x) = [y(x) - R(x)]\int_0^1 y(s)\,ds.$$

Se arată că soluţia $y(x)$ depinde de ecuaţia

$$\int_0^1 \varepsilon(x)\sqrt{1 - cK(x)}\,dx = 1$$

în necunoscuta c pentru $K(x)$ dat. Aici $\varepsilon(x)$ este funcţie salt prescrisă luînd valorile ± 1. Obiectul acestei lucrări este de a determina intervalul căruia trebuie să aparţină c, astfel ca funcţia $K(x)$ să varieze într-o anumită clasă. Aceasta dă naştere la următoarea problemă. Să se găsească maximum c pentru care

$$\int_0^1 \varepsilon(x)\sqrt{1 - cK(x)}\,dx = 1 \quad \text{şi} \quad \int_0^1 K^r(x)\,dx = 1,$$

unde r este un întreg pozitiv fixat şi $\varepsilon(x)$ este o funcţie salt prescrisă luînd valorile ± 1. Pentru cazul $r = 1$, se găseşte că cel mai mare c este dat de expresia $-(1-m)/(1+m)$, $m = \int_0^1 \varepsilon(x)\,dx$.

Dacă se ia $\varphi(x) = -cK(x)$, problema revine la a minimiza $\int_0^1 \varphi^r(x)\,dx$, supusă condiţiei $\int_0^1 \varepsilon(x)\sqrt{1 + \varphi(x)}\,dx = 1$. Folosind regula de multiplicare Euler-Lagrange, se arată că funcţia minimizantă trebuie să fie funcţie salt.

Pentru cazul r impar, $r > 1$, minimum $\int_0^1 \varphi^r(x)\,dx$ este $p\left(\frac{1}{p^2} - 1\right)^r - n$, dacă $p^2 \leqslant \frac{1}{1+X}$ şi $-1 + \frac{X^r + 1}{1+X}$ dacă $p^2 > \frac{1}{1+X}$, unde p este măsura

mulţimii pentru care $\varepsilon(x) = -1$ şi X este soluţia unică a ecuaţiei $(2r-1)x^r + 2rx^{r-1} = 0$.

Pentru cazul r par, se obţin rezultate analoge cu cele precedente, dar problema şi rezultatele sînt mult mai complicate. Aici problema se reduce la a determina minimul expresiei

$$\text{Cazul I.} \quad \int_0^1 \varphi^r(x)\,dx = p\left|\frac{(1+n\theta)^2}{p^2} - 1\right|^r + n(1-\theta), \quad 0 \leqslant \theta \leqslant 1$$

$$\text{Cazul II.} \quad \int_0^1 \varphi^r(x)\,dx = p\left[\frac{(1+n\theta)^2}{p^2} - 1\right]^2 + n(1-\theta^2)^r, \quad 0 \leqslant \theta \leqslant 1$$

ca funcţie de θ pentru $0 \leqslant \theta \leqslant 1$ şi apoi la a lua drept minim pentru $\int_0^1 \varphi^r(x)\,dx$ supusă la $\int_0^1 \varepsilon(x)\sqrt{1 + \varphi(x)}\,dx = 1$, r par, cea mai mică dintre cele două valori. În secţiunea 10, cazul r par este dus pînă la capăt folosind diagrame. O parte din calcule apar sub formă tabelară la sfîrşitul articolului.