MEAN CURVATURE OF IMMERSED SURFACES

BY

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1. Introduction. Let X be a closed orientable surface, differentiable of class C^∞, and let $f: X \rightarrow E^3$ be a C^∞ — immersion of X into euclidean space of three dimensions. We denote by $\tau_1(X, f)$ the real number given by

$$\tau_1(X, f) = \frac{1}{2\pi} \int_X H^2 \, d\mu,$$

where H is the mean curvature of $f(X)$ and $d\mu$ is the element of surface area of X. We define $\tau_1(X)$ by

$$\tau_1(X) = \inf_{f \in \mathcal{J}} \tau_1(X, f),$$

where the infimum is taken over the space $\mathcal{J}$ of all C^∞-immersions of X in E^3. In this note we prove the following:

Theorem 1. *For all $f \in \mathcal{J}$ and all X, we have $\tau_1(X, f) \geqslant 2$, equality holding if and only if X is embedded as a euclidean sphere.*

A particular case of our theorem was proved in [4] where X was assumed to be diffeomorphic to the 2-sphere and moreover the immersions were restricted to be embeddings. In order to prove our theorem for X of arbitrary genus, it will be necessary to consider an application of the theory of Morse to local properties of surfaces in E^3. In particular we shall require the following

Lemma. *Let $K > 0$ denote the open set of all $x \in X$ at which the Gaussian curvature $K > 0$. Then we have*

$$\frac{1}{2\pi} \int_{K<0} K \, d\mu \geqslant 2.$$

2. Proof of the lemma. The lemma is contained as equation (3.7) of [2] by N. H. Kuiper, and we give here an outline of Kuiper's analysis.

A differentiable real-valued function Φ on X has a *critical point* at $x_0 \in X$ if $d\Phi = 0$ at x_0. The critical point is called *non-degenerate* if local coordinates Φ_1, Φ_2 exist such that in some neighbourhood of x_0 we have

$$\Phi = \Phi(x_0) \pm \Phi_1^2 \pm \Phi_2^2. \tag{1}$$

The non-degenerate critical point is said to be of index k if there are k negative signs in (1). We denote the number of non-degenerate critical points of index k of the function Φ by $\beta_k(X, \Phi)$.

Following Kuiper, let $z \in S^2$ be a unit vector in E^3 considered as a point on the unit sphere S^2, and let zf denote the real-valued function on X which assigns to $x \in X$ the value of the scalar-product $z.f(x)$, where f is now considered as a vector-valued function.

Let z, z_1, z_2 be three points on S^2 such that they form an orthonormal frame, with z orthogonal to the tangent plane at $f(x_0)$ and z_1, z_2 parallel to the direction of principal curvatures $\varkappa_1$, $\varkappa_2$ of the surface at $f(x_0)$. In some neighbourhood of x_0 we may take as coordinates the functions $\Phi_1 = z_1.f$, $\Phi_2 = z_2.f$. At x_0 the line element is given by

$$ds^2 = d\Phi_1^2 + d\Phi_2^2.$$

The 2-jet of zf at x_0 (i. e. the equivalence class of all functions defined in a neighbourhood of x_0 with the same values, first and second derivatives as the function zf) may be represented by the equation

$$\mathfrak{G}^2(zf) = z.f(x_0) + \varkappa_1 d\Phi_1^2 + \varkappa_2 d\Phi_2^2.$$

It is well-known that the Gaussian curvature K of $f(X)$ at x_0 is given by $K = \varkappa_1\varkappa_2$.

We see that K is positive if and only if zf has a non-degenerate critical point of index 0 or 2 at x_0. K is negative if and only if x_0 has index 1. K is 0 at x_0 if and only if x_0 is a degenerate critical point of zf.

Denote by γ the Gauss normal map which sends $x \in X$ to the pair of unit normal vectors orthogonal to the surface $f(X)$ at $f(x)$.

Let U be a connected neighbourhood of x for which these normal vectors can be classified according to the two „sides" of $f(U)$. Then if U is sufficiently small and $K \neq 0$ at all points $x \in U$, then $\gamma | U$ is 1—1, and the historical definition of Gaussian curvature gives

$$\int_{[\gamma(U)]} d\sigma = \int_U |K|\, d\mu,$$

where $d\sigma$ is the volume element on S^2 and the integral is evaluated over just one component of $\gamma(U)$. Taking both components together, we get

$$\int_{\gamma(U)} d\sigma = 2\int_U |K|\, d\mu.$$

The integral on the left hand member is taken over all $z \in \gamma(U)$, and these are precisely those points for which the function zf restricted to the open set U has a critical point. Owing to the restrictions on U, this critical point will be unique and its index will be 0 or 2 if $K > 0$ and 1 if $K < 0$.

Hence

$$\int_{z \in \gamma(U)} [\beta_0(U, zf) + \beta_2(U, zf)] \frac{d\sigma}{4\pi} = \int_{U|K>0} \frac{K d\mu}{2\pi}. \tag{2}$$

The set of points z for which the corresponding function zf has degenerate critical points with corresponding value of $K = 0$, form a set of measure zero according to Sard's theorem [see (3)]. Thus we may replace U in (2) by X to get

$$\int_{z \in S^2} [\beta_0(X, zf) + \beta_2(X, zf)] \frac{d\sigma}{4\pi} = \int_{K>0} \frac{K d\mu}{2\pi}. \tag{3}$$

Similarly we have

$$\int_{z \in S^2} \beta_1(X, zf) \frac{d\sigma}{4\pi} = \int_{K<0} \frac{-K d\mu}{2\pi}. \tag{4}$$

Now, except for a set of z of measure 0, the function zf has only non-degenerate critical points. In particular, this continuous function on the compact space X must attain a maximum (point of index 2) and a minimum (point of index 0). So, for almost every z, we have

$$\beta_0(X, zf) \geqslant 1 \text{ and } \beta_2(X, zf) \geqslant 1.$$

Thus, from (3) we have

$$\int_{K>0} \frac{K d\mu}{2\pi} \geqslant 2,$$

thus completing the proof of the lemma.

3. Proof of Theorem 1. We have $H^2 = K + \frac{1}{4}(\varkappa_1 - \varkappa_2)^2$, so that

$$\tau_1(X, f) = \frac{1}{2\pi}\int_X H^2 d\mu = \frac{1}{2\pi}\int K d\mu + \frac{1}{8\pi}\int (\varkappa_1 - \varkappa_2)^2 d\mu.$$

Hence

$$\tau_1(X,f) \geqslant \frac{1}{2\pi}\int_X K d\mu = \frac{1}{2\pi}\int_{K>0} K d\mu +$$

$$+ \frac{1}{2\pi}\int_{K<0} K d\mu \geqslant \frac{1}{2\pi}\int_{K>0} K d\mu \geqslant 2$$

from the lemma.
Moreover, if $\tau_1(X,f) = 2$, then

$$\frac{1}{2\pi}\int_{K<0} K d\mu = 0.$$

Hence, by the Gauss-Bonnet theorem,

$$\frac{1}{2\pi}\int K d\mu = \frac{1}{2\pi}\int_{K>0} K d\mu + \frac{1}{2\pi}\int_{K<0} K d\mu = 2,$$

and the Euler characteristic of X is equal to 2. Thus X is homeomorphic (and therefore diffeomorphic) to the 2-sphere S^2.

The total absolute curvature is then

$$\frac{1}{2\pi}\int |K|\, d\mu = \frac{1}{2\pi}\int_{K>0} K d\mu - \frac{1}{2\pi}\int_{K<0} K d\mu = 2.$$

It follows from theorem 1 of Chern and Lashof [1], that such an immersion must be an imbedding. Moreover, we have $H^2 = K$ except at set of measure zero. In this case almost every point is an umbilic and it follows that H and K are constant [see, for example, (5) page 128]; and the continuity of H and K guarantees that X is embedded as a euclidean sphere. This completes the proof of the theorem.

For $X = S^2$, the infimum of 2 is attained. An open question is the extent to what this infimum value can be improved for surfaces of genus p. There are strong reasons [see (4)] for conjecturing that for a surface of genus 1, the corresponding value of $\tau_1(X)$ is π, but even this case remains open.

REFERENCES

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CURBURA MEDIE A SUPRAFEŢELOR IMERSE

Rezumat

Dacă $f: X \to E^3$ este o C^∞ — imersiune a suprafeţei de clasă C^∞ închise şi orientabile X în spaţiul euclidian E^3, se notează

$$\tau_1(X,f) = \frac{1}{2\pi}\int_X H^2 d\mu$$

şi $\tau_1(X)$ infimumul lui $\tau_1(X,f)$ în raport cu f.

În lucrare se demonstrează că $\tau_1(X,f) \geqslant 2$, egalitatea avînd loc dacă $f(X)$ este o sferă.