pect to t between the limits 0 and ∞ and evaluating the integrals with the help of the formulae (2) and (3), we get (4).

In particular, if we take $y = 0$ and $z = 0$ in (4) we obtain.

$$G^{q+1,p}_{s,r+2}\left[x \,\middle|\, \begin{matrix} \beta_1, \ldots, \beta_s \\ \lambda, \alpha_1, \ldots, \alpha_r, -\lambda \end{matrix}\right] = \tag{6}$$

$$= \frac{\prod_{j=1}^{p} \Gamma(1 - \beta_j + \lambda) \prod_{j=1}^{q} \Gamma(\alpha_j + \lambda)}{\prod_{j=q+1}^{r} \Gamma(1 - \alpha_j + \lambda) \prod_{j=p+1}^{s} \Gamma(\beta_j - \lambda)\, \Gamma(2\lambda + 1)}\, x^{\lambda}$$

$${}_sF_{r+1}[1-\beta_1+\lambda, \ldots, 1-\beta_s+\lambda;\ 1-\alpha_1+\lambda, \ldots, 1-\alpha_r+\lambda, 2\lambda+1;\ (-1)^{p+q-s+1}x]$$

If we take $q = 0$, $p = s$ in (6) it reduces to a known result [5, p. 215].

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O NOUĂ DEZVOLTARE PENTRU FUNCȚIA G

Rezumat

Se demonstrează unele formule de dezvoltare pentru funcții hipergeometrice generalizate.

ON SOME SYSTEMS OF AUTOMATIC CONTROL THEORY

BY

A. N. V. RAO

1. Introduction: This paper is concerned with the absolute stability of some systems of equations that arise in the Automatic Control Theory. In particular, we shall consider the case when the relation between the error signal and the state variables is given in the form of a convolution integral. It is shown that the study of such systems can be reduced to the study of integral equation of the type

$$\sigma(t) = \psi(t) + \int_0^t k(t-u)\, \varphi(\sigma(u))\, du,$$

which has been studied by Corduneanu [2], [3], [4], Nohel, Levin [6], [7] and others ([5]).

The main results of this paper are stated in the form of Theorems (1), (2) and (3).

2. Consider the equations

$$\dot{x}(t) = A\, x(t) + b\, \varphi(\sigma(t)), \tag{1}$$

$$\sigma(t) = f(t) + \int_0^t c^T(t-s)\, x(s)\, ds \tag{2}$$

where A is a constant $n \times n$ matrix, b and c are constant column n-vectors (c^T denotes the transpose of c) and σ and f are scalar functions. These equations arise in the theory of automatic control. It is usual to consider $\sigma(t)$ to be the scalar product of a constant vector c and $x(t)$. However, we shall consider here the case when the relation between $\sigma(t)$ and $x(t)$ is given by a convolution integral.

3. From (1) we get

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} b \varphi(\sigma(s))\, ds$$

where $x(0) = x_0$. Substituting this in (2) one has

$$\sigma(t) = f(t) + \int_0^t c^T(t-s) e^{As} x_0\, ds +$$

$$+ \int_0^t c^T(t-s) \int_0^s e^{A(s-u)} b \varphi(\sigma(u))\, du\, ds =$$

$$= \psi(t) + \int_0^t c^T(t-s) \int_0^s e^{A(s-u)} b \varphi(\sigma(u))\, du\, ds$$

where $\psi(t) = f(t) + \int_0^t c^T(s) e^{A(t-s)} x_0\, ds.$

Using the property of the convolution integral, we have

$$\int_0^t c^T(t-s) \int_0^s e^{A(s-u)} b \varphi(\sigma(u))\, du\, ds =$$

$$= \int_0^t c^T(s) \int_0^{t-s} e^{A(t-s-u)} b \varphi(\sigma(u))\, du\, ds =$$

$$\int_0^t \int_0^{t-s} c^T(s) e^{A(t-s-u)} b \varphi(\sigma(u))\, du\, ds = \tag{3}$$

$$= \int_0^t \int_0^{t-u} c^T(s) e^{A(t-s-u)} b \varphi(\sigma(u))\, ds\, du.$$

Now define $k(t) = \int_0^t c^T(s) e^{A(t-s)} b\, ds$. Then

$$\int_0^t k(t-u) \varphi(\sigma(u))\, du = \int_0^t \int_0^{t-u} c^T(s) e^{A(t-u-s)} b\, ds\, \varphi(\sigma(u))\, du.$$

which is the same as (3). Therefore the equation for $\sigma(t)$ can be written as

$$\sigma(t) = \psi(t) + \int_0^t k(t-u) \varphi(\sigma(u))\, du. \tag{4}$$

Instead of the system of equations (1) and (2), suppose one considers now the equations

$$\dot{x}(t) = A x(t) + \int_0^t b(t-s) \varphi(\sigma(s))\, ds, \tag{5}$$

$$\sigma(t) = f(t) + \int_0^t c^T(t-s) x(s)\, ds. \tag{6}$$

These too, by a similar procedure, can be reduced to an integral equation of form (4), by defining

$$\psi(t) = f(t) + \int_0^t c^T(t-s) e^{As} x_0\, ds$$

and

$$k(t) = \int_0^t c^T(s) k_1(t-s)\, ds, \text{ where } k_1(t) = \int_0^t e^{As} b(t-s)\, ds.$$

The frequency method enables us to find the stability conditions for the non-linear integral equations of the form (4). In fact, it has been shown, ([2], [3]), that the integral equation (4) has a solution $\sigma(t)$, tending to zero as $t \to \infty$, under the following conditions.

Theorem (1*)
(A) $\psi(t), \psi'(t) \in L_1(0, \infty)$,
(B) $k(t), k'(t) \in L_1(0, \infty) \cap L_2(0, \infty)$.
(C) $\varphi(\sigma)$ is continuous and bounded for all $\sigma \in R$ and $\sigma\varphi(\sigma) > 0$ for $\sigma \neq 0$,
(D) There exists a $q \geqslant 0$ such that $Re\{(1 + i\omega q)\, \tilde{k}(i\omega)\}$ $\leqslant 0$ for all real, where $\tilde{k}(i\omega) = \int_0^\infty k(t) e^{-i\omega t} dt.$

Now, we proceed by stating our main results.

Theorem 1. *Suppose the system of equations* (1) *and* (2) *satisfies the following conditions:*

(i) *the matrix A is stable, i.e., all the characteristic roots have negative real parts;*

(ii) (a) *the vector function $c(t)$ is defined for all $t \geqslant 0$ and is such that*

$$c(t) \in L_1(0, \infty) \cap L_2(0, \infty);$$

(b) $f(t)$ is defined for $t \geqslant 0$, $f(t), f'(t) \in L_1(0, \infty)$;

(iii) $\varphi(\sigma)$ *is continuous and bounded for all $\sigma \in R$ and $\sigma\varphi(\sigma) > 0$ for $\sigma \neq 0$;*

(iv) *there exists a $q \geqslant 0$ such that*

$$Re\{(1 + i\omega q)\tilde{c}^T(i\omega)(i\omega I - A)^{-1}b\} \leqslant 0, \text{ where } \tilde{c}^T(i\omega) = \int_0^\infty c^T(t)\, e^{i\omega t}\, dt$$

Then the system of equations (1) *and* (2) *is absolutely stable.*

Proof: We show that the assumption of Theorem (1*) are satisfied. Indeed, we have

$$\psi(t) = f(t) + \int_0^t c^T(s)\, e^{A(t-s)}\, x_0\, ds. \quad \psi(t) \in L_1(0, \infty)$$

because of (i) $f(t) \in L_1(0, \infty)$ by hypothesis, (ii) in the convolution product, $\int_0^t c^T(s)\, e^{A(t-s)}\, x_0\, ds$, e^{At} (as A is stable) and $c^T(s)$, both belong to $L_1(0, \infty)$. Differentiating $\psi(t)$ we get $\psi'(t) = f'(t) + \int_0^t c^T(s)\, A\, e^{A(t-s)}\, x_0 ds + c^T(t)\, x_0$. This belongs to $L_1(0, \infty)$ as each term belongs to $L_1(0, \infty)$.
Thus the condition (A) of (1*) is satisfied.

Consider now $k(t) = \int_0^t c^T(s)\, e^{A(t-s)}\, b\, ds$. Then

$$k'(t) = \int_0^t c^T(s)\, A e^{A(t-s)}\, b\, ds + c^T(t)\, b.$$

As $c^T(t) \in L_1(0, \infty)$ and $e^{At} \in L_2(0, \infty)$, the convolution product, $k(t)$, belongs to $L_1(0, \infty) \cap L_2(0, \infty)$ (see [9]). Thus (B) of (1*) is verified. The function $\varphi(\sigma)$, by hypothesis of Theorem 1, satisfies (C) of (1*). It is easy to show that $\tilde{k}(i\omega) = \tilde{c}^T(i\omega)(i\omega I - A)^{-1}b$. From assumption (iv) of Theorem 1, we have

$$Re\{(1 + i\omega q)\, k(i\omega)\} \leqslant 0, \text{ which is (D) of (1*).}$$

Thus from Theorem (1*), the absolute stability of the system (1). (2) follows.

Theorem 2. *Suppose the equation* (5) *and* (6) *satisfy the following conditions:*

(i) *The matrix A is stable.*

(ii) (a) *The vector function $c(t)$ defined for $t \geqslant 0$, is such that*

$$c(t) \in L_1(0, \infty) \cap L_2(0, \infty).$$

(b) *$f(t)$ is defined for $t \geqslant 0$, $f(t), f'(t) \in L_1(0, \infty)$*

(iii) $b(t) \in L_1(0, \infty) \cap L_2(0, \infty)$.

(iv) $\varphi(\sigma)$ *is continuous for $\sigma \in R$ and $\sigma\varphi(\sigma) > 0$ for $\sigma \neq 0$.*

(v) *There exists a $q \geqslant 0$ such that*

$$Re\{(1 + i\omega q)\, \tilde{c}^T(i\omega)(i\omega I - A)^{-1}\tilde{b}(i\omega)\} \leqslant 0 \text{ where}$$

$$\tilde{c}^T(i\omega) = \int_0^\infty c^T(t)\, e^{i\omega t}\, dt \quad \text{and} \quad \tilde{b}(i\omega) = \int_0^\infty b(t)\, e^{i\omega t}\, dt.$$

Then the system of equations (5), (6) *is absolutely stable.*

Proof: Again, we verify the conditions of the Theorem (1*) are satisfied.

$\psi(t)$ is the same as in Theorem 1 and so (A) of (1*) is satisfied.

$$k(t) = \int_0^t c^T(s)\, k_1(t - s)\, ds$$

and

$$k_1(t) = \int_0^t e^{A(t-s)}\, b(s)\, ds.$$

Hence

$$k'(t) = \int_0^t c^T(s)\, k_1'(t - s)\, ds + c^T(t)\, k_1(0).$$

and

$$k_1'(t)=\int_0^t A\, e^{A(t-s)}\, b(s)\, ds + b(t).$$

Since $b(t)$ and e^{At} both belong to $L_1(0,\infty)\cap L_2(0,\infty)$, $k_1(t)$ belongs to $L_1(0,\infty)\cap L_2(0,\infty)$. Again since both $k_1(t)$ and $c^T(t)$ belong $L_1(0,\infty)\cap L_2(0,\infty)$, $k(t)$ also belongs to $L_1(0,\infty)\cap L_2(0,\infty)$. Further

$$\tilde{k}(i\omega)=\tilde{c}^T(i\omega)(i\omega I - A)^{-1}\tilde{b}(i\omega).$$

The hypothesis (5) of Theorem (2) implies $Re\{(1+i\omega q)\, k(i\omega)\}\leqslant 0$ which is the assumption (D) of Theorem (1*). So it follows that the system of equation (5) and (6) is absolutely stable.

If one of the characteristic roots of the matrix A is zero, then the control equations, corresponding to (5) and (6), can be written as (see [1])

(7)
$$\dot{x}=A_1 x(t)+\int_0^t b(t-s)\,\varphi(\sigma(s))\,ds$$

$$\dot{\xi}=\varphi(\sigma)\,;\ \sigma(t)=f(t)+\int_0^t c^T(t-s)\,x(s)\,ds-\gamma\,\xi,\ \gamma>0$$

This system can be reduced to the integral equation of the form (4), if we define

$$\psi(t)=f(t)+\int_0^t c^T(t-s)\, e^{A_1 s}\, x_0\, ds-\gamma\,\xi_0 .$$

and

$$k(t)=\int_0^t c^T(s)\, k_1(t-s)\, ds-\gamma,\ \text{where } k_1(t)=\int_0^t e^{A_1(t-s)}\, b(s)\, ds.$$

as shown in Appendix A.

The stability of integral equations of the form (4) with the above type of kernel, has been discussed by Corduneanu [2]. We quote his theorem without proof.

Theorem (2*). *Consider the non-linear equation*

(E)
$$\sigma(t)=\psi(t)+\int_0^t k(t-u)\,\varphi(\sigma(u))\,du$$

under the following assumptions:

(A) $\psi(t)$ *is defined for* $t\geqslant 0$ *and* $\psi'(t),\ \psi''(t)\in L_1(0,\infty)$

(B) $k(t)=h(t)-\gamma$ $(\gamma>0)$ *and* $h(t)$ *is defined for* $t>0$ *and is such that* $h(t)$ *and* $h'(t)\in L_1(0,\infty)\cap L_2(0,\infty)$.

(C) $\varphi(\sigma)$ *is continuous for* $\sigma\in R$ *and* $\sigma\varphi(\sigma)>0$ for $\sigma\neq 0$.

(D) If $\tilde{h}(i\omega)=\int_0^\infty h(t)\,e^{i\omega t}dt$ *and* $g(i\omega)=\tilde{h}(i\omega)-(i\omega)^{-1}$, *there exists a* $q>0$ *such that*

$$Re\{(1+i\omega q)\,\tilde{g}(i\omega)\}\leqslant 0.$$

Then there is at least one solution $\sigma(t)$ *of* (E) *and* $\sigma(t)\to 0$ *as* $t\to\infty$.

Theorem 3. *Let the system of equations* (7) *satisfy the following conditions*

(i) *The matrix* A_1 *is stable.*

(ii) (a) *The function* $c(t)$ *is defined for* $t\geqslant 0$ *and is such that*
$c(t)\in L_1(0,\infty)\cap L_2(0,\infty)$, $c'(t)\in L_1(0,\infty)$.

(b) $f(t)$ *is defined for* $t\geqslant 0$ and $f'(t),\ f''(t)\in L_1(0,\infty)$.

(iii) $b(t)$ *is defined for* $t\geqslant 0$ *and* $\in L_1(0,\infty)\cap L_2(0,\infty)$.

(iv) $\varphi(\sigma)$ *is continuous for* $\sigma\in R$ *and* $\sigma\varphi(\sigma)>0$ *for* $\sigma\neq 0$.

(v) *There exists a* $q>0$ *such that*

$$Re[\{(1+i\omega q)\ \tilde{c}^T(i\omega)(i\omega I-A_1)^{-1}\tilde{b}(i\omega)\}-\gamma(i\omega)^{-1}]\leqslant 0.$$

Then the system of equations (7) *is absolutely stable.*

Proof: We verify that the conditions of Theorem (2*) are satisfied. We have

$$\psi(t)=f(t)+\int_0^t c^T(s)\, e^{A_1(t-s)}\, x_0\, ds-\gamma\,\xi_0 ,$$

$$\psi'(t)=f'(t)+\int_0^t c^T(s)\, A_1 e^{A_1(t-s)}\, x_0\, ds+c^T(t)\, x_0 ,$$

$$\psi''(t)=f''(t)+\int_0^t c^T(s)\, A_1^2\, e^{A_1(t-s)}\, x_0\, ds+c^T(t)\, A_1\, x_0+\frac{d}{ds}(c^T(t))\, x_0$$

Thus $\psi'(t) \in L_1(0, \infty)$ as $f'(t)$, $c(t)$, e^{At}, all belong to $L_1(0, \infty)$, and $\psi''(t) \in L_1(0, \infty)$ as $f''(t)\, c(t)$, $c'(t)$, $e^{At} \in L_1(0, \infty)$. So (A) of (2*) is satisfied. Further,

$$h(t) = k(t) + \gamma = \int_0^t c^T(s)\, k_1(t-s)\, ds \quad \text{and}$$

$$k_1(t) = \int_0^t e^{A(t-s)}\, b(s)\, ds.$$

$k_1(t) \in L_1(0, \infty) \cap L_2(0, \infty)$, as e^{At} and $b(t)$ both belong to $L_1(0, \infty) \cap L_2(0, \infty)$. Since $k_1(t)$ and $c^T(t)$ belong to $L_1(0, \infty) \cap L_2(0, \infty)$, $h(t)$ belongs $L_1(0, \infty) \cap L_2(0, \infty)$. Further

$$\tilde{h}(i\omega) = \tilde{c}^T(i\omega)\,(i\omega I - A)^{-1}\,\tilde{b}(i\omega).$$

By hypothesis (v) of Theorem 3, we have

$$Re\,\{(1 + i\omega q)\,(\tilde{h}(i\omega) - \gamma(i\omega)^{-1})\} \leqslant 0 \text{ which is (D) of (2*).}$$

Thus all the conditions of Theorem (2*) are satisfied and so we conclude that the system (7) is absolutely stable.

If instead of (7), suppose we have

$$\dot{x} = A_1 x(t) + b\varphi(\sigma(t)),$$

(8) $$\dot{\xi} = \varphi(\sigma)\,;\ \sigma = f(t) + \int_0^t c^T(t-s)\, x(s)\, ds - \gamma\xi,\quad \gamma > 0,$$

where b is a constant n-vector. We can reduce system (8) to the integral equation (5), if we define

$$\psi(t) = f(t) + \int_0^t c^T(t-s)\, e^{As} x_0\, ds - \gamma\xi_0$$

and

$$k(t) = \int_0^t c^T(s)\, e^{A(t-s)}\, b\, ds.$$

If $c(t) \in L_1(0, \infty) \cap L_2(0, \infty)$ and $c'(t) \in L_1(0, \infty)$, and the frequency condition

$$Re\,\{(1 + i\omega q)\,\tilde{c}^T(i\omega)\,(i\omega I - A)^{-1} b - \gamma(i\omega)^{-1})\} \leqslant 0,\ q > 0,$$

is satisfied (f and φ are as in Th. 3), then by using the same method as in Theorem (3), we can prove that the system of equations (8) is absolutely stable.

Remark: It is possible to replace the frequency condition by the other types of conditions, similar to those given by L e v i n and N o h e l [6], [7].

The author wishes to thank Professor C. Corduneanu for introducing me to this problem and for his valuable suggestions.

APPENDIX A

The reduction of system (7) is as follows:

$$\dot{x} = A_1 x(t) + \int_0^t b(t-s)\, \varphi(\sigma(s))\, ds,$$

$$\dot{\xi} = \varphi(\sigma)\,;\ \sigma(t) = f(t) + \int_0^t c^T(t-s)\, x(s)\, ds - \gamma\xi,\quad \gamma > 0.$$

From the first equation it follows.

$$x(t) = e^{At} x_0 + \int_0^t e^{A_1(t-s)} \int_0^s b(s-u)\, \varphi(\sigma(u))\, du\, ds =$$

$$= e^{At} x_0 + \int_0^t k_1(t-u)\, \varphi(\sigma(u))\, du.$$

Where $$k_1(t) = \int_0^t e^{A_1 s}\, b(t-s)\, ds =$$

$$= \int_0^t e^{A_1(t-s)}\, b(s)\, ds.$$

$$x(t) = e^{At} x_0 + \int_0^t k_1(t-u)\,\varphi(\sigma(u))\,du,$$

$$\sigma(t) = f(t) + \int_0^t c^T(t-s)\,e^{As} x_0 +$$

$$+ \int_0^t c^T(t-s) \int_0^s k_1(s-u)\,\varphi(\sigma(u))\,du\,ds - \int_0^t \varphi(\sigma(s))\,ds - \gamma\,\xi_0 =$$

$$= f(t) + \int_0^t c^T(t-s)\,e^{As} x_0 - \gamma\,\xi_0 +$$

$$+ \int_0^t c^T(t-s) \int_0^s k_1(s-u)\,\varphi(\sigma(u))\,du\,ds - \int_0^t \varphi(\sigma(s))\,ds =$$

$$= \psi(t) + \int_0^t c^T(s) \int_0^{t-s} k_1(t-s-u)\,\varphi(\sigma(u))\,du\,ds -$$

$$- \gamma \int_0^t \varphi(\sigma(u))\,du.$$

Where $$\psi(t) = f(t) + \int_0^t c^T(t-s)\,e^{As} x_0 - \gamma\,\xi_0 =$$

$$= \psi(t) + \int_0^t \int_0^{t-u} c^T(s)\,k_1(t-s-u)\,ds\,\varphi(\sigma(u))\,du -$$

$$- \gamma \int_0^t \varphi(\sigma(s))\,ds =$$

$$= \psi(t) + \int_0^t \left[\int_0^{t-u} c^T(s)\,k_1(t-s-u)\,ds - \gamma\right] \varphi(\sigma(u))\,du.$$

Define $$k(t) = \int_0^t c^T(s)\,k_1(t-s)\,ds - \gamma.$$

One has

$$\int_0^t k(t-u)\,\varphi(\sigma(u))\,du = \int_0^t \left[\int_0^{t-u} c^T(s)\,k_1(t-u-s)\,ds - \gamma\right] \varphi(\sigma(u))\,du,$$

which is the same as above.

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ASUPRA UNOR SISTEME ALE TEORIEI CONTROLULUI

Rezumat

Autorul consideră sisteme de forma (1), (2) sau (5), (6) şi arată că fiecare din aceste sisteme poate fi redus la o ecuaţie integrală de forma (4). Apoi se aplică metoda transformatei Fourier pentru stabilirea comportării asimptotice a soluţiilor. Rezultatul principal este stabilitatea absolută a sistemelor considerate, în condiţiile precizate în lucrare.