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DESPRE FUNCŢII H DE DOUĂ VARIABILE (II)

Rezumat

Sînt studiate unele proprietăţi ale funcţiei H de două variabile, pornind de la funcţia H a lui Fox [2]. Apoi, în ultima parte, sînt deduse unele integrale noi şi generale.

FOURIER SERIES OF MEIJER'S G-FUNCTION OF HIGHER ORDER

BY

R. S. DAHIYA and BHAGAT SINGH

1. Introduction. In 1965, Agarwal [2] defined a G-function of higher order (i. e. with more parameters) in two variables, which includes not only the Meijer's G-function of one variable as a particular case, but also most of the known functions of two variables, e. g. the Appell's functions F_1, F_2, F_3 and F_4, the Whittaker functions of two variables, etc.

The following notations will be used throughout the present paper.

The symbol (α_p, m) denotes the sequence of $p-m+1$ parameters α_m, $\alpha_{m+1}, \ldots, \alpha_p$ but when $m=1$, we shall denote it by (α_p) instead of by (α_1, p).

The Meijer's G-function of two variables due to Agarwal [2] is defined by the relation

$$G^{n_1,\, v_1,\, v_2,\, m_1,\, m_2}_{p,\,[t:t'],\, s,\, [q:q']}\left[\begin{array}{c|l} x & (\varepsilon_p) \\ & (\gamma_t);\ (\gamma'_{t'}) \\ & (\delta_s) \\ y & (\beta_q);\ (\beta'_{q'}) \end{array}\right] = \tag{1.1}$$

$$= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Phi(\xi + n)\, \psi(\xi, n)\, x^{\xi} y^{n}\, d\xi\, dn,$$

where

$$\Phi(\xi + n) = \Gamma\left[\begin{array}{l} 1-(\varepsilon_{n_1}) + \xi + n\,; \\ (\varepsilon_{n_1+1}, p) - \xi - n,\ (\delta_s) + \xi + n \end{array}\right],$$

$$\psi(\xi, n) = \Gamma\left|\begin{array}{l}(\beta_{m_1}) - \xi,\ (\gamma_{v_1}) + \xi,\ (\beta'_{m_2}) - n,\ (\gamma'_{v_2}) + n\,; \\ 1 - (\beta_{m_1+1},\ q) + \xi,\ 1 - (\gamma_{v_1+1},\ t) - \xi,\ 1 - (\beta'_{m_2+1},\ q') + \\ \qquad + n, 1 - (\gamma'_{v_2+1},\ t' - n)\end{array}\right|$$

$$0 \leq m_1 \leq q,\quad 0 \leq m_2 \leq q',\quad 0 \leq v_1 \leq t,\quad 0 \leq v_2 \leq t',\quad 0 \leq n_1 \leq p.$$

The integral (1.1) converges if $p + q + s + t < 2(m_1 + v_1 + n_1)$, $p + q' + s + t' < 2(m_2 + v_2 + n_1)$, $|\arg x| < \pi[m_1 + v_1 + n_1 - \frac{1}{2}(p + q + s + t)]$

and $|\arg y| < \pi[m_2 + v_2 + n_1 - \frac{1}{2}(p + q' + s + t')]$.

The form in (1.1) is slightly variant from that of [2], though in essence the function is the same. Now, we have

$$(1.2)\quad G^{n_1,\, v_1,\, v_2,\, m_1,\, m_2}_{p,\,[t:t'],\,s,\,[q:q']}\left[\begin{array}{c|l} x & (\varepsilon_p) \\ & (\gamma_t)\,;\ (\gamma'_{t'}) \\ & (\delta_s) \\ y & (\beta_q)\,;\ (\beta'_{q'})\end{array}\right] = \sum_{h=1}^{m_1}\sum_{k=1}^{m_2} x^{\beta_h} y^{\beta'_k}\, \frac{\prod_{j=1}^{v_1}\Gamma(\gamma_j + \beta_h)}{\prod_{j=v_1+1}^{t}\Gamma(1 - \gamma_j - \beta_h)}$$

$$\frac{\prod_{j=1}^{v_2}\Gamma(\gamma'_j + \beta'_k)\ \prod_{j=1}^{m_2}{}'\,\Gamma(\beta'_j - \beta'_k)\ \prod_{j=1}^{n_1}\Gamma(1 - \varepsilon_j + \beta_h + \beta'_k)\ \prod_{j=1}^{m_1}{}'\,\Gamma(\beta_j - \beta_h)}{\prod_{j=v_2+1}^{t'}\Gamma(1 - \gamma'_j - \beta'_k)\ \prod_{j=m_1+1}^{q}\Gamma(1 - \beta_j + \beta_h)\ \prod_{j=m_2+1}^{q'}\Gamma(1 - \beta'_k + \beta'_k)\ \prod_{j=n_1+1}^{p}\Gamma(\varepsilon_j - \beta_h - \beta'_k)\ \prod_{j=1}^{s}\Gamma(\delta_j + \beta_h + \beta'_k)}$$

$$F\left[\begin{array}{c|ll} p & (1 - \varepsilon_p + \beta_h + \beta'_k) & (-)^{m_1 + p - n_1 + t + v_1}\, x \\ t & (\gamma_t + \beta_h)\,;\ (\gamma'_{t'} + \beta'_k) & \\ s & (\delta_s + \beta_h + \beta'_k) & \\ q-1 & (1 - \beta_q + \beta_h)^*\,;\ (1 - \beta'_{q'} + \beta'_k)^* & (-)^{m_2 + p - n_1 + t' - v_2}\, y\end{array}\right]$$

where the prime π' indicates the omission of the factor of the type $\Gamma(\beta_j - \beta_j)$; the asterisk in the F denotes the omission of the parameters of the type $(1 - \beta_h + \beta_h)$. (1.2) converges absolutely for $p + t < s + q$ or $p + t = s + q$ and $p + t' < s + q'$ or $p + t' = s + q'$; $|x| + |y| < \min(1, 2^{s-p+1})$.

The Fourier series of the above mentioned G-function will be stated and proved in Section 2; while particular cases will be deduced in Section 3. It may be noted that the constants and the parameters are such that the functions involved exist.

2. Fourier series. The Fourier series to be established are:

$$\sum_{r=0}^{\infty} G^{n_1+m'+1,\, v_1,\, v_2,\, m_1,\, m_2}_{p+u+1,\,[t:t'],\,s+m+1,\,[q:q']}\left[\begin{array}{c|l} x & \frac{1}{2} - \rho - a_1,\ 1 - \rho - (b_{m'}),\ (\varepsilon_p),\ 1 - \rho - (b_{m'+1},\ u) \\ & (\gamma_t)\,;\ (\gamma'_{t'}) \\ & (\delta_s),\ \rho + (a_{n'+1},\ m),\ 1 + \rho + a_1 + r \\ y & (\beta_q)\,;\ (\beta'_{q'})\end{array}\right]$$

$$(2.1)\qquad \cdot \sin(2r + 1)\,\theta = \frac{\sqrt{\pi}}{2}\,(\sin\theta)^{2a_1 - 2\rho - 1}$$

$$\cdot\, G^{n_1+m',\, v_1,\, v_2,\, m_1,\, m_2}_{p+u,\,[t:t'],\,s+m,\,[q:q']}\left[\begin{array}{c|l} \dfrac{x}{\sin^2\theta} & 1 - \rho - (b_{m'}),\ (\varepsilon_p),\ 1 - \rho - (b_{m'+1},\ u) \\ & (\gamma_t)\,;\ (\gamma'_{t'}) \\ \dfrac{y}{\sin^2\theta} & (\delta_s),\ 1 + (a_{n'+1},\ m) \\ & (\beta_q)\,;\ (\beta'_{q'})\end{array}\right],$$

where $0 \leq \theta \leq \pi$ and valid under the conditions of (1.1).

$$G^{n_1+m'+1,\, v_1,\, v_2,\, m_1,\, m_2}_{p+u+1,\,[t:t'],\,s+m+1,\,[q:q']}\left[\begin{array}{c|l} x & \frac{3}{2} - \rho - a_1,\ 1 - \rho - (b_{m'}),\ (\varepsilon_p),\ 1 - \rho - (b_{m'+1},\ u) \\ & (\gamma_t)\,;\ (\gamma'_{t'}) \\ & (\delta_s),\ \rho + (a_{n'+1},\ m),\ \rho + a_1 \\ y & (\beta_q)\,;\ (\beta'_{q'})\end{array}\right] +$$

$$(2.2)\qquad + 2\sum_{r=1}^{\infty} G^{n_1+m'+1,\, v_1,\, v_2,\, m_1,\, m_2}_{p+u+1,\,[t:t'],\,s+m+1,\,[q:q']}\left[\begin{array}{c|l} x & \frac{3}{2} - \rho - a_1,\ 1 - \rho - (b_{m'}),\ (\varepsilon_p),\ 1 - \rho - (b_{m'+1},\ u) \\ & (\gamma_t)\,;\ (\gamma'_{t'}) \\ & (\delta_s),\ \rho + (a_{n'+1},\ m),\ \rho + a_1 + r \\ y & (\beta_q)\,;\ (\beta'_{q'})\end{array}\right]\cdot$$

$$\cos r\theta = \sqrt{\pi}\left(\sin\frac{1}{2}\theta\right)^{2a_1 - 2\rho - 2}.$$

$$\cdot\, G^{n_1+m',\, v_1,\, v_2,\, m_1,\, m_2}_{p+u,\,[t:t'],\,s+m,\,[q:q']}\left[\begin{array}{c|l} \dfrac{x}{\sin^2\left(\frac{1}{2}\theta\right)} & 1 - \rho - (b_{m'}),\ (\varepsilon_p),\ 1 - \rho - (b_{m'+1},\ u) \\ & (\gamma_t)\,;\ (\gamma'_{t'}) \\ \dfrac{y}{\sin^2\left(\frac{1}{2}\theta\right)} & (\delta_s),\ 1 + (a_{n'+1},\ m) \\ & (\beta_q)\,;\ (\beta'_{q'})\end{array}\right]$$

where $0 \leqslant \theta \leqslant \pi$ and holds under the conditions of (1.1).

Proofs : Since [4, p. 103]

$$(2.3) \qquad \sum_{r=0}^{\infty} Z^{-r} G^{m'+1,\, n'}_{m+1,\, u+1}\left(Z \,\middle|\, \begin{matrix} a_1,\ a_2+r, \ldots, a_m+r,\ a_1+1+2r \\ a_1+\frac{1}{2}+r,\ b_1+r, \ldots, b_u+r \end{matrix} \right) \cdot$$

$$\cdot \sin(2r+1)\,\theta = \frac{\sqrt{\pi}}{2} (\sin\theta)^{2a_1-1}\, G^{m',\, n'}_{m,\, u}\left(\frac{Z}{\sin^2\theta} \,\middle|\, \begin{matrix} a_1, \ldots, a_m \\ b_1, \ldots, b_u \end{matrix} \right)$$

where $0 \leqslant \theta \leqslant \pi$, $|\arg Z| < \left(m' + n' - \frac{1}{2}m - \frac{1}{2}u \right)\pi$.

We have

$$\sum_{r=0}^{\infty} \left\{ \int_0^{\infty} Z^{\rho-r-1} f(Z)\, G^{m'+1,\, n'}_{m+1,\, u+1}\left(Z \,\middle|\, \begin{matrix} a_1,\ a_2+r, \ldots, a_m+r,\ a_1+1+2r \\ a_1+\frac{1}{2}+r,\ b_1+r, \ldots, b_u+r \end{matrix} \right) dZ \right\} \cdot$$

$$(2.4) \qquad \cdot \sin(2r+1)\,\theta = \frac{\sqrt{\pi}}{2} (\sin\theta)^{2a_1-1} \int_0^{\infty} Z^{\rho-1} f(Z)\, G^{m',\, n'}_{m,\, u}\left(\frac{Z}{\sin^2\theta} \,\middle|\, \begin{matrix} a_m \\ b_u \end{matrix} \right) dZ$$

provided the integrals involved exist.

Now if we take

$$f(Z) = G^{n_1,\, v_1,\, v_2,\, m_1,\, m_2}_{p+n',\, [t:t'],\, s,\, [q:q']} \left[\begin{matrix} n'xZ \\ \\ n'yZ \end{matrix} \,\middle|\, \begin{matrix} (\varepsilon_p),\ 1-\rho-(a_{n'}) \\ (\gamma_t);\ (\gamma'_{t'}) \\ (\delta_s) \\ (\beta_q);\ (\beta'_{q'}) \end{matrix} \right]$$

in (2.4) and evaluate the integrals their in to get the desired result (2.1). Thus (2.1) is proved. Similarly (2.2) can be proved, if we start with [4, eqn (2.4), p. 103] instead (2.3).

3. Particular cases. The following particular cases have been obtained by proper choice of parameters.

$$(3.1) \qquad \sum_{r=0}^{\infty} G^{2,\,1,\,1,\,1,\,1}_{2,\,1,\,2,\,1} \left[\begin{matrix} x \\ \\ y \end{matrix} \,\middle|\, \begin{matrix} \frac{1}{2}-\rho-a_1,\ 1-\rho-b_1 \\ \gamma_1;\ \gamma'_1 \\ \rho+a_1,\ 1+\rho+a_1+r \end{matrix} \right] \sin(2r+1)\,\theta$$

$$= \frac{\sqrt{\pi}}{2} (\sin\theta)^{2(a_1-\rho-\beta_1-\beta'_1)-1} x^{\beta_1} y^{\beta'_1} \frac{\Gamma(\gamma_1+\beta_1)\,\Gamma(\gamma'_1+\beta'_1)\,\Gamma(\rho+b_1+\beta_1+\beta'_1)}{\Gamma(1+a_1+\beta_1+\beta'_1)}$$

$$\cdot F_1\left[\rho+b_1+\beta_1+\beta'_1;\ \gamma_1+\beta_1,\ \gamma'_1+\beta'_1;\ 1+a_1+\beta_1+\beta'_1;\ -\frac{x}{\sin^2\theta},\ -\frac{y}{\sin^2\theta} \right],$$

$0 \leqslant \theta < \pi$.

$$(3.2) \qquad \sum_{r=0}^{\infty} G^{1,\,2,\,2,\,1,\,1}_{1,\,2,\,2,\,1} \left[\begin{matrix} x \\ \\ y \end{matrix} \,\middle|\, \begin{matrix} \frac{1}{2}-\rho-a_1 \\ \gamma_1;\ \gamma'_1:\ \gamma_2;\ \gamma'_2 \\ 1+\rho+a_1+r,\ \rho+a_1 \\ \beta_1;\ \beta'_1:\ \beta_2;\ \beta'_2 \end{matrix} \right] \sin(2r+1)\,\theta$$

$$= \frac{\sqrt{\pi}}{2} (\sin\theta)^{2(a_1-\rho-\beta_1-\beta'_1)-1} x^{\beta_1} y^{\beta'_1} \frac{\Gamma(\gamma_1+\beta_1)\,\Gamma(\gamma_2+\beta_1)\,\Gamma(\gamma'_1+\beta'_1)\,\Gamma(\gamma'_2+\beta'_1)}{\Gamma(1+a_1+\beta_1+\beta'_1)}$$

$$\cdot F_3\left[\gamma_1+\beta_1,\ \gamma'_1+\beta'_1,\ \gamma_2+\beta_1,\ \gamma'_2+\beta'_1;\ 1+a_1+\beta_1+\beta'_1;\ -\frac{x}{\sin^2\theta},\ -\frac{y}{\sin^2\theta} \right],$$

where $0 \leqslant \theta \leqslant \pi$.

$$(3.3) \qquad \sum_{r=0}^{\infty} G^{n_1+1,\,0,\,0,\,1,\,1}_{p+2,\,0,\,s+2,\,1} \left[\begin{matrix} x \\ \\ y \end{matrix} \,\middle|\, \begin{matrix} \frac{1}{2}-\rho-a_1,\ (\varepsilon_p),\ 1-\rho-b_1 \\ \cdots\cdots\cdots\cdots \\ (\delta_s),\ \rho+a_1,\ 1+\rho+a_1+r \\ \beta_1;\ \beta'_1 \end{matrix} \right] \sin(2r+1)\,\theta$$

$$= \frac{\sqrt{\pi}}{2} (\sin\theta)^{2(a_1-\rho-\beta_1-\beta'_1)-1} x^{\beta_1} y^{\beta'_1} \cdot$$

$$\cdot \frac{\prod_{j=1}^{n_1} \Gamma(1-\varepsilon_j+\beta_1+\beta'_1)\,[\Gamma(1+a_1+\beta_1+\beta'_1)]^{-1}}{\prod_{j=1}^{s} \Gamma(\delta_j+\beta_1+\beta'_1) \prod_{j=n_1+1}^{p} \Gamma(\varepsilon_j-\beta_1-\beta'_1)\,\Gamma(1-\rho-b_1-\beta_1-\beta'_1)}$$

$$ {}_{p+1}F_{s+1}\left[\begin{matrix} (1-\varepsilon_p+\beta_1+\beta'_1),\ \rho+b_1+\beta_1+\beta'_1 \\ (\delta_s+\beta_1+\beta'_1),\ 1+a_1+\beta_1+\beta'_1 \end{matrix};\ (-)^{n_1-p}(x+y)\sin^2\theta \right],$$

where $0 \leqslant \theta \leqslant \pi$ and $p+1 \leqslant s+2$.

Similarly, we can obtain the cosine series in terms of F_1, F_3 and ${}_pF_s$, etc. Besides these particular cases, many more can be obtained as, $G[{}^x_y]$ is a generalized function which yields many particular cases as mentioned in Section 1.

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SERII FOURIER ALE G-FUNCŢIEI LUI MEIJER DE ORDIN SUPERIOR

Rezumat

Pornind de la formula (1.1) a unei G-funcţii de ordin superior autorul stabileşte în § 2 seriile Fourier pentru acestea, iar în § 3 dă unele cazuri particulare ale acestora.

SUM OF A SERIES INVOLVING LAGUERRE POLYNOMIALS AND GENERALIZED FUNCTIONS OF TWO VARIABLES

BY

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1. Introduction. In a recent paper [5] the author has defined generalized function of two variables as follows:

$$S\left[x, y \,\middle\|\, \begin{bmatrix} m_1, & 0 \\ p_1, & q_1 \end{bmatrix} \begin{matrix} a_{p_1} \\ b_{q_1} \end{matrix} \middle| \begin{pmatrix} n_2, & m_2 \\ p_2, & q_2 \end{pmatrix} \begin{matrix} c_{p_2} \\ d_{q_2} \end{matrix} \middle| \begin{pmatrix} n_3, & m_3 \\ p_3, & q_3 \end{pmatrix} \begin{matrix} e_{p_3} \\ f_{q_3} \end{matrix} \right\|\right] =$$

$$= \frac{1}{(2\pi i)^2} \int_{c_1}\int_{c_2} \frac{\prod_{j=1}^{m_1} \Gamma(a_j+s+t) \prod_{j=1}^{m_2} \Gamma(1-c_j+s)}{\prod_{j=m_1+1}^{p_1} \Gamma(1-a_j-s-t) \prod_{j=1}^{q_1} \Gamma(b_j+s+t)} \times \tag{1}$$

$$\frac{\prod_{j=1}^{n_2} \Gamma(d_j-s) \prod_{j=1}^{m_3} \Gamma(1-e_j+t) \prod_{j=1}^{n_3} \Gamma_j(f-t) x^s y^t}{\prod_{j=m_2+1}^{p_2} \Gamma(c_j-s) \prod_{j=n_2+1}^{q_2} \Gamma(1-d_j+s) \prod_{j=m_3+1}^{p_3} \Gamma(e_j-t) \prod_{j=n_3+1}^{q_3} \Gamma(1-f_j+t)} \, ds\, dt,$$

where c_1 and c_2 are two suitable contours and the positive integers p_1, p_2, p_3, q_1, q_2, q_3, m_1, m_2, m_3, n_2 and n_3 satisfy the following inequalities: $q_2 \geq 1$, $q_3 \geq 1$, $p_1 \geq 0$, $q_1 \geq 0$, $0 \leq m_1 \leq p_1$, $0 \leq m_2 \leq p_2$, $0 \leq n_2 \leq q_2$, $0 \leq m_3 \leq p_3$, $0 \leq n_3 \leq q_3$, $p_1 + p_2 \leq q_1 + q_2$, $p_1 + p_3 \leq q_1 + q_3$. The values $x = 0$, $y = 0$ are excluded.