ON A GENERALIZATION OF THE HOPF FIBRATION

BY

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1. In [2] a generalization of the Hopf fibration $\pi' : S^{2n+1} \to PC^n$ was introduced as a canonical example of a manifold with an f-structure and some of its geometric properties were studied. Let Δ denote the diagonal map; then define a principal toroidal bundle H^{2n+s} over PC^n by the following diagram.

$$\begin{tikzcd} H^{2n+s} \arrow[r, "\hat{\Delta}"] \arrow[d, "\pi"'] & S^{2n+1} \times \cdots \times S^{2n+1} \arrow[d, "\pi' \times \cdots \times \pi'"] \\ PC^n \arrow[r, "\Delta"] & PC^n \times \cdots \times PC^n \end{tikzcd}$$

We first show that for s even, H^{2n+s} carries a complex structure but cannot carry a Kaehler structure. Similarly for s odd, H^{2n+s} carries a natural normal almost contact structure which is not quasi-Sasakian and cannot carry a cosymplectic structure nor a Sasakian structure if $s > 1$. Finally we note that $H^{2n+s} \times H^{2m+t}$, s, t of the same parity, carries a complex structure but cannot carry a Kaehler structure.

Section 2 reviews the notion of an f-structure introduced by K. Yano [10] and section 3 discusses the geometry of H^{2n+s}. The main results are then given in section 4.

2. An *f-structure* on a C^∞ manifold M^{2n+s} is a tensor field f of type (1,1) and rank $2n$ satisfying $f^3+f=0$; the existence of an f-structure is equivalent to a reduction of the structural group of the tangent bundle to $\mathcal{U}(n)\times\mathcal{O}(s)$ (K. Yano [10]). Almost complex ($s=0$) and almost contact ($s=1$) structures are well-known examples of f-structures. An f-structure with $s=2$ has arisen in the study of hypersurfaces in almost contact spaces [4]; this structure has been further studied by S. I. Goldberg and K. Yano [5].

If there exists on M^{2n+s} vector fields ξ_x, $x=1,\dots,s$ such that if η_x are dual 1-forms,

$$\eta_x(\xi_y)=\delta_{xy},$$

$$f\xi_x=0,\quad \eta_x\circ f=0,$$

$$f^2=-I+\sum \xi_x\otimes\eta_x,$$

we say that the f-structure has *complemented frames*. If M^{2n+s} has an f-structure with complemented frames, then there exists on M^{2n+s} a Riemannian metric g such that

$$g(X,Y)=g(fX,fY)+\sum \eta_x(X)\,\eta_x(Y),$$

where X, Y are vector fields on M^{2n+s} [10] and we say M^{2n+s} has a *metric f-structure*. Define the *fundamental 2-form* F by

$$F(X,Y)=g(X,fY).$$

Further we say an f-structure is *normal* if it has complemented frames and

$$[f,f]+\sum \xi_x\otimes d\eta_x=0$$

where $[f,f]$ is the Nijenhuis torsion of f [8].

It is also shown in [8], that if an f-structure is normal, then $[\xi_x,\xi_y]=0$ and $\mathcal{L}_{\xi_x}f=0$ where $\mathcal{L}$ denotes Lie differentiation. Finally a metric f-structure which is normal and has closed fundamental 2-form is called a *$\mathcal{K}$-structure* and M^{2n+s} a *$\mathcal{K}$-manifold* [2].

In [2] it is shown that on a $\mathcal{K}$-manifold the vector fields $\xi_1,\dots,\xi_s$ are Killing.

Let M^{2n+s} be a Riemannian manifold with global linearly independent 1-forms $\eta_1,\dots,\eta_s$ such that $d\eta_1=\dots=d\eta_s$ and

$$\eta_1\wedge\cdots\wedge\eta_s\wedge(d\eta_x)^n\neq 0.$$

Let $L(m)=\{X\in M^{2n+s}_m,\ m\in M^{2n+s}\mid \eta_x(X)=0,\ x=1,\cdots,s\}$ then L determines a distribution which together with its complement reduces the structural group to $\mathcal{O}(2n)\times\mathcal{O}(s)$. Now if $\xi_1,\cdots,\xi_s$ are vector fields dual to $\eta_1,\cdots,\eta_s$ and $X_1,\cdots,X_{2n}$ linearly independent vector fields in L, then

$$(\eta_1\wedge\cdots\wedge\eta_s\wedge(d\eta_x)^n)(\xi_1,\cdots,\xi_s,X_1,\cdots,X_{2n})=(d\eta_x)^n(X_1,\cdots,X_{2n})\neq 0$$

giving L a symplectic structure. Thus the structural group can be reduced to $\mathcal{U}(n)\times\mathcal{O}(s)$ and M^{2n+s} has a metric f-structure with complemented frames $\eta_1,\cdots,\eta_s$ and fundamental 2-form $F=d\eta_x$. If this structure is a $\mathcal{K}$-structure, we will call it an *$\mathcal{S}$-structure.*

3. In [2] the following result was obtained.

Proposition. *Let M^{2n+s} be the bundle space of a principal toroidal bundle over a Kaehler manifold N^{2n} and let $\gamma=(\eta_1,\cdots,\eta_s)$ be a Lie algebra valued connexion form on M^{2n+s} such that $d\eta_x=\pi^*\Omega$, $x=1,\cdots,s$ where π is the projection map and Ω the fundamental 2-form on N^{2n}. Then M^{2n+s} is an $\mathcal{S}$-manifold.*

It is well-known that the canonical example of a normal contact metric (Sasakian) manifold, the odd-dimensional sphere S^{2n+1}, is a circle bundle over complex projective space PC^n by the Hopf-fibration. Let $\pi'\colon S^{2n+1}\to PC^n$ denote the Hopf-fibration, then using the diagonal map Δ we define a principal toroidal bundle over PC^n by the following diagram

$$\begin{tikzcd}
H^{2n+s} \arrow[r, "\widehat{\Delta}"] \arrow[d, "\pi"'] & S^{2n+1}\times\cdots\times S^{2n+1} \arrow[d, "\pi'\times\cdots\times\pi'"] \\
PC^n \arrow[r, "\Delta"] & PC^n\times\cdots\times PC^n
\end{tikzcd}$$

that is $H^{2n+s}=\{(p_1,\cdots,p_s)\in S^{2n+1}\times\cdots\times S^{2n+1}\mid \pi'(p_1)=\cdots\pi'(p_s)\}$.

Now let η'_x be the contact form on S^{2n+1}_x and define η_x on H^{2n+s} by $\eta_x=\widehat{\Delta}^*|_{S^{2n+1}_x}\eta_x\equiv\widehat{\Delta}^*_x\eta'_x$. Then

$$d\eta_x=d\widehat{\Delta}^*_x\eta'_x=\widehat{\Delta}^*_x d\eta'_x=\widehat{\Delta}^*_x\pi'^*_x\Omega_x=\pi^*\Delta^*_x\Omega_x=\pi^*\Omega,$$

where Ω_x is the fundamental 2-form on PC^n_x and Ω that on PC^n. Further $\gamma=(\eta_1,\cdots,\eta_s)$ is equivariant and fibre preserving, hence by the Proposition the space H^{2n+s} is an $\mathcal{S}$-manifold.

Recall that PC^n has constant holomorphic sectional curvature 1 (Fubini-Study metric) and that S^{2n+1} (as a Sasakian manifold with the constant curvature metric) has constant curvature $\frac{1}{4}$. In [2] the notion of f-sectional curvature was introduced and it was shown that H^{2n+s} has constant f-sectional curvature $1-\frac{3s}{4}$. Moreover for $X,Y\in L(m)$, $m\in H^{2n+s}$, $s\geq 2$, the sectional curvatures $K(X,Y)$ satisfy $1-\frac{3s}{4}\leq K(X,Y)\leq\frac{1}{4}$.

Finally we note that since $H^{2n+s} = \{(p_1, \cdots, p_s) \in S^{2n+1} \times \cdots \times S^{2n+1} | \pi'(p_1) = \cdots = \pi'(p_s)\}$, π' being the Hopf-fibration, H^{2n+s} is diffeomorphic to $S^{2n+1} \times T^{s-1}$ where T^{s-1} denotes the $(s-1)$-dimensional torus.

4. Let M^{2n+s} be a manifold with a normal f-structure; following S. I. Goldberg and K. Yano [6] we can define a tensor field $\tilde{f}$ of type (1,1) giving rise to a complex or normal almost contact structure on M^{2n+s}. $\tilde{f}$ was introduced locally by T. Ogata [9].

We now show that for s even H^{2n+s} is a complex manifold which is not Kaehlerian.

Proposition 4.1. *Let (f, ξ_x, η_x), $x = 1, \cdots, s$ denote the f-structure of section 3 on H^{2n+s} and suppose s is even. Then*

$$\tilde{f} = f + \sum_{i=1}^{s/2} (\eta_i \otimes \xi_{i*} - \eta_{i*} \otimes \xi_i), \; i^* = i + s/2,$$

defines a (non-Kaehlerian) complex structure on H^{2n+s}.

Proof: The proof is a computation showing that $\tilde{f}^2 = -I$ and $[\tilde{f}, \tilde{f}] = 0$. Since $\tilde{f}X = fX + \sum_{i=1}^{s/2} (\eta_i(X) \xi_{i*} - \eta_{i*}(X)\xi_i)$, we have

$$\tilde{f}^2 X = f^2 X + \sum_{i=1}^{s/2} (-\eta_{i*}(X)\xi_{i*} - \eta_i(X)\xi_i) = -X.$$

The proof that $[\tilde{f}, \tilde{f}] = 0$ is more complicated and is done by cases. First ince $\tilde{f}\xi_x = \pm \xi_{x*}$ and $[\xi_x, \xi_y] = 0$, $x, y = 1, \cdots, s$, $[\tilde{f}, \tilde{f}](\xi_x, \xi_y) = -[\xi_x, \xi_y] \pm [\xi_{x*}, \xi_{y*}] \pm \tilde{f}[\xi_x, \xi_{y*}] \pm \tilde{f}[\xi_{x*}, \xi_y] = 0$.

Now let $X \in L$, then

$$\begin{aligned}
[\tilde{f}, \tilde{f}](X, \xi_i) &= -[X, \xi_i] + [fX, \xi_{i*}] - \tilde{f}[X, \xi_{i*}] - \tilde{f}[fX, \xi_i] \\
&= -[X, \xi_i] - f[\xi_{i*}, X] \\
&\quad - f[X, \xi_{i*}] - \textstyle\sum_j (\eta_j([X, \xi_{i*}])\xi_{j*} - \eta_{j*}([X, \xi_{i*}])\xi_j) \\
&\quad - f[fX, \xi_i] - \textstyle\sum_j (\eta_j([fX, \xi_i])\xi_{j*} - \eta_{j*}([fX, \xi_i])\xi_j) \\
&= 0
\end{aligned}$$

using $\mathfrak{L}_{\xi_x} f = 0$ and the coboundary formula $0 = d\eta_x(X, \xi_{i*}) = -\eta([X, \xi_{i*}])$. Similarly $[\tilde{f}, \tilde{f}](X, \xi_{i*}) = 0$. Finally for $X, Y \in L$,

$$\begin{aligned}
[\tilde{f}, \tilde{f}](X, Y) &= -[X, Y] + [fX, fY] - \tilde{f}[X, fY] - \tilde{f}[fX, Y] \\
&= [f, f](X, Y) - \textstyle\sum_x \eta_x([X, Y])\xi_x - \sum_i (\eta_i([X, fY])\xi_{i*} \\
&\quad - \eta_{i*}([X, fY])\xi_i) - \textstyle\sum_i (\eta_i([fX, Y])\xi_{i*} - \eta_{i*}([fX, Y])\xi_i) \\
&= -\textstyle\sum_x d\eta_x(X, Y)\xi_x + \sum_x d\eta_x(X, Y)\xi_x + \sum_i d\eta_i(X, fY)\xi_{i*} \\
&\quad - \textstyle\sum_i d\eta_{i*}(X, fY)\xi_i + \sum_i d\eta_i(fX, Y)\xi_{i*} - \sum_i d\eta_{i*}(fX, Y)\xi_i
\end{aligned}$$

where we have used normality, the coboundary formula and the fact that $d\eta_x(fX, Y) = -d\eta_x(X, fY)$.

We note that the metric g is Hermitian for $\tilde{f}$, i. e.

$$g(\tilde{f}X, \tilde{f}Y) = g(fX, fY) + \sum_x \eta_x(X)\eta_x(Y) = g(X, Y).$$

Setting $\tilde{F}(X, Y) = g(X, \tilde{f}Y)$ we see that $\tilde{F} = F - \sum_i \eta_i \wedge \eta_{i*}$. A short computation then shows $d\tilde{F} = \sum_i F \wedge (\eta_i - \eta_{i*})$. Thus $d\tilde{F} \neq 0$ and the Hermitian structure $(\tilde{f}, g)$ is not Kaehlerian. More generally we prove the following theorem:

Theorem 4.2. *The space H^{2n+s} for s even cannot carry a Kaehler structure.*

Proof. Since H^{2n+s} is diffeomorphic to $S^{2n+1} \times T^{s-1}$ we see that the first betti number is $\binom{s-1}{1} = s - 1$ which is odd. Thus since the odd dimensional betti numbers of a compact Kaehler manifold are even, H^{2n+s} cannot be Kaehlerian.

We remark that the 1-forms $\eta_x - \eta_y$ are harmonic and $s-1$ of them are linearly independent. Clearly $d(\eta_x - \eta_y) = d\eta_x - d\eta_y = 0$ and $\delta\eta_x = 0$ since η_x is Killing.

We consider next the space H^{2n+s} with s odd, say $s = 2a + 1$. Label the η_x's, $\eta_0, \eta_i, \eta_{i*}$, $i = 1, \cdots, a$ or η_0, η_A, $A = 1, \cdots, 2a$ and similarly for the ξ_x's.

Theorem 4.3. *On H^{2n+s} with s odd, the tensors ξ_0, η_0 and*

$$\tilde{f} = f + \sum_{i=1}^{a} (\eta_i \otimes \xi_{i*} - \eta_{i*} \otimes \xi_i)$$

define a normal almost contact structure which is not quasi-sasakian.

Proof. First note that

$$\tilde{f}^2 X = f^2 X + \sum_{i=1}^{a} (-\eta_{i*}(X)\xi_{i*} - \eta_i(X)\xi_i) = -X + \eta_0(X)\xi_0$$

while $\eta_0(\xi_0) = 1$, $\tilde{f}\xi_0 = 0$, $\eta_0 \circ \tilde{f} = 0$ are immediate. The proof that this structure is normal is a matter of showing that $[\tilde{f}, \tilde{f}] + \xi_0 \otimes d\eta_0 = 0$. This

is a lengthy computation analogous to the one given in the even-dimen-
sional case. As before we see that the metric g is compatible with the
structure, i. e.

$$g(X, \xi_0) = g(-\tilde{f}^2 X + \eta_0(X)\,\xi_0, \xi_0) = \eta_0(X)$$

$$g(\tilde{f}X, \tilde{f}Y) = g(fX, fY) + \sum_{A=1}^{2a} \eta_A(X)\,\eta_A(Y) = g(X, Y) - \eta_0(X)\,\eta_0(Y).$$

we again have $\tilde{F} = F - \sum_i \eta_i \wedge \eta_{i*}$ and $d\tilde{F} =$
Setting $\tilde{F}(X, Y) = g(X, \tilde{f}Y)$ we mean a
$= \sum_i F \wedge (\eta_i - \eta_{i*}) \neq 0$. Now by a *quasi-Sasakian structure* whose fundamental 2-form is clo-
normal almost contact metric structure on H^{2n+s} is not quasi-Sasakian.
sed [1]. Thus the structure $(\tilde{f}, \xi_0, \eta_0, g)$ has rank $2n+1$, i. e. $\eta_0 \wedge (d\eta_0)^n \neq 0$,
We do note here that η_0
$(d\eta_0)^{n+1} = 0$.

Although the above structure on H^{2n+s} is not quasi-Sasakian, H^{2n+s}
does carry a quasi-Sasakian structure. Since H^{2n+s}, s odd is diffeomorphic
to $S^{2n+1} \times T^{s-1}$, we consider H^{2n+s} as the product of the Sasakian mani-
fold S^{2n+1} and the Kaehler manifold T^{s-1}. These structures naturally in-
duce a quasi-Sasakian structure on $S^{2n+1} \times T^{s-1}$ [1]. We do show however
that H^{2n+s}, s odd cannot carry a cosymplectic structure, i. e. a quasi-Sa-
sakian structure such that its (almost) contact form η is closed. Moreover
H^{2n+s}, s odd > 1 cannot carry a Sasakian structure.

Theorem 4.4. *The space H^{2n+s}, $s = 2a+1$, cannot carry a cosym-
plectic structure.*

Proof. Suppose H^{2n+s} is cosymplectic, then its fundamental 2-form Φ
is harmonic and $\eta \wedge \Phi^{n+a}$ is a volume element [3]. By de Rham's theo-
rem Φ determines an element in $H^2(H^{2n+s}, R)$, but since H^{2n+s} is diffeo-
morphic to $S^{2n+1} \times T^{s-1}$ its cohomology is given by that of T^{s-1} by the
Kunneth theorem. Hence $\Phi^{a+1} = 0$, a contradiction.

Theorem 4.5. *The space H^{2n+s}, s odd, $s \geqslant 3$, cannot carry a Sasa-
kian structure.*

Proof. Using the $\mathcal{S}$-structure (f, ξ_x, η_x, g) of section 3 on H^{2n+s}, we
see that since η_x is Killing, $d\eta_x(X, Y) = 2(\nabla_X \eta_x)(Y)$ where ∇ denotes Rie-
mannian covariant differentiation. Now since $d\eta_x = d\eta_y$, $x \neq y$ for an $\mathcal{S}$-
structure, $\eta_x - \eta_y$ is a non-trivial covariant constant 1-form on H^{2n+s}. How-
ever it is shown in [3] that on a compact Sasakian manifold M^{2N+1} there
are no non-trivial covariant constant p-forms for $1 \leqslant p \leqslant 2N$. Thus H^{2n+s}
cannot be Sasakian.

In [7] A. Morimoto proved that the product of two normal
almost contact manifolds carries a complex structure. We now give a gene-
ralization of the Calabi-Eckmann manifolds.

Theorem 4.6. *$H^{2n+s} \times H^{2m+t}$, $s = 2a + c$, $t = 2b + c$, $c = 0,1$ has a
complex structure. Moreover $H^{2n+s} \times H^{2m+t}$ cannot carry a Kaehler structure.*

Proof. The first statement is an immediate consequence of Proposi-
tion 4.1 for $c=0$ and of Theorem 4.3 and the Morimoto result for $c=1$.
Now suppose $M = H^{2n+s} \times H^{2m+t}$ is Kaehlerian, then, as is well-known its
fundamental 2-form Ω is harmonic and Ω^N, $N = n + m + (s+t)/2$, is a
volume element. By de Rham's theorem Ω determines an element in $H^2(M, R)$.
Now M is diffeomorphic to $S^{2n+1} \times S^{2m+1} \times T^e$, $e = 2(a+b) - 2$ or $2(a+b)$
according as $c = 0$ or 1. By the Kunneth theorem the second cohomology
group of M is given by that of T^e. Thus $\Omega^{(e/2)+1} = 0$, a contradiction.

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ASUPRA UNEI GENERALIZĂRI A FIBRĂRII LUI HOPF

Rezumat

În [2] s-a introdus următoarea generalizare a fibrării $\pi' : S^{2n+1} \to PC^n$
a lui Hopf. Se defineşte o fibrare principală toroidală H^{2n+s} peste PC^n
prin diagrama din § 1 în care Δ este aplicaţia diagonală.

În prezenta Notă se obţin anumite proprietăţi geometrice ale lui
H^{2n+s} şi se arată că pentru s par, H^{2n+s} are o structură complexă, iar pen-
tru s impar are o structură aproape de contact normală. Se arată apoi că
$H^{2n+s} \times H^{2n+t}$, unde s şi t sînt impari, are o structură complexă. Aceste spaţii
generalizează deci varietăţile lui Calabi-Eckmann.