Indeed, the matrix $(a^{i_1}_1\, a^{i_2}_2 \ldots a^{i_n}_n)$ can be reduced to a triangular matrix by means of left and right multiplication by permutation matrices P_{ij} for which $d(P_{ij}) = -1$ or $\det(P_{ij}) = -1$ and in view of lemma 2 we obtain

$$d(a^{i_1}_1\, a^{i_2}_2 \ldots a^{i_n}_n) = \det(a^{i_1}_1\, a^{i_2}_2 \ldots a^{i_n}_n), \quad \forall i_1, i_2, \ldots, i_n;$$

hence,

$$d(A) = \det(A).$$

Corollary. *A functional definition of the determinant for matrices over a general commutative ring with unity is obtained by the following axioms:*

1) d is a multiplicative morphism from $R^{n,n}_c$ onto R_c,

2) $d(a_1 \ldots ca_p + v_p \ldots a_n) = c\,.\,d(a_1 \ldots a_p \ldots a_n) + d(a_1 \ldots v_p \ldots a_n)$,

$\forall c \in R_c$, $\forall p \in \{1, 2, \ldots, n\}$ *and* $\forall v_p \in R^{n,1}_c$.

REFERENCES

1. Hitotumatu S. — *A characterization of the determinant,* Comm. Math. Univ. Sti Pauli 13, (1965) 45 — 50.
2. Wolffhardt K. — *Über eine Charakterisierung der Determinante,*-Mathematische Zeitschrift 103, (1968) 259 — 267.
3. Taussky O. and Wielandt H. — *On the role of the determinant in semigroup of matrices,* Quart. J. Math. Oxford, II ser. 14, (1963) 123 — 130.
4. Gaspar G. — *Die Charakterisierung der Determinanten über einen unendlichen Integritätsbereich mittels Functionalgleichungen,* Public. Math. Debrecen 10 (1963) 244 — 253.
5. Bergmann A. — *Ein Axiomensystem für Determinanten,* Arch. Math. vol X, (1959) 243 — 257.

Seminarie voor Algebra
Rijksuniversiteit Gent
Belgium

MORFISME MULTIPLICATIVE ALE INELELOR DE MATRICE

Rezumat

Generalizînd unele rezultate ale lui S. Hitotumatu şi K. Wolffhardt, se demonstrează că dacă R_c este un inel comutativ cu unitate atunci un morfism multiplicativ de la inelul de matrice $R^{n,n}_c$ în R_c, care este o funcţie liniară de coloane, este, în mod necesar determinantul. În acelaşi timp se obţine o definiţie funcţională a determinantului, care a fost dată numai pentru inele comutative speciale.

E-REGULAR SEMIGROUPS (I)

BY

R. J. WARNE

Let E be an idempotent semigroup (a band). The collection $E(R)$ of $\mathfrak{R}$-classes of E may be partially ordered by the following rule: If R_1, $R_2 \in E(R)$, $R_1 < R_2$ if and only if $e < f$ ($e \leq f$ if and only if $ef = fe = e$) for all $e \in R_1$ and $f \in R_2$. If $E(R)$, under this order, is order isomorphic to I^0, the non-negative integers, under the reverse of the usual order, E is called a naturally ordered band. A regular semigroup is called an E-regular semigroup if its idempotents form a naturally ordered band. The purpose of this paper is to describe simple E-regular semigroups mod E-chains of right groups in terms of a simple multiplication. In [7], we use this result to describe simple E-regular semigroups mod groups. In [8], we determine the structure of non-simple E-regular semigroups mod groups.

Naturally ordered bands have played a role in the decomposition theory of regular semigroups [5].

Unless otherwise specified, we use the notation and terminology of [1].

The structure theorem is a consequence of sixteen lemmas.

The following lemma includes a result of W. D. Munn [4, p. 56].

Lemma 1. *Let S be a simple E-regular semigroup. Then, S has a finite number, d, of $\mathfrak{D}$-classes, d is the smallest positive integer such that* $(E_0, E_d) \in \mathfrak{D}$.

Proof. Each $\mathfrak{D}$-class of S is a regular bisimple semigroup [5, lemma 1.1]. By well known results [1], each $\mathfrak{D}$-class of S is a right group or an E-bisimple semigroup. Let D_0 denote the $\mathfrak{D}$-class containing E_0. Suppose that $\mathfrak{D}_0$ is a right group. Let $T = S \setminus_{D_0}$ and let $x \in D_0$ and $y \in T$. Thus, $y \in R_{e_r} \cap L_{f_s}$, say, with $e_r \in E_r$ and $f_s \in E_S$ and $r, s > 0$. We have $x \mathfrak{L} f_0$ for some $f_0 \in E_0$. Thus, $xe_r \mathfrak{L} e_r$ and, hence, xe_r and y are in the same $\mathfrak{D}$-class. Thus, $xy = x(e_r y) = (xe_r) y \in T$. Similarly, $yx \in T$. Next, suppose that $z \in T$, i. e.

$z \in R_{e_n} \cap L_{f_m}$ where $e_n \in E_n$ and $f_m \in E_m$ with $m, n > 0$. Since $y \mathfrak{R} e_r$, $f_m y \mathfrak{R} f_m e_r$. If $m > r$, $f_m y$ and z are the same $\mathfrak{D}$-class and $zy = (zf_m) y = z(f_m y) \in T$. Next, suppose that $r \geqslant m$. Since $z \mathfrak{L} f_m$, $ze_r \mathfrak{L} e_r \mathfrak{R} y$ and $zy = z(e_r y) = (ze_r) y \in T$. Hence, T is an ideal of S and the simplicity of S is contradicted. Thus, D_0 is an E-bisimple semigroup. Let d be the smallest positive integer such that $E_d \subseteq D_0$ and let $e_d \in E_d$. It follows from, [3, lemma 1] and its proof that there exists an isomorphism θ of $e_0 Se_0$ onto $e_d Se_d$ such that $(x, x\theta) \in \mathfrak{D}$ for all $x \in e_0 Se_0$. Thus, $(E_k, E_{d+k}) \in \mathfrak{D}$ for all $k \in I^0$, and, hence, by induction, $(E_k, E_{k+nd}) \in \mathfrak{D}$ for all $n, k \in I^0$. Hence, S has at most d $\mathfrak{D}$-classes. Suppose $(E_i, E_j) \in \mathfrak{D}$ with $0 \leqslant i < j \leqslant d$. Thus, $E_0 \geqslant E_i > E_j > E_d$. Choose $e_i \in E_i$ and $e_j \in E_j$. As above, there exists an isomorphism φ of $e_i Se_i$ onto $e_j Se_j$ such that $(E_{d+(i-j)}, E_{d+(i-j)}\varphi = E_d) \in \mathfrak{D}$. Hence, we have a contradiction and S contains precisely d $\mathfrak{D}$-classes.

Remark 1. From the proof of lemma 1, we may label the $\mathfrak{D}$-classes of S as $D_0, D_1, \ldots, D_{d-1}$ where $E_{D_i} = \{E_{i+nd}\colon n \in I^0\}$. From each E_i, choose an e_i. The $\mathfrak{R}$-classes of S contained in D_i may be labeled $\{R_{e_{i+nd}}\colon n \in I^0\}$ and the $\mathfrak{L}$-classes of S contained in D_i are $\{L_f\colon f \in E_{i+nd}\colon n \in I^0\}$. It is easily seen that each $\mathfrak{R}$-class ($\mathfrak{L}$-class) of S contained in D_i is an $\mathfrak{R}$-class ($\mathfrak{L}$-class) of D_i and conversely. By [5, lemma 1.1], each D_i is an E-bisimple semigroup [6], and, hence, its structure is known mod groups [6, theorem 1.1]. In [6], we defined $t_{(i+nd,\, i+md)} = \bigcup (R_{e_{i+nd}} \cap L_f\colon f \in E_{i+md})$. If $a \in R_e \cap L_f$ where $e, f \in E_S$, there exists a unique inverse $a^{-1} \in R_f \cap L_e$ such that $aa^{-1} = e$ and $a^{-1}a = f$ [1, 2]. This fact will be used repeatedly in the sequel. All lemmas involve simple *E-regular* semigroup.

Lemma 2. *Let* $a \in R_{e_0} \cap L_{e_d}$. *Then,* $e_i a \in t_{(i,\, i+d)}$ *and* $a^{-1} e_i \in t_{(i+d,\, i)}$.

Proof. Since $e_i aa^{-1} e_i = e_i$, $e_i a \in R_{e_i}$ and $a^{-1} e_i \in L_{e_i}$. Thus, $a^{-1} e_i$, $e_i a \in D_i$ and $a^{-1} e_i a = a^{-1} e_i a \in D_i$. Let us suppose that $e_{i+d} > a^{-1} e_i a$. Hence, $ae_{i+d} a^{-1} \geqslant e_i$. Since $a \mathfrak{L} e_d$ and $a^{-1} \mathfrak{R} e_d$, $ae_{i+d} \mathfrak{L} e_{i+d}$ and $e_{i+d} a^{-1} \mathfrak{R} e_{i+d}$. Thus, $ae_{i+d} a^{-1} = (ae_{i+d})(e_{i+d} a^{-1}) \in D_i$ and, hence, $ae_{i+d} a^{-1} = e_i$ or $e_{i+d} = a^{-1} e_i a$. Thus, we have a contradiction and $a^{-1} e_i a \in E_i \cup E_{i+d}$. If $a^{-1} e_i a = f_i \in E_i$, $f_i e_d = f_i$ and we again have a contradiction. Thus, since $a^{-1} e_i \mathfrak{L} e_i$, $a^{-1} e_i a \mathfrak{L} e_i a$ and $e_i a \in t_{(i,\, i+d)}$. Since $e_i a \mathfrak{R} e_i$, $a^{-1} e_i a \mathfrak{R} a^{-1} e_i$ and $a^{-1} e_i \in t_{(i+d,\, i)}$.

Remark 2. By lemma 2 and its proof $(e_i a)^{-1}$, $a^{-1} e_i \in R_{e_{i+d}} \cap L_{e_i}$. Since $a^{-1} e_i$ is an inverse of $e_i a$, $(e_i a)^{-1} = a^{-1} e_i$ by [2, 1, theorem 2.18]. Let $a_i = e_i a$, and, then, $a_i^{-1} = a^{-1} e_i$. Utilizing [6, lemma 1.5], we obtain that $a_i^n = e_i a^n$, for all $n \in I^0$. (We define $a^0 = e_0$ and $a_i^0 = e_i$). By [6, lemma 1.7], if $c \in R_{e_i}$, $(c^{-1})^n = (c^n)^{-1}$ for all positive integers n. We define $c^{-n} = (c^{-1})^n = (c^n)^{-1}$. Thus, we obtain $a_i^{-n} = a^{-n} e_i$ for all $n \in I^0$.

Lemma 3. $t_{(i+kd,\, i+nd)}\, t_{(i+rd,\, i+sd)} \leqslant t_{(i+(k+r-\min(n,r))d,\, i+(n+s-\min(n,r))d)}$.

Proof. The lemma is an immediate consequence of [6, lemma 1.9] and [6, lemma 1.15].

Lemma 4. If $0 \leqslant j < d$ *and* $g \in t_{(j,\, j)}$, *then* $ga^{-r} \in t_{(rd,\, 0)}$ *where* r *is a positive integer.*

Proof. Let us first establish the result for $r = 1$. We have $ag^{-1} ga^{-1} = afa^{-1} = ae_d fa^{-1} = e_0$, $(f \in E_j,\ 0 \leqslant j < d)$. Thus, $ga^{-1} \mathfrak{L} e_0$. Hence, by [6, lemma 1.9] $ga^{-1} = a^{-n} g_0$ for some $n \in I^0$ and $g_0 \in t_{(0,0)}$. Thus, $ga^{-1} e_j = ga_j^{-1} \in t_{(j+d,\, j)}$ by remark 2 and [6, lemma 1.10]. Hence, $a^{-n} g_0 e_j \in t_{(j+d,\, j)}$. Since $g_0 \mathfrak{L} f$ for some $f \in E_0$, $g_0 e_j \mathfrak{L} e_j$, and, hence, $g_0 e_j \in t_{(j+kd,\, j)}$ for some $k \in I^0$ by remark 1. Thus, $a^{-n} g_0 e_j = (a^{-n} e_j)(g_0 e_j) \in t_{(j+(n+k)d,\, j)}$ by lemma 2, remark 2, and lemma 3. Hence, $j + (n+k)d = j + d$ and $n + k = 1$. Suppose $n = 0$ and $k = 1$. Then $ga^{-1} \in t_{(0,0)}$. Thus, $ga^{-1} \mathfrak{R} e_0$, $g^{-1} ga^{-1} \mathfrak{R} g^{-1} e_0$, $fa^{-1} \mathfrak{R} g^{-1} e_0$, and $a^{-1} \mathfrak{R} g^{-1} e_0$. Since the lemma is valid for $j = 0$ by [6, lemma 1.10], we may assume that $j > 0$.

Thus, $e_0 gg^{-1} e_0 = e_j$ and $e_j \in Sg^{-1} e_0$. However, $g^{-1} e_0 e_j = g^{-1} e_j e_0 = g^{-1} e_0$ and $g^{-1} e_0 \in Se_j$. Thus, $g^{-1} e_0 \mathfrak{L} e_j$ and $a^{-1} \in D_j$, a contradiction. Thus, $n = 1$ and $ga^{-1} \in t_{(d,0)}$ by [6, lemma 1.9]. If $r > 1$, $ga^{-r} = ga^{-1} a^{-(r-1)}$ by [6, lemma 1.7], and $ga^{-r} \in t_{(rd,0)}$ by lemma 3.

Lemma 5. *If* $g \in t_{(j+kd,\, j+kd)}$ *and* $r > k$, *then* $ga^{-r} \in t_{(rd,0)}$.

Proof. We first note that

$$a^r g^{-1} ga^{-r} = a^r f_{j+kd} e_{rd} a^{-r} = a^r e_{rd} a^{-r} = a^r a^{-r} = e_0, \text{ where } f_{j+kd} \in E_{j+kd}.$$

Thus, $e_0 \in Sga^{-r}$ and, hence, $ga^{-r} \mathfrak{L} e_0$. Thus, $ga^{-r} \in D_0$. Let us suppose that $ga^{-r} \in t_{(nd,0)}$ for some non-negative integer n. Thus, by [6, lemma 1.9], $ga^{-r} = a^{-n} g_0$ where $g_0 \in t_{(0,0)}$. By remark 2 and [6, lemma 1.10], $ga^{-r} e_j = ga_j^{-r} \in t_{(j+rd,\, j)}$. Clearly, $g_0 e_j g_0 \mathfrak{R} g_0 e_j$. Hence, since $g_0 e_j \mathfrak{L} e_j$, $(g_0 e_j)^2 \mathfrak{R} g_0 e_j$ by lemma 3. Hence, by lemma 3, $g_0 e_j \in t_{(j,j)}$. Thus, $a^{-n}(g_0 e_j) = (a^{-n} e_j)(g_0 e_j) = a_j^{-n}(g_0 e_j) \in t_{(j+nd,\, j)}$ by lemma 3. Thus, $n = r$ and $ga^{-r} \in t_{(rd,0)}$.

Lemma 6. *If* $g \in t_{(j+rd,\, j+rd)}$, *then* $ga^{-r} \in t_{(j+rd,\, j)}$.

Proof. Since $a^{-r} \mathfrak{R} e_{rd}$, $ga^{-r} \mathfrak{R} ge_{rd} = gf_{j+rd} e_{rd} = gf_{j+rd} = g$ where $f_{j+rd} \in E_{j+rd}$. (We may assume that $j > 0$ as the lemma is valid for $j = 0$ by [6, lemma 1.10]. Thus, $ga^{-r} \mathfrak{R} e_{j+rd}$. Hence, $ga^{-r} \in D_j$ and $ga^{-r} \in t_{(j+rd,\, j+nd)}$ for some $n \in I^0$. Thus, by [6, lemma 1.9], $ga^{-r} = a_j^{-r} a_j^n h$ for some $h \in t_{(j+nd,\, j+nd)}$. If $n > 0$, we have $ga_j^{-r} = ga^{-r} e_j$ (remark 2) $= a_j^{-r} a_n^j he_j = a_j^{-r} a_j^n hf_{j+nd} e_j\ (f_{j+nd} \in E_{j+nd}) = a_j^{-r} a_j^n h$.

However, $ga_j^{-r} \in t_{(j+rd,\, j)}$ by [6, lemma 1.10]. Hence, we have a contradiction and $n = 0$. Thus, $ga^{-r} \in t_{(j+rd,\, j)}$ as desired.

Lemma 7. *If* $g \in t_{(j+kd,\, j+kd)}$, $ga^s \in t_{(j+kd,\, j+(k+s)d)}$.

Proof. By [6, lemma 1.10], we may assume that $j > 0$. We have

$$ga^s a^{-s} g^{-1} = ge_0 g^{-1} = ge_0 e_{j+kd} g^{-1} = gg^{-1} = e_{j+kd}.$$

Thus, $e_{j+kd} \in ga^s S$. However, $ga^s \in gS = e_{j+kd} S$ and $ga^s \in R_{e_{j+kd}}$. Hence, $ga^s \in D_j$. We suppose that $ga^s \in t_{(j+kd,\, j+nd)}$ where $n \in I^0$. Hence, $ga^s = a_j^{-k} a_j^n h$ for some $h \in t_{(j+nd,\, j+nd)}$ by [6, lemma 1.9]. However, if $k > 0$,

$$ga^s = gf_{j+kd}\, e_j\, a^s = ge_j\, a^s = ga_j^s \in t_{(j+kd,\, j+(k+s)d)}, \quad (g \in L_{f_{j+kd}}),$$

(Remark 2 and [6, lemma 1.10]).

Thus, if $k > 0$, $n = k + s$. If $k = 0$,

$$ga^s = a_j^n h, \qquad g = a_j^n h a^{-s},$$

$$ge_0 = a_j^n h a_j^{-s}, \quad ge_i = a_j^n h a_i^{-s} \text{ (remark 2)}.$$

We note that $ge_j \in t_{(j,j)}$. However, by lemma 3, $a_j^n h a_j^{-s} \in t_{(j,j+nd)(j+sd,j)}$. Since both the cases $n > s$ and $s > n$ lead to a contradiction, $n = s$ and we again have the desired result.

Lemma 8. *Every element of S may be uniquely expressed in the form $a^{-n} a^k g_{kj}$ where $g_{kj} \in t_{(j+kd,j+kd)}$ with $0 \leqslant j < d$ and $a^{-n} a^k g_{kj} \in t_{(j+nd,j+kd)}$.*

Proof. Let $x \in S$, $x \in D_j$, say. Suppose that $x \in t_{(j+nd,j+kd)}$. Hence, by [6, lemma 1.9], $x = a_j^{-n} a_j^k g_{kj}$ where $g_{kj} \in t_{(j+kd,j+kd)}$. Thus, by remark 2, $x = a^{-n} e_j a^k g_{kj}$. We note that

$$g_{kj}^{-1} a^{-k} a^k g_{kj} = g_{kj}^{-1} f_{kd}\, g_{kj} = g_{kj}^{-1} f_{kd}\, e_{kd+j}\, g_{kj} = g_{kj}^{-1} g_{kj} = f_{j+kd}, \; (g_{kj} \in L_{f_{j+kd}}, f_{j+kd} \in$$

$\in E_{j+kd})$. Thus, $f_{j+kd} \in Sa^k g_{kj}$. However, $a^k g_{kj} \in Sg_{kj} = Sf_{j+kd}$. Hence, $a^k g_{kj}\, \mathfrak{L}\, f_{j+kd}$ and $a^k g_{kj} \in D_j$. Thus, $x = a^{-n} a^k g_{kj}$. By [6, lemma 1.9] and the above, $a^{-n} a^k g_{kj} \in t_{(j+nd,j+kd)}$. We next establish uniqueness. Suppose that $x = a^{-n} a^k g_{kj} = a^{-r} a^s h_{si}$ where $g_{kj} \in t_{(j+kd,j+kd)}$ and $h_{si} \in t_{(i+sd,i+sd)}$. Thus, $j = i$ and, hence, $n = r$, $k = s$ and $g_{kj} = h_{si}$.

Lemma 9. $t_{(k,k)}\, t_{(r,r)} \leqslant t_{(\max(k,r),\, \max(k,r))}$.

Proof. Let $k = i + nd$ and $r = j + md$ where $n, m \in I^0$ and $0 \leqslant i, j < d$. Suppose that $k > r$. Let $g \in t_{(r,r)}$ and $h \in t_{(k,k)}$. Hence, $g\, \mathfrak{L}\, f_{j+md}$ for some $f_{j+md} \in E_{j+md}$. Thus, $ge_{i+nd}\, \mathfrak{L}\, e_{i+nd}$, $ge_{i+nd} \in D_i$, and $ge_{i+nd} \in t_{(i+sd,i+nd)}$ for some $s \in I^0$. Hence, $ge_{i+nd} = a^{-s} a^n h$, where $h \in t_{(i+nd,i+nd)}$, by lemma 8. We have $ga^{-(n+1)} = ge_{i+nd}\, a^{-(n+1)} = a^{-s} a^n (ha^{-(n+1)})$. Hence, by lemma 5 and lemma 3, $ga^{-(n+1)} \in t_{((s+1)d,0)}$. However, $ga^{-(n+1)} \in t_{((n+1)d,0)}$ by lemma 5. Thus, $s = n$ and $ge_{i+nd} \in t_{(i+nd,i+nd)} = t_{(k,k)}$. Hence, $gh = g(e_k h) = (ge_k) h \in t_{(k,k)}$ since $t_{(k,k)}$ is a semigroup by [6, lemma 1.12]. Suppose that $g \in R_{e_r} \cap L_{f_r}$ with $f_r \in E_r$ and $h \in R_{e_k} \cap L_{p_k}$ with $p_k \in E_k$. Hence, there exists a unique inverse g^* of g contained in $R_{e_r} \cap L_{e_r}$ and a unique inverse h^* of h contained in $R_{e_k} \cap L_{e_k}$. Thus, by the above, $g^* h^* \in t_{(k,k)}$. Since hg is an inverse of $g^* h^*$, $hg \in t_{(k,k)}$ by [6, lemma 1.11]. Hence, if $k > r$, $t_{(k,k)}\, t_{(r,r)} \subseteq t_{(k,k)}$ and $t_{(r,r)}\, t_{(k,k)} \subseteq t_{(k,k)}$.

Remark 3. By [6, lemma 1.12], each $t_{(k,k)}$ is a right group and $t_{(i+kd,i+kd)} \cong t_{(i+sd,i+sd)}$ for all $s, k > 0$. Clearly, $E_T = E_S$ where $T = \bigcup (t_{(k,k)} : k \in I^0)$.

For each $s, r \in I^0$, define $g\alpha_{(r,s)} = a^{-r} a^s g a^{-s} a^r$ for $g \in T$.

Lemma 10. *For each $r, s \in I^0$, $\alpha_{(r,s)}$ is an endomorphism of T and*

1) $t_{(i+kd,i+kd)}\, \alpha_{(r,s)} \subseteq t_{(rd,rd)}$ *if* $s > k$,

2) $t_{(i+kd,i+kd)}\, \alpha_{(r,s)} \subseteq t_{(i+(k+r-s)d,\, i+(k+r-s)d)}$ *if* $k \geqslant s$.

$\alpha_{(s,s)}$ is an inner right translation of T determined by an idempotent of $t_{(sd,sd)}$.

Proof. Clearly, $\alpha_{(r,s)}$ is a mapping of T into S. Let $g_k \in t_{(i+kd,i+kd)}$ and $h_q \in t_{(j+qd,j+qd)}$. Hence,

$$g_k \alpha_{(r,s)}\, h_q \alpha_{(r,s)} = (a^{-r} a^s g_k a^{-p} a^r)(a^{-r} a^s h_q a^{-s} a^r) = a^{-r} a^s g_k f_{sd} (h_q a^{-s}) a^r =$$

$$= a^{-r} a^s g_k h_q a^{-s} a^r \text{ (lemma 5)} = (g_k h_q)\, \alpha_{(r,s)}.$$

Hence, $\alpha_{(r,s)}$ is a homomorphism of T into S. If $s > k$, $g_k a^{-s} \in t_{(sd,0)}$ by lemma 5. Thus, $g_k \alpha_{(r,s)} = a^{-r} a^s (ga^{-s}) a^r \in t_{(rd,rd)}$ by lemma 3 and (1) is valid. We next consider the case $k \geqslant s$. We have

$$g_k a^{-s} a^s g_k^{-1} = g_k f_{sd}\, g_k^{-1} = g_k e_{sd}\, e_{i+kd}\, g_k^{-1} = g_k g_k^{-1} = e_{i+kd}, \; (f_{sd} \in E_{sd}).$$

Hence, $e_{i+kd} \in g_k a^{-s} S$. However, $g_k a^{-s} \in g_k S = e_{i+kd} S$ and $g_k a^{-s}\, \mathfrak{R}\, e_{i+kd}$. Similarly, $a^s g_k\, \mathfrak{L}\, f_{i+kd}$ where $f_{i+kd} \in E_{i+kd}$. Thus, $a^s g_k$ and $g_k a^{-s}$ are contained in D_i. Hence, if $g_k a^{-s} \in L_{f_{i+nd}}$ for some $n \in I^0$ and $n > 0$,

$$a^{-r} a^s g_k a^{-s} a^r = a^{-r} e_i a^s g_k a^{-s} f_{i+nd} a^r = a_i^{-r} a_i^s g_k a^{-s} f_{i+nd}\, e_i a^r \text{ (remark 2)}$$

$$= a_i^{-r} a_i^s g_k a_i^{-s} a_i^r.$$

Thus, if $n > 0$ and $k \geqslant s$, $g_k \alpha_{(r,s)} \in t_{(i+(k+r-s)d,\, i+(k+r-s)d)}$ by [6, lemma 1.13]. Suppose $n = 0$. Thus, $g_k a^{-s} \in t_{(i+kd,i)}$. Hence, $g_k a^{-s}\, \mathfrak{L}\, f_i$ and $g_k f_{sd}\, \mathfrak{L}\, f_i a^s$. Thus, since $f_i a^s \in t_{(i,i+sd)}$ by lemma 7, and since $g_k e_{sd} \in t_{(i+kd,i+kd)}$ by lemma 9, $s = k$. Hence, by lemma 7 and lemma 3, $g_k \alpha_{(r,s)} = a_i^{-r} a_i^s (g_k a^{-s})(f_i a^r) \in t_{(i+rd,i+rd)}$. Hence, (2) is valid. In addition, we have shown that $\alpha_{(r,s)}$ is an endomorphism of T. We note that $g_k \alpha_{(s,s)} = a^{-s} a^s (g_k a^{-s}) a^s = gf_{sd}$, for $f_{sd} \in E_{sd}$, by virtue of lemma 5 and, hence, the last statement of the lemma is valid.

Lemma 11. $a^{-k} a^p a^{-r} a^s = a^{-(k+r-\min(p,r))}\, a^{p+s-\min(p,r)}$.

Proof. This is [6, lemma 1.14].

Lemma 12. *If $g_{pi} \in t_{(i+pd,i+pd)}$ and $h_{sj} \in t_{(j+sd,j+sd)}$,*

$$(a^{-k} a^p g_{pi})(a^{-r} a^s h_{sj}) = a^{-(k+r-\min(p,r))}\, a^{p+s-\min(p,r)}$$

$$(g_{pi}\, \alpha_{(s,r)})\, h_{sj} \textit{ and } g_{pi}\, \alpha_{(s,r)}\, h_{sj} \in t_{(j+sd,j+sd)} \textit{ if } r > p;$$

$$t_{(i+(p+s-r)d,\, i+(p+s-r)d)} \textit{ if } p > r; \; t_{(i+sd,i+sd)} \textit{ if } p = r \text{ and } i \geqslant j;$$

$$t_{(j+sd,j+sd)} \textit{ if } p = r \textit{ and } j > i.$$

Proof :

$$(a^{-k} a^{p} g_{pi})(a^{-r} a^{s} h_{sj}) = a^{-k} a^{p} (a^{-r} a^{s} a^{-s} a^{r} (g_{pi} a^{-r}) a^{s}) h_{sj} =$$
$$= a^{-k} a^{p} a^{-r} a^{s} (a^{-s} a^{r} g_{pi} a^{-r} a^{s}) h_{sj} =$$
$$= a^{-(k+r-\min(r,p))} a^{p+s-\min(r,p)} g_{pr} \alpha_{(s,r)} h_{sj}.$$

The last statement of the theorem is a consequence of lemma 10 and 9.

Lemma 13. *The mapping $(n, r) \to \alpha_{(n,r)}$ is an anti-homomorphism of C, the bicyclic semigroup, into the semigroup of endomorphisms of T.*

Proof. Let $g_{ki} \in t_{(i+kd,i+kd)}$. Then, by lemma 11,

$$g_{ki}\, \alpha_{(r,s)}\, \alpha_{(n,p)} = a^{-n} a^{p} (a^{-r} a^{s} g_{ki} a^{-s} a^{r}) a^{-p} a^{n} =$$
$$= a^{-(n+r-\min(r,p))} a^{p+s-\min(r,p)} g_{ki}\, a^{-(s+p-\min(r,p))} a^{r+n-\min(r,p)} =$$
$$= g_{ki}\, \alpha_{((n+r-\min(r,p)),\, p+s-\min(r,p))} = g_{ki}\, \alpha_{(n,p)(r,s)}.$$

Lemma 14. $S \cong \{((n, k), g_{ki}) : g_{ki} \in t_{(i+kd,i+kd)},\ n, k \in I^{0};\ 0 \leqslant i < d\}$ *under the multiplication* $((n, k), g_{ki})((r, s), h_{sj}) = ((n, k)(r, s), g_{ki}\, \alpha_{(s,r)} h_{sj})$ *where juxtaposition denotes multiplication in C and T.*

Proof. We consider the mapping $(a^{-n} a^{k} g_{ki})\varphi = ((n, k), g_{ki})$. By lemma 8, φ is a bijection. By lemma 12, φ is a homomorphism.

Let T be a semigroup which is a disjoint union of right groups $\{T_k : k \in I^{0}\}$ such that $T_k T_r \subseteq T_{\max(k,r)}$ and such that E_T is a naturally ordered band. We call T an E-chain of right groups.

Lemma 15. *Let $T = \{T_{i+sd};\ s \in I^{0},\ 0 \leqslant i < d,\ d$ a positive integer$\}$ be an E-chain of right groups and let C denote the bicyclic semigroup. Let $(n, r) \to \alpha_{(n,r)}$ denote an antihomomorphism of C into the semigroup of endomorphism of T such that for each $s \in I^{0}$, $\alpha_{(s,s)}$ is an inner right translation of T determined by an idempotent of T_{sd} (i.e., for all $g \in T$, $g\alpha_{(s,s)} = ge$ for some $e \in T_{sd}$) and for $k, s, r \in I^{0}$, $T_{i+kd}\, \alpha_{(r,s)} \subseteq T_{rd}$ if $s > k$; $T_{i+kd}\, \alpha_{(r,s)} \subseteq T_{i+(k+r-s)d}$ if $k \geqslant s$. Let $S = \{((n, k), g_{ki}) :\ g_{ki} \in T_{i+kd},\ n, k \in I^{0},\ 0 \leqslant i < d\}$ under the multiplication $((n, k), g_{ki})((r, s), h_{sj}) = ((n, k)(r, s), g_{ki}\, \alpha_{(s,r)} h_{sj})$ where juxtaposition denotes multiplication in C and T. Then, S is a simple E-regular semigroup with d $\mathfrak{D}$-classes.*

Proof. Closure is immediate and associativity is easily verified. We next show that $E_S = \{(k, k), g_{ki}\} :\ g_{ki} \in E_{T_{i+kd}},\ 0 \leqslant i < d,\ k \in I^{0}$. Suppose that $((n, k), g_{ki}) \in E_S$. Hence, $((n, k), g_{ki})((n, k), g_{ki}) = ((n, k)(n, k), g_{ki}\, \alpha_{(k,n)} g_{ki})$. Thus, $n = k$ and $g_{ki}\, \alpha_{(k,k)} g_{ki} = g_{ki}^{2} = g_{ki}$. Conversely, if $g_{ki} \in E_{T_{i+kd}}$, $((k,k), g_{ki})(k, k), g_{ki}) = ((k, k), g_{ki} g_{ki}) = ((k, k), g_{ki})$. Let $((k, k), g_{ki}),\ ((s, s), h_{sj}) \in E_S$. Thus, $((k, k), g_{ki})((s, s), h_{sj}) = ((k, k)(s, s), g_{ki} h_{sj})$ and $((k, k), g_{ki})\varphi = g_{ki}$ defines an isomorphism of E_S onto E_T. To show S is simple, let $((n, k), g_{ki}),\ ((r, s), h_{sj}) \in S$. Let $e_{n+1,j} \in T_{j+(n+1)d}$ with $e_{n+1,j}^{2} = e_{n+1,j}$ and let $e_{n+1,j}\, \alpha_{(k,n)} = e_{k+1,j}^{*} \in T_{j+(k+1)d}$. Choose $t_{sj} \in T_{j+sd}$ such that $(e_{k+1,j}^{*} g_{ki})\, \alpha_{(s,k+1)} t_{sj} = h_{sj}$. Then, $((r, n+1), e_{n+1,j})((n, k) g_{ki})((k+1, s) t_{sj}) = ((r, s), h_{sj})$. Let us now determine the $\mathfrak{D}$-structure of S. We first note that $((n, k), g_{ki})\,\mathfrak{L}\,((r, s), h_{sj})$ if and only if $g_{ki}\,\mathfrak{L}\,h_{sj}$ (in T) (this implies $i = j$) and $k = s$. First suppose that $k = s$ and $g_{ki}\,\mathfrak{L}\,h_{sj}$ (in T). Since $(h_{ki} g_{ki}^{-1})\, \alpha_{(n,k)} \in T_{i+nd}$, let $(h_{ki} g_{ki}^{-1})\, \alpha_{(n,k)} = x_{ni}$. Hence,

$$x_{ni}\, \alpha_{(k,n)} = (h_{ki} g_{ki}^{-1})\, \alpha_{(n,k)}\, \alpha_{(k,n)} = (h_{ki} g_{ki}^{-1})\, \alpha_{(k,k)} = h_{ki} g_{ki}^{-1} e_{kd}$$

for some $e_{kd} \in E_{T_{kd}}$. Thus,

$$((r, n), x_{ni})((n, k), g_{ki}) = ((r, k), x_{ni}\, \alpha_{(k,n)} g_{ki}) =$$
$$= ((r, k), h_{ki} g_{ki}^{-1} g_{ki}) = ((r, k), h_{ki}).$$

Since $(g_{ki} h_{ki}^{-1})\, \alpha_{(r,k)} \in T_{i+rd}$, let $(g_{ki} h_{ki}^{-1})\, \alpha_{(r,k)} = y_{ri}$. Then, as above, $((n, r)\, y_{ri})((r, k), h_{ki}) = ((n, k), g_{ki})$. Hence, $((n, k), g_{ki})\,\mathfrak{L}\,(r, s), h_{sj})$. By a routine calculation, $((n, k), g_{ki})\,\mathfrak{L}\,((r, s), h_{sj})$ implies that $k = s$ and $g_{ki}\,\mathfrak{L}\,h_{sj}$. We next show that $((n, k), g_{ki})\,\mathfrak{R}\,((r, s), h_{sj})$ if and only if $n = r$ and $i = j$. Suppose that $i = j$ and $n = r$. Since $g_{ki}\, \alpha_{(s,k)} \in T_{i+sd}$, choose t_{si} such that $g_{ki}\, \alpha_{(s,k)} t_{si} = h_{si}$. Since $h_{si}\, \alpha_{(k,s)} \in T_{i+kd}$, choose v_{ki} such that $h_{si}\, \alpha_{(k,s)} v_{ki} = g_{ki}$. Hence, $((n, k), g_{ki})((k, s), t_{si}) = ((n, s), h_{si})$, $((r, s), h_{si})((s, k), v_{ki}) = ((r, k), g_{ki})$, and $((n, k), g_{ki})\,\mathfrak{R}\,((r, s), h_{sj})$. Next, suppose that $((n, k), g_{ki})\,\mathfrak{R}\,((r, s), h_{sj})$. Then, there exists $((x, y), z_{yp})$ and $((a, b), t_{bq})$ such that $((n, k), g_{ki})((x, y) z_{yp}) = ((r, s), h_{sj})$ and $((r, s), h_{sj})((a, b), t_{bq}) = ((n, k), g_{ki})$. Thus, $n + x - \min(k, x) = r$ and $r + a - \min(s, a) = n$. Hence, $n = r$. We also have $g_{ki}\, \alpha_{(y,x)} z_{y,p} = h_{sj}$ and $h_{sj}\, \alpha_{(b,a)} t_{bq} = q_{ki}$. If $x > k$, $n + x - k = n$, a contradiction. If $a > s$, $r + a - s = r$, a contradiction. Thus, $k \geqslant x$ and $s \geqslant a$. Hence, $sd + j \geqslant (k + y - x) d + i$ and $kd + i \geqslant (s + b - a)\, d + j$. However, $k + y - x = s$ and $s + b - a = k$. Thus, $sd + j \geqslant sd + i$ and $kd + i \geqslant kd + j$. Hence, $i = j$. Hence, it is easily seen that $((n, k), g_{ki})\,\mathfrak{D}\,((r, s), h_{sj})$ if and only if $i = j$. Thus, the $\mathfrak{D}$-classes of S are $\{D_i\}_{i=0}^{d-1}$ where $D_i = \{((n, k), g_{ki}) :\ n, k \in I^{0},\ g_{ki} \in T_{i+kd}\}$. Since each D_i is a regular $\mathfrak{D}$-class [6], S is a regular semigroup. Let $\beta_{(n,r)} = \alpha_{(r,n)}$.

Theorem: *Let S be a simple E-regular semigroup. Then, there exists a positive integer d, an E-chain T of right groups $\{T_{i+kd} :\ k \in I^{0},\ 0 \leqslant i < d\}$, a homomorphism $(n, r) \to \beta_{(n,r)}$ of C, the bicyclic semigroup, into the semigroup of endomorphisms of T such that*

1) *For each $k \in I^{0}$, there exists $e_{kd} \in E_{T_{kd}}$ such that $g\beta_{(k,k)} = ge_{kd}$ for all $g \in T$;*

2) *For each $k, r, s \in I^{0}$,*

$$T_{i+kd}\, \beta_{(s,r)} \subseteq \begin{matrix} T_{rd} & \text{if } s > k, \\ T_{i+(k+r-s)d} & \text{if } k \geqslant s. \end{matrix}$$

Furthermore, $S \cong \{((n, k), g_{ki}) :\ g_{ki} \in T_{i+kd},\ n, k \in I^{0},\ 0 \leqslant i < d\}$ under the multiplication;

3) *$((n, k), g_{ki})((r, s), h_{sj}) = ((n, k)(r, s), g_{ki}\, \beta_{(r,s)} h_{sj})$ where juxtaposition denotes multiplication in C and T. S has precisely d $\mathfrak{D}$-classes.*

Conversely, let d be a positive integer and let T be an E-chain of right groups $\{T_{i+kd}: k \in I^0, 0 \leqslant i < d\}$ *and* $(n,r) \to \beta_{(n,r)}$ *be a homomorphism of C into the semigroup of endomorphisms of T such that* (1) *and* (2) *are valid. Then* $\{((n,k), g_{ki}): g_{ki} \in T_{i+kd}, n, k \in I^0, 0 \leqslant i < d\}$ *under the multiplication* (3) *is a simple E-regular semigroup with d $\mathfrak{D}$-classes.*

Proof. The theorem is valid by lemma 9, remark 3, lemma 10, lemma 13, lemma 14, lemma 1, remark 1 and lemma 15.

REFERENCES

1. Clifford A. H. and Preston G. B. — *The algebraic theory of semigroups*. Math. Surveys of the American Math. Soc., 7, Vol. I, Providence, R. I., 1961.
2. Miller D. D. and Clifford A. H. — *Regular D-classes in semigroups*. Trans. Amer. Math. Soc. 82 (1956), pp. 270—280.
3. Munn W. D. — *The idempotent separating congruences on a regular o-bisimple semigroup*. Proc. Edinburgh Math. Soc. 15 (1967), pp. 233—240.
4. Munn W. D. — *Regular ω-semigroups*. Glasgow Math. Journal, 9 (1968), pp. 46—66.
5. Warne R. J. — *Regular D-classes whose idempotents obey certain conditions*. Duke Math. Journal, 33 (1966), pp. 187—195.
6. Warne R. J. — *E-bisimple semigroups*, Journal of Natural Sciences, ord. Math., 10, № 2 (1970).
7. Warne R. J. — *E-regular-semigroups*, II, to appear.
8. Warne R. J. — *E-regular-semigroups, III*, to appear in La Ricerca.

SEMIGRUPURI *E*-REGULATE

Rezumat

În lucrare sînt studiate semigrupurile simple *E*-regulate. Rezultatul principal îl constituie o teoremă de structură pentru această categorie de semigrupuri.

Rezultatul este utilizat de către autor într-o altă Notă aflată sub tipar, în care sînt studiate semigrupurile simple *E*-regulate modulo un grup. Din acest punct de vedere prezenta lucrare constituie o verigă a unui lanţ de Note, unele deja apărute, consacrate studiului semigrupurilor *E*-regulate.

RELATIONS ENTRE LES FONCTIONS ET LES SUITES PRESQUE-AUTOMORPHES

PAR

ANCA PRECUPANU

Communication présentée à la session scientifique jubilaire „A. Myller", Jassy, 20—25 août 1970

Le but de cette Note est de mettre en évidence deux propriétés de liaison entre les fonctions et les suites presque-automorphes, en généralisant les théorèmes 1.27 et 1.28 de [1], donnés par N. Gheorghiu [3] pour les fonctions presque-périodiques.

Dans ce qui suit, nous utiliserons les notations et les notions générales concernant la théorie des fonctions presque-automorphes (abrégé p. a.) de [4].

On désigne par: R — l'ensemble des nombres réels, Z — l'ensemble des nombres entiers, G — un sous-groupe de R, C — l'ensemble des nombres complexes.

Si $f: G \to C$ est une fonction sur G avec des valeurs complexes, on désigne par ${}_{\alpha}f(x) = f(x+\alpha)$, une translatée de f. Si $\alpha = \{\alpha_i\}_{i\in\Gamma}$ est une suite généralisée de $G \subseteq R$ telle que $\lim\limits_{i\in\Gamma} {}_{\alpha_i}f = g$ au sens de la convergence ponctuelle sur G, alors on la désigne par $g = T_{\alpha} f$ et de même s'il existe $\lim\limits_{i\in\Gamma} {}_{-\alpha_i}g = h$ (ponctuellement sur G) on la désigne par $h = T_{-\alpha} g$.

Définition 1. *On dit qu'une fonction $f: G \to C$ est une fonction presque-automorphe au sens de Bochner, si chaque suite généralisée $\alpha' = \{\alpha'_i\}_{i\in\Gamma'}$, d'éléments de G contient une sous-suite généralisée $\alpha = \{\alpha_i\}_{i\in\Gamma}$ telle que:*

$$T_{-\alpha} T_{\alpha} f = f \tag{1}$$

soit remplie sur G.

En particulier, si $G = Z$, on obtient la définition d'une suite presque-automorphe au sens de Bochner.