GLOBAL RESULTS AND STABILITY OF MOTION[1])

BY

V. LAKSHMIKANTHAM and S. LEELA

Presented at the „A. Myller" Memorial Scientific Session, Iaşi, August 20–25, 1970

1. Introduction: The proofs of many results in the theory of stability and boundedness basically depend on dividing the vicinity of some kind of invariant set (or other convenient set) into suitable subsets and then trying either to prove that solutions cannot leave such sets or to estimate the escape time. This observation makes it possible to give some global results in terms of arbitrary sets which can be employed as tools in dealing with various problems of stability and boundedness. In applications, these tools enlarge the class of useful Lyapunov-like functions and also offer more flexibility.

In this paper, we shall first discuss two global theorems of a very general character by means of several Lyapunov functions and the theory of differential inequalities. We then give, as applications, a variety of results on the stability and boundedness of motions. Also, introducing a new notion called restrictive stability which is of practical interest in certain situations, we obtain sufficient conditions for such a concept to hold as a consequence of main global results. We notice, however, that the full force of our main results has not been fully tapped and requires a further study.

2. Main Global Results. Let us consider the differential system

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C[R^+ \times E, R^n]$, E being an open set in R^n. Let $x(t, t_0, x_0)$ be any solution of (2.1). We shall also be concerned with the auxiliary differential system

$$u' = g(t, u), \quad u(t_0) = u_0, \tag{2.2}$$

---

1) Research sponsored by N.S.F. Grant GP-9647.

where $g \in C[R^+ \times R^N, R^N]$. Let $u(t, t_0, u_0)$ be any solution of (2.2). The following theorem gives a very general set of sufficient conditions for preventing the solutions that start in a given set of R^n through any given part of its boundary.

Theorem 2.1. *Assume that*

(i) $V \in C[R^+ \times E, R^N]$ *and* $V(t, x)$ *is locally Lipschitzian in* x;

(ii) *the set* $H \subset R^n$ *is such that* $\bar{H} \subset E$ *and* $G \subset \partial H$;

(iii) *there exist two sets* A, B *such that* $A \subset R^n$ $B \subset R^N$, $A \cap H \neq \emptyset$ *and* $x \in A$ *implies* $u = V(t, x) \in B$;

(iv) $a \in C[R^+, R]$ *and for* $(t, x) \in R^+ \times G$, $\sum_{i=1}^{N} V_i(t, x) \geq a(t)$;

(v) $x_0 \in A \cap H$ *and* $\sum_{i=1}^{N} V_i(t_0, x_0) < a(t_0)$;

(vi) $g \in C[R^+ \times R^N, R^N]$, $g(t, u)$ *is quasimonotone nondecreasing in* u *for each* $t \in R^+$ *and for* $(t, x) \in R^+ \times H$,

$$D^+ V(t, x) \equiv \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)] \leq g(t, V(t, x));$$

(vii) *any solution* $u(t, t_0, u_0)$ *of* (2.2) *satisfies*

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < a(t), \quad t \geq t_0,$$

provided $u_0 \in B$ *and* $\sum_{i=1}^{N} u_{i0} < a(t_0)$;

then there exist no $t^* > t_0$ *such that* $x(t, t_0, x_0) \in H$, $t \in [t_0, t^*)$ *and* $x(t^*, t_0, x_0) \in G$.

Proof: Suppose that there exists a $t^* > t_0$ satisfying $x(t, t_0, x_0) \in H$, $t \in [t_0, t^*)$ and $x(t^*, t_0, x_0) \in G$. This, in view of assumption (iv), implies that

$$\sum_{i=1}^{N} V_i(t^*, x(t^*, t_0, x_0)) \geq a(t^*). \tag{2.3}$$

Set $u_0 = V(t_0, x_0)$ so that (i) and (vi) yield on the basis of a known result (See theorem 4.1.1 in [1]), the inequality

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0), \quad t \in [t_0, t^*), \tag{2.4}$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.2). We now let $x_0 \in A \cap H$ and $\sum_{i=1}^{N} V_i(t_0, x_0) < a(t_0)$. This means, using (iii) and the fact $u_0 = V(t_0, x_0)$ that $u_0 \in B$ and $\sum_{i=1}^{N} u_{i0} < a(t_0)$. It therefore follows readily from (2.4) and (vii) that

$$\sum_{i=1}^{N} V_i(t^*, x(t^*, t_0, x_0)) \leq \sum_{i=1}^{N} r_i(t^*, t_0, u_0) < a(t^*),$$

which contradicts (2.3). Hence the theorem is proved.

A special case of theorem 2.1 that is itself important in many applications may be stated as follows:

Corollary 2.1. *Assume that*

(i) $V \in C[R^+ \times E, R]$ *and* $V(t, x)$ *is locally Lipschitzian in* x;

(ii) *the set* $H \subset R^n$ *is such that* $\bar{H} \subset E$ *and* $G \subset \partial H$;

(iii) $a \in C[R^+, R]$ *and for* $(t, x) \in R^+ \times G$, $V(t, x) \geq a(t)$;

(iv) $x_0 \in H$ *and* $V(t_0, x_0) < a(t_0)$;

(v) $g \in C[R^+ \times R, R]$ *and for* $(t, x) \in R^+ \times H$, $D^+ V(t, x) \leq g(t, V(t, x))$;

(vi) *any solution* $u(t, t_0, u_0)$ *of scalar differential equation*

$$u' = g(t, u), \quad u(t_0) = u_0, \tag{2.5}$$

satisfies $u(t, t_0, u_0) < a(t)$, $t \geq t_0$, *provided* $u_0 < a(t_0)$;

then the assertion of theorem 2.1 *is true.*

A further specialization results if the function $a(t)$ verifies the differential inequality

$$D^+ a(t) \geq g(t, a(t)), \tag{2.6}$$

in which case, two situations arise. The corresponding results are stated below as corollaries.

Corollary 2.2. *Let the assumptions* (i) *to* (v) *of Corollary* 2.1 *hold. Then if one of the following inequalities*

$$D^+ V(t, x) \leq g(t, (t, x)),$$

$$D^+ a(t) \geq g(t, a(t)),$$

is strict, the conclusion of Theorem 2.1 *remains the same.*

Corollary 2.3. *Let the assumptions* (i) *to* (v) *of Corollary* 2.1 *hold together with* (2.6). *Suppose further that for each* $t_1 \in (t_0, \infty)$ *and* $t \in [t_0, t_1]$, $g(t, u)$ *satisfies the condition*

$$g(t, u_1) - g(t, u_2) \geq -h(t_1 + t_0 - t, u_1 - u_2), \quad u_1 \geq u_2,$$

where $h \in C[R^+ \times R, R]$ *and* $r(t) \equiv 0$ *is the maximal solution of*

$$r' = h(t, r), \quad r(t_0) = 0.$$

Then the conclusion of Theorem 2.1 *remains the same.*

The proof of Corollary 2.3 is a consequence of a known result in differential inequalities (See Theorem 1.4.3 in [1]). A result equivalent to Corollary 2.2 is contained in the work of N. Rouche and Dang-Chau Phien [2]. In fact, our ideas stem from their work.

Suppose that the solutions that start in a given set are required to reach another given set in a finite time and remain there for all future time. The theorem that follows offers a set of general sufficient conditions for such a behaviour.

Theorem 2.2. *Assume that*

(i) $V \in C[R^+ \times E, R^N]$ *and* $V(t, x)$ *is locally Lipschitzian in* x;

(ii) $g \in C[R^+ \times R^N, R^N]$, $g(t, u)$ *is quasi-monotone nondecreasing in* u *for each* $t \in R^+$ *and for* $(t, x) \in R^+ \times E$,

$$D^+ V(t, x) \leqslant g(t, V(t, x));$$

(iii) *there exist sets* A, B *such that* $A \subset R^n$, $B \subset R^N$ *and* $x \in A$ *implies* $u = V(t, x) \in B$;

(iv) *there exists a set* $F \subset E$ *such that* $F \cap A \neq \emptyset$ *and* $x_0 \in F \cap A$ *implies* $x(t, t_0, x_0) \in E$, $t \geqslant t_0$;

(v) *the set* $D \subset E$ *is such that* $\bar{D} \subset E$ *and for* $(t, x) \in R^+ \times E \setminus D$,

$$\sum_{i=1}^{N} V_i(t, x) \geqslant a(t),$$

where $a \in C[R^+, R]$;

(vi) *there exists a* $\tau = \tau(t_0, u_0) > 0$, *such that if* $u_0 \in B$,

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < a(t), \quad t \geqslant t_0 + \tau,$$

where $u(t, t_0, u_0)$ *is any solution of* (2.2);

then there exists a $T = T(t_0, x_0) > 0$ *such that* $x(t, t_0, x_0) \in D$ *for* $t \geqslant t_0 + T$ *whenever* $x_0 \in F \cap A$.

Proof: Let $x_0 \in F \cap A$ so that, by (iv), $x(t, t_0, x_0) \in E$, $t \geqslant t_0$. Also, because of (iii), we have $u_0 \in B$, setting $u_0 = V(t_0, x_0)$. Define $T(t_0, x_0) = \tau(t_0, V(t_0, x_0))$. Let $\{t_k\}$ be a sequence such that $t_k \geqslant t_0 + T$ and $t_k \to \infty$ as $k \to \infty$. Suppose that, if possible, $x(t_k, t_0, x_0) \in E \setminus D$. Then, in view of (v), we would have

$$\sum_{i=1}^{N} V_i(t_k, x(t_k, t_0, x_0)) \geqslant a(t_k).$$

This however is not true, because of the inequality

$$\sum_{i=1}^{N} V_i(t_k, x(t_k, t_0, x_0)) \leqslant \sum_{i=1}^{N} r_i(t_k, t_0, u_0) < a(t_k)$$

which results on account of (i), (ii) and (vi), as in the proof of Theorem 2.1. Hence the proof is complete.

Again, a special case of Theorem 2.2 that is sufficient for many applications is given below as a corollary.

Corollary 2.4. *Assume that*

(i) $V \in C[R^+ \times E, R]$ *and* $V(t, x)$ *is locally Lipschitzian in* x;

(ii) $g \in C[R^+ \times R, R]$ *and for* $(t, x) \in R^+ \times E$, $D^+ V(t, x) \leqslant g(t, V(t, x))$;

(iii) *there exists a set* $F \subset E$ *such that* $x_0 \in F$ *implies* $x(t, t_0, x_0) \in E$ $t \geqslant t_0$;

(iv) *the set* $D \subset E$ *is such that* $\bar{D} \subset E$ *and for* $(t, x) \in R^+ \times E \setminus D$, $V(t, x) \geqslant a(t)$, *where* $a \in C[R^+, R]$;

(v) *there exists a* $\tau = \tau(t_0, u_0) > 0$ *such that, for any solution* $u(t, t_0, u_0)$ *of* (2.5), *the relation* $u(t, t_0, u_0) < a(t)$, $t \geqslant t_0 + \tau$, *holds.*

Then there exists a $T = T(t_0, x_0) > 0$ *such that* $x_0 \in F$ *implies* $x(t, t_0, x_0) \in D$ *for* $t \geqslant t_0 + T$.

3. Applications. To begin with, let us show, as a simple application, that a result on uniform boundedness may be obtained from Corollary 2.1. For this purpose, let $f \in C[R^+ \times R^n, R^n]$,

$$S(\alpha) = [x \in R^n : \|x\| < \alpha] \text{ and } Z_\rho = [x \in R^n : \|x\| \geqslant \rho].$$

Theorem 3.1. *Assume that*

(i) $V \in C[R^+ \times Z_\rho, R]$ *and* $V(t, x)$ *is locally Lipschitzian in* x;

(ii) $b \in C[[\rho, \infty), R]$ *and for* $(t, x) \in R^+ \times Z_\rho$, $V(t, x) \geqslant b(\|x\|)$;

(iii) *for every* $(t, r) \in R^+ \times (\rho, \infty)$ *there exists a* $\beta(r) > \rho$ *such that* $\|x\| \leqslant r$ *implies* $V(t, x) < b(\beta(r))$;

(iv) $g \in C[R^+ \times R, R]$ *and for* $(t, x) \in R^+ \times Z_\rho$, $D^+ V(t, x) \leqslant g(t, V(t, x))$

(v) *every solution* $u(t, t_1, u_0)$ *of the scalar differential equation*

$$u' = g(t, u), \; u(t_1) = u_0, \; t_1 \in R^+,$$

verifies the inequality $u(t, t_1, u_0) < b(\beta(r))$, $t \geqslant t_1$, *for every* $r \in [\rho, \infty]$, *provided* $u_0 < b(\beta(r))$;

then, the differential system (2.1) *is uniformly bounded.*

Proof. Let $\alpha \in (\rho, \infty)$ and $t_0 \in R^+$. By (iii), we have $V(t_0, x_0) < b(\beta(\alpha))$ if $\|x_0\| \leqslant \alpha$. Set $E = Z_\rho$, $H = S(\beta) \setminus S(\alpha)$, $G = \partial S(\beta)$ and $a(t) = b(\beta)$. Then $\|x_0\| \leqslant \alpha$ implies $x(t_1, t_0, x_0) \in S(\beta)$, $t \geqslant t_0$. To prove this statement, we appeal to Corollary 2.1. First of all, observe that only those solutions $x(t, t_0, x_0)$ of (2.1) with $\|x_0\| \leqslant \alpha$ that reach $\partial S(\alpha)$ for some $t_1 > t_0$ and leave G remaining in H thereafter, need to be considered. Hence for such solutions, if $\|x_1\| = \alpha$, where $x_1 = x(t_1, t_0, x_0)$, we have $V(t_1, x_1) < b(\beta(\alpha))$, by (iii). All the assumptions of Corollary 2.1 are thus verified and, as a consequence, the theorem is proved.

Let M be a compact set in R^n and suppose that it is self invariant with respect to the system (2.1). For any set M, let $S(M, \rho) = [x \in R^n : d(x, M) < \rho]$. The theorem that follows gives general criteria for the stability of the set M.

Theorem 3.2. *Assume that*

(i) $V \in C[R^+ \times S(M, \rho) \setminus M, R]$, $V(t, x)$ *is locally Lipschitzian in* x *and* $V(t, x) \to -\infty$ *when* $d(x, M) \to 0$ *for each* $t \in R^+$;

(ii) $b \in C[R^+ \times (0, \rho), R]$ *and for* $(t, x) \in R^+ \times S(M, \rho) \setminus M$,

$$V(t, x) \geqslant b(t, d(x, M));$$

(iii) $g \in C[R^+ \times R, R]$ and for $(t, x) \in R^+ \times S(M, \rho)\setminus M$,

$$D^+ V(t, x) \leq g(t, V(t, x));$$

(iv) any solution $u(t, t_0, u_0)$ of (2.5) satisfies $u(t, t_0, u_0) < b(t, r)$, $t \geq t_0$ for every $r \in (0, \rho)$, provided $u_0 < b(t_0, r)$;

then the self invariant set M is stable.

Proof. For every $(t_0, \varepsilon) \in R^+ \times (0, \rho)$, there exists a $\delta = \delta(t_0, \varepsilon)$ such that $x_0 \in S(M, \delta)\setminus M$ implies $V(t_0, x_0) < b(t_0, \varepsilon)$, in view of assumption (i). Setting $E = S(M, \rho)\setminus M$, $H = S(M, \varepsilon)\setminus M$, $G = \partial S(M, \varepsilon)$ and $a(t) = b(t, \varepsilon)$, we see all the hypotheses of Corollary 2.1 are verified. Hence the conclusion follows.

Remark. Notice that the Lyapunov-like function used in Theorem 3.2 is neither positive definite nor defined on the set M.

Let M and Y be any two subsets of R^n such that $M \subset Y$. Let us assume that Y is conditionally invariant on M with respect to the differential system (2.1). For the definition of conditionally invariant set and its stability see [1, 3]. We then have.

Theorem 3.3. *Assume that*

(i) $V \in C[R^+ \times S(Y, \rho), R]$ *and* $V(t, x)$ *is locally Lipschitzian in* x

(ii) $g \in C[R^+ \times R, R]$ *and for* $(t, x) \in R^+ \times S(Y, \rho)\setminus M$,

$$D^+V(t, x) \leq g(t, V(t, x));$$

(iii) $b \in C[R^+ \times [0, \rho), R]$ *and for* $(t, x) \in R^+ \times S(Y, \rho)\setminus M$, $V(t, x) \geq b(t, d(x, Y))$;

(iv) *whenever* $x \in M$, *for every* $(t, r) \in R^+ \times (0, \rho)$, $V(t, x) < b(t, r)$;

(v) *every solution* $u(t, t_0, u_0)$ *of* (2.5) *satisfies, for every* $r \in (0, \rho)$, *the inequality* $u(t, t_0, u_0) < b(t, r)$, $t \geq t_0$, *if* $u_0 < b(t_0, r)$.

Then the conditionally invariant set Y on M is stable.

Proof. Let $(t_0, \varepsilon) \in R^+ \times (0, \rho)$. By (iv), there exists a $\delta = \delta(t_0, \varepsilon)$ such that $x_0 \in S(M, \delta)\setminus M$ implies $V(t_0, x_0) < b(t_0, \varepsilon)$. Stated result follows by appealing to Corollary 2.1 when we set $E = S(Y, \rho)$, $H = S(Y, \varepsilon)\setminus M$, $G = \partial S(Y, \varepsilon)$ and $a(t) = b(t, \varepsilon)$.

Remark. If the sets M and Y are bounded, the stability information obtained in the foregoing theorem is of real practical interest. Observe that the definition of conditional invariant set is so general, it includes many situations. For example, a case of interest is where M is a compact self invariant set which is unstable and Y is an unstable subspace of R^n. In such a case, although the above result prevents the scattering of solutions throughout the space R^n and channelizes along the neighbourhoods of Y, it is of little practical significance. A more useful concept in such situations, may be defined as follows:

Definition. *The set Y is said to be*

(i) *restrictively stable on* M, *if for every* $\varepsilon > 0$ *and* $t_0 \in R^+$, *there exist positive numbers* $\delta = \delta(t_0, \varepsilon)$ *and* $\beta = \beta(t_0, \varepsilon)$ *such that* $x_0 \in S(M, \delta)$ *implies*

$$x(t, t_0, x_0) \in S(Y, \varepsilon) \cap S(M, \beta), \ t \geq t_0;$$

(ii) *restrictively asymptotically stable on* M, *if* (i) *holds and for every* $t_0 \in R^+$, *there exist positive numbers* $\delta_0 = \delta(t_0)$ *and* $\beta_0 = \beta(t_0)$ *such tha* $\lim_{t\to\infty} x(t, t_0, x_0) = x^0 \in Y \cap S(M, \beta_0)$, *uniformly on* $x_0 \in S(M, \delta_0)$.

A simple set of conditions that guarantee restrictive stability of Y on M is given in

Theorem 3.4. *Assume that*

(i) $V \in C[R^+ \times S(Y, \rho), R^2]$, $V(t, x)$ *is locally Lipschitzian in* x *and or* $(t, x) R^+ \times S(Y, \rho)\setminus M$

$$D^+ V(t, x) \leq 0;$$

(ii) $b \in C[R^+, R^2]$ *and for* $(t, x) \in R^+ \times S(Y, \rho)\setminus M$,

$$V_1(t, x) \geq b_1(d(x, Y)),$$

$$V_2(t, x) \geq b_2(d(x, M));$$

(iii) *whenever* $x \in M$, *for every* $(t, r) \in R^+ \times (0, \rho)$, $V_1(t, x) < b_1(r)$ *ant for every* $(t, r) \in R^+ \times (0, \rho)$, *there exists a* $\beta = \beta(t, r) > 0$ *such that* $x \in S(M, rf$ *implies* $V_2(t, x) < b_2(\beta)$.

Then the set Y is restrictively stable on M.

Proof. Let $(t_0, \varepsilon) \in R^+ \times (0, \rho)$ be given. By assumption (iii), there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $x_0 \in S(M, \delta)$ implies $V_1(t_0, x_0) < b_1(\varepsilon)$ and $V_2(t_0, x_0) < b_2(\beta)$, where $\beta = \beta(t_0, \delta)$. We set $E = S(Y, \rho)$ and $H = [S(Y, \varepsilon) \cap S(M, \beta)]\setminus M$. First of all, let us write $G_1 = \partial H \cap \partial S(Y, \varepsilon)$ and $a(t) = b_1(\varepsilon)$. An appeal to Corollary 2.1 with $g(t, u) = 0$ yields that solutions $x(t, t_0, x_0)$ with $x_0 \in S(M, \delta)$ cannot leave G_1 for $t \geq t_0$. Similarly, by setting $G_2 = \partial H \cap \partial S(M, \beta)$ and $a(t) = b(\beta)$, it follows, by a repeated application of Corollary 2.1, that solutions $x(t, t_0, x_0)$ with $x_0 \in S(M, \delta)$ cannot leave G_2 either for $t \geq t_0$. Hence the theorem is proved.

The applications so far considered centered around Corollary 2.1. We shall next discuss a result on conditional stability [1] utilizing Theorem 2.1. Let us assume, to that end, that the set M is self invariant.

Theorem 3.5. *Assume that*

(i) $V \in C[R^+ \times S(M, \rho), R^N]$ *and* $V(t, x)$ *is locally Lipschitzian in* x;

(ii) $g \in C[R^+ \times R^N, R^N]$, $g(t, u)$ *is quasimonotone nondecreasing in* u *for each* $t \in R^+$ *and for* $(t, x) \in R^+ \times S(M, \rho)\setminus M$,

$$D^+ V(t, x) \leq g(t, V(t, x));$$

(iii) *there exist two sets* A, B *such that* $M \subset A \subset R^n$, $\{0\} \subset B \subset R^N$ *and* $x \in A$ *implies* $u = V(t, x) \in B$;

(iv) $b \in C[R^+ \times (0, \rho), R]$ *and for* $(t, x) \in R^+ \times S(M, \rho)\setminus M$,

$$\sum_{i=1}^{N} V_i(t, x) \geq b(t, d(x, M));$$

(v) *whenever* $x \in M$, *for* $(t, r) \in R^+ \times (0, \rho)$,

$$\sum_{i=1}^{N} V_i(t, x) < b(t, r)$$

provided $x \in A$;

(vi) *every solution* $u(t, t_0, u_0)$ *of* (2.2) *satisfies*

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < b(t, r), \quad t \geq t_0,$$

for all $r \in (0, \rho)$, *provided* $u_0 \in B$ *and* $\sum_{i=1}^{N} u_{i0} < b(t_0, r)$;

then the self invariant set M *is conditionally stable.*

Proof. Let $(t_0, \varepsilon) \in R^+ \times (0, \rho)$. By assumption (v), there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$x_0 \in S(M, \delta) \cap A \text{ implies } \sum_{i=1}^{N} V_i(t_0, x_0) < b(t_0, \varepsilon).$$

Setting $E = S(M, \rho)$, $H = S(M, \rho) \setminus M$, $G = \partial S(M, \varepsilon)$ and $a(t) = b(t, \varepsilon)$, we see that all the assumptions of Theorem 2.1 are verified and consequently the proof is complete.

Remark. Notice that the foregoing theorem offers more general criteria for conditional stability than known results [1].

We shall now discuss some applications of Theorem 2.2 and its Corollary 2.4. First of all let us consider the criteria for asymptotic stability of a self invariant set M.

Theorem 3.6. *Suppose that the hypotheses of Theorem 3.2 hold. Assume further that* $b(t, u)$ *is nondecreasing in* u *for each* $t \in R^+$ *and that there exists a* $\tau = \tau(t_0, u_0) > 0$ *such that every solution* $u(t, t_0, u_0)$ *of* (2.5) *satisfies the relation* $u(t, t_0, u_0) < b(t, r)$, $t \geq t_0 + \tau$ *for all* $r \in (0, \rho)$. *Then the self invariant set* M *is asymptotically stable.*

Proof. Since, by Theorem 3.2, the set M is stable, we have, for $\varepsilon = \rho$, a $\delta_0 = \delta(t_0, \rho)$ such that $x_0 \in S(M, \delta_0)$ implies $x(t, t_0, x_0) \in S(M, \rho)$, $t \geq t_0$. We therefore set $F = S(M, \delta_0)$ so that the condition (iii) of Corollary 2.4 is verified. Let $(t_0, \varepsilon) \in R^+ \times (0, \rho)$. Write $D = S(M, \varepsilon)$ and observe that for $(t, x) \in E \setminus D$ (recall that $E = S(M, \rho) \setminus M$), $V(t, x) \geq b(t, \varepsilon)$ because of monotonicity of $b(t, u)$). If we choose $a(t) = b(t, \varepsilon)$, this shows that assumption (iv) of Corollary 2.4 is satisfied. The rest of the hypotheses are checked already in the proof of Theorem 3.2. Hence the conclusion follows from Corollary 2.4.

We now merely state two theorems on other kinds of asymptotic stability whose proofs can similarly be reduced to Theorem 2.2 or Corollary 2.4.

Theorem 3.7. *Assume that*

(i) $V \in C[R^+ \times S(Y, \rho), R]$ *and* $V(t, x)$ *is locally Lipschitzian in* x;

(ii) $g \in C[R^+ \times R, R]$ *and for* $(t, x) \in R^+ \times S(Y, \rho) \setminus M$,

$$D^+ V(t, x) \leq g(t, V(t, x));$$

(iii) $b \in C[R^+ \times [0, \rho), R]$, $b(t, u)$ *is nondecreasing in* u *for each* $t \in R^+$ *and for* $(t, x) \in R^+ \times S(Y, \rho) \setminus M$, $V(t, x) \geq b(t, d(x, Y))$;

(iv) *there exists a* $\tau = \tau(t_0, u_0) > 0$ *such that any solution* $u(t, t_0, u_0)$ *of* (2.5) *satisfies the inequality*

$$u(t, t_0, u_0) < b(t, r), \quad t \geq t_0 + \tau,$$

for every $r \in (0, \rho)$;

then (1) *the restrictive stability of* Y *on* M *implies the restrictive asymptotic stability of* Y *on* M;

(2) *the stability of conditionally invariant set* Y *on* M *implies the asymptotic stability of* Y *on* M.

Theorem 3.8. *Let the hypotheses of Theorem 3.5 hold. Assume further that* $b(t, u)$ *is nondecreasing in* u *for each* $t \in R^+$ *and that there exists a* $\tau = \tau(t_0, u_0) > 0$ *such that for any solution* $u(t, t_0, u_0)$ *of* (2.2), *the relation* $\sum_{i=1}^{N} u_i(t, t_0, u_0) < b(t, r)$, $t \geq t_0 + \tau$, *holds for every* $r \in (0, \rho)$; *then the self-invariant set* M *is conditionally asymptotically stable.*

REFERENCES

1. Lakshmikantham V. and Leela S. — *Differential and integral inequalities, theory and applications*, Vol. I. Academic Press, New York 1969.
2. Rouche N. and Dang-Chau Phien — *Sufficient stability conditions based on global lemmas.* To appear in the Proc. International Symp. Nonlinear Oscillations, Kiev, U.S.S.R., 1969.
3. Leela S. — *Lyapunov stability and conditional invariant sets.* To appear.

University of Rhode Island
Kingston, Rhode Island
and
State University College of New York
Geneseo, New York

REZULTATE GLOBALE ŞI STABILITATEA MIŞCĂRII

Rezumat

Se discută în primul rînd două teoreme globale privind teoria stabilităţii cu caracter foarte general, cu ajutorul unor funcţii Liapunov şi al teoriei inegalităţilor diferenţiale. Se dau apoi, drept aplicaţii, diferite rezultate asupra stabilităţii şi mărginirii mişcărilor. De asemenea, introducînd noţiunea de stabilitate restrictivă care e de interes practic în anumite situaţii, se obţin condiţii suficiente pentru ca un astfel de concept să fie valabil ca o consecinţă a unor rezultate globale.