LYAPUNOV STABILITY OF CONDITIONAL INVARIANT SETS

BY

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1. Introducing the notion of conditional invariant sets, Kayande and Lakshmikantham [2] discussed the Lyapunov stability criteria of such invariant sets. This interesting work made us realize that the classification of invariant sets and the study of Lyapunov stability with respect to these different invariant sets, broadens the outlook and applicability of Lyapunov's second method. However, the sufficient conditions given in [2] are not completely satisfactory in the sense that one can not deduce from them, results parallel to those of Lyapunov's original theorems. Moreover, the problem of converse theorems is also open.

In this paper, we shall obtain necessary and sufficient conditions for the stability criteria of conditional invariant sets and also discuss a perturbation result. The treatment is parallel to that of Lyapunov's theory and also contains it as a special case. Furthermore, the type of Lyapunov functions employed in the development demand less restrictive assumptions concerning the decrescent character, which is a definite advantage. The important and interesting result of the paper is, perhaps, theorem 3.1, which shows that no matter whether the given system enjoys the stability properties or not, there exist Lyapunov functions having all the usual properties, except the positive definiteness and theorem 3.2 corroborates this claim. In fact, the remark that follows implies that even for obtaining positive definiteness stability is superfluous and all that is important is an estimate of the type (3.11) which does not necessarily imply any stability.

2. We shall use the following notation in the sequel:

$$\mathcal{J} = [t_0, \infty), t_0 \geq 0,$$

R^k = euclidian space of k dimension,

$\|x\|$ = any convenient norm of $x \in R^k$,

$R_+ =$ non-negative real line,

$d(a, B) = \inf[\|a - b\|, b \in B]$, B being any set in R^k,

$S(B, \rho) = [x \in R^k : d(x, B) < \rho]$,

$C[\mathcal{J} \times R^k, R^k] =$ the class of functions defined and continuous on $\mathcal{J} \times R^k$, taking values in R^k,

$\mathcal{K} =$ the class of functions $a \in C[[0, \rho), R_+]$ such that $a(0) = 0$ and $a(u)$ is increasing in u,

$L =$ the class of functions $\sigma \in C[R_+, R_+]$ such that $\sigma(s)$ is decreasing in s and $\sigma(s) \to 0$ as $s \to \infty$.

Let us consider the differential system

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{2.1}$$

and let $x(t, t_0, x_0)$ denote a solution of (2.1). Suppose that A, B are any two subsets of R^k such that $A \subset B$.

Definiton 2.1. *The set B is said to be conditionally invariant with respect to the set A and the differential system (2.1) if $x_0 \in A$ implies $x(t, t_0, x_0) \in B$, $t \in \mathcal{J}$. In the special case when $B = A$, B is said to be self-invariant.*

Definition 2.2. *The conditionally invariant set B is said to be uniformly stable (i) if, given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $x_0 \in S(A, \delta)$ implies $x(t, t_0, x_0) \in S(B, \varepsilon)$, $t \in \mathcal{J}$, or (ii) if there exists a $\beta \in \mathcal{K}$ such that*

$$d(x(t, t_0, x_0), B) \leqslant \beta(d(x_0, A)), \quad t \in \mathcal{J}.$$

Definition 2.3. *The conditionally invariant set B is said to be uniformly asymptotically stable (i) if it is uniformly stable and if, given any $\varepsilon > 0$, there exist positive numbers δ_0, $T(\varepsilon)$ such that $x_0 \in S(A, \delta_0)$ implies $x(t, t_0, x_0) \in S(B, \varepsilon)$, $t \geqslant t_0 + T(\varepsilon)$, or (ii) if there exist functions $\beta \in \mathcal{K}$ and $\sigma \in L$ such that*

$$d(x(t, t_0, x_0), B) \leqslant \beta(d(x_0, A))\, \sigma(t - t_0), \quad t \in \mathcal{J}.$$

Let us now state the following known results [1] that we shall be using later.

Theorem 2.1. *Assume that*

(i) *$f \in C[\mathcal{J} \times R^k, R^k]$ and $V \in C[\mathcal{J} \times R^k, R_+]$, $V(t, x)$ is locally Lipschitzian in x and for $(t, x) \in \mathcal{J} \times R^k$,*

$$D^+ V(t, x) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x + h f(t, x)) - V(t, x)] \leqslant$$
$$\leqslant g(t, V(t, x)), \tag{2.2}$$

where $g \in C[\mathcal{J} \times R_+, R]$,

(ii) *$r(t, t_0, u_0)$ is the maximal solution of*

$$u' = g(t, u), \quad u(t_0) = u_0 \geqslant 0 \tag{2.3}$$

existing on $\mathcal{J}$.

Then $V(t_0, x_0) \leqslant u_0$ implies

$$V(t, x(t, t_0, x_0)) \leqslant r(t, t_0, u_0),$$

as far as $x(t, t_0, x_0)$ exists.

Theorem 2.2. *Let $f \in C[\mathcal{J} \times R^k, R^k]$ and for (t, x), $(t, y) \in \mathcal{J} \times R^k$*

$$\|f(t, x) - f(t, y)\| \leqslant L(t)\|x - y\|,$$

where $L \in C[\mathcal{J}, R_+]$. Then, for $t \in \mathcal{J}$

$$\|x_0 - y_0\| \exp\Big[-\int_{t_0}^{t} L(s)\, ds\Big] \leqslant \|x(t, t_0, x_0) - x(t, t_0, y_0)\| \leqslant \tag{2.4}$$
$$\leqslant \|x_0 - y_0\| \exp\Big[\int_{t_0}^{t} L(s)\, ds\Big],$$

where $x(t, t_0, x_0)$, $x(t, t_0, y_0)$ are the solutions of (2.1).

3. Let us begin with the necessary conditions for the stability criteria of the conditionally invariant set B. Before proceeding to that, we shall, first of all, prove the following result, which may be considered as a basic tool in discussing the converse theorems.

Theorem 3.1. *Assume that*

(i) *$f \in C[\mathcal{J} \times R^k, R^k]$ and for $(t, x), (t, y) \in \mathcal{J} \times R^k$*

$$\|f(t, x) - f(t, y)\| \leqslant L(t)\|x - y\|,$$

where $L \in C[\mathcal{J}, R_+]$;

(ii) *$g \in C[\mathcal{J} \times R_+, R]$, $g(t, 0) \equiv 0$ and for $t \in \mathcal{J}$, $u, v \geqslant 0$,*

$$g(t, u) - g(t, v) \leqslant M(t)(u - v), \quad u \geqslant v,$$

where $M \in C[\mathcal{J}, R]$.

Then there exists a function $V(t, x)$ such that

(a) *$V \in C[\mathcal{J} \times S(B, \rho), R_+]$ and $V(t_0, x) = d(x, A)$;*

(b) *$|V(t, x) - V(t, y)| \leqslant K(t)\|x - y\|$, where $K \in C[\mathcal{J}, R_+]$ and $K(t_0) = 1$;*

(c) *$D^+ V(t, x) \leqslant g(t, V(t, x))$, $(t, x) \in \mathcal{J} \times S(B, \rho)$.*

Proof. We notice that the assumptions (i) and (ii) guarantee the uniqueness and continuous dependence of solutions of (2.1) and (2.3) so that letting $x = x(t, t_0, x_0)$, we have $x_0 = x(t_0, t, x)$.

If $u(t, t_0, u_0)$ is the solution of (2.3), we define

$$V(t, x) = u(t, t_0, d(x(t_0, t, x), A)). \tag{3.1}$$

Notice that, because of (i), $x(t_0, t, x)$ is well defined for $(t, x) \in \mathcal{J} \times S(B, \rho)$ and therefore (a) is verified.

Denoting $x(t+h, t_0, x_0)$ by x^*, we observe that

$$V(t+h, x^*) = u(t+h, t_0, d(x(t_0, t+h, x^*), A)) = u(t+h, t_0, d(x(t_0, t, x)\, A)),$$

in view of uniqueness of solutions of (2.1). As a result, it follows that

$$\text{(3.2)} \quad \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x^*) - V(t,x)] = \limsup_{h\to 0^+} \frac{1}{h}[u(t+h, t_0, d(x(t_0, t, x), A)) - u(t, t_0, d(x(t_0, t, x), A)] = g(t, V(t,x)).$$

Furthermore, as a consequence of theorem 2.2, the assumption (i) yields the estimate

$$\text{(3.3)} \qquad \|x_0 - y_0\| \leqslant \|x - y\| \exp\left|\int_{t_0}^{t} L(s)\,ds\right|, \quad t \in \mathcal{J},$$

where $x = x(t, t_0, x_0)$ and $y = x(t, t_0, y_0)$. Similarly, we get

$$\text{(3.4)} \qquad |u(t, t_0, u_0) - u(t, t_0, v_0)| \leqslant |u_0 - v_0| \exp\left|\int_{t_0}^{t} M(s)\,ds\right|, \quad t \in \mathcal{J},$$

in view of (ii). Hence for $t \in \mathcal{J}, x, y \in S(B, \rho)$, we obtain

$$|V(t,x) - V(t,y)| = |u(t, t_0, d(x(t_0, t, x), A)) - u(t, t_0, d(x(t_0, t, y), A))| \leqslant$$

$$\leqslant |d(x(t_0, t, x), A) - d(x(t_0, t, y), A)| \exp\left(\int_{t_0}^{t} M(s)\,ds\right) \leqslant$$

$$\text{(3.5)} \quad \|x(t_0, t, x) - x(t_0, t, y)\| \exp\left(\int_{t_0}^{t} M(s)\,ds\right) \leqslant \|x - y\| \exp\left(\int_{t_0}^{t} (L(s) + M(s))\,ds\right) \equiv$$

$$\equiv K(t)\|x - y\|.$$

Here, we have used the relations (3.1), (3.3), (3.4) and

$$|d(x, A) - d(y, A)| \leqslant \|x - y\|.$$

Observe that $K(t_0) = 1$. This proves (b).

It is now easy to prove (c), from (3.2) and (3.5), using the standard arguments as in [1]. The proof is complete.

Having the foregoing theorem at our disposal, one can readily obtain the necessary conditions in a unified way.

Theorem 3.2. *Assume the hypotheses of theorem 3.1. Suppose that one of the following conditions hold, in addition:*

(1), (3.6) $\qquad d(x(t, t_0, x_0), B) \leqslant \beta(d(x_0, A)), \quad t \in \mathcal{J}, \quad \beta \in \mathcal{K}$

and

(3.7) $\qquad \gamma(u_0) \leqslant u(t, t_0, u_0), \quad t \in \mathcal{J}, \quad \gamma \in \mathcal{K};$

(2). (3.8) $\qquad d(x(t, t_0, x_0), B) \leqslant \beta(d(x_0, A))\,\sigma(t - t_0), \quad t \in \mathcal{J}$

and

(3.9) $\qquad \beta(u_0)\,\sigma(t - t_0) \leqslant u(t, t_0, u_0), \quad t \in \mathcal{J}, \quad \beta \in \mathcal{K}, \quad \sigma \in L;$

(3). (3.10) $\qquad d(x(t, t_0, x_0), B) \leqslant L d(x_0, A) \exp[\rho(t_0) - \rho(t)], \quad t \in \mathcal{J},$

where $\rho \in \mathcal{K}$ *such that* $\rho'(t)$ *exists and* $\rho(t) \to \infty$ *as* $t \to \infty$.
Then, the function $V(t, x)$, *in addition to having all the properties of theorem 3.1, is positive definite with respect to the set* B.

Proof. Assume (3.6) and (3.7). Then, because of (3.1), it follows that

$$V(t, x) = u(t, t_0, d(x(t_0, t, x), A)) \geqslant \gamma(d(x(t_0, t, x), A)) \geqslant$$

$$\geqslant \gamma\beta^{-1}(d(x, B)) \equiv b(d(x, B)), \quad b \in \mathcal{K}.$$

This proves that $V(t, x)$ satisfies the additional property of positive definiteness with respect to the set B, if (1) holds.

Similar conclusion is immediate, when (2) holds. In the case of (3), we take $g(t, u) = -\rho'(t)\,u$ so that $u(t, t_0, u_0) = u_0 \exp[\rho(t_0) - \rho(t)]$. Hence (3.1) reduces to

$$V(t, x) = d(x(t_0, t, x), A) \exp[\rho(t_0) - \rho(t)]$$

and consequently the positive definiteness follows at once. The function $K(t)$, occurring in (3.5), now becomes equal to

$$\exp\left[\rho(t_0) - \rho(t) + \int_{t_0}^{t} L(s)\,ds\right].$$

Remark. Instead of one of the conditions (1), (2), (3) assumed in theorem 3.2, an estimate of the form

(3.11) $\qquad d(x(t, t_0, x_0), B) \leqslant u(t, t_0, d(x_0, A)), \quad t \in \mathcal{J}$

is all that is required to guarantee the additional positive definiteness of $V(t, x)$ with respect to B. This is clear from the proof of theorem 3.2. It is important to note that the estimate (3.11) need not necessarily imply any type of stability of the conditionally invariant set B. This shows the surprising fact, mentioned in the introduction, that for the existence of Lyapunov functions stability behaviour is superfluous.

4. In this section, we shall state a direct theorem which gives sufficient conditions for the stability criteria of the conditional invariant set B. We make use of the special type of Lyapunov functions obtained in the earlier section.

Theorem 4.1. *Assume that*

(i) *there exists a function* $V \in C[\mathcal{J} \times S(B, \rho), R_+]$ *such that* $V(t, x)$ *is locally Lipschitzian in* x, *positive definite with respect to the set* B *and* $V(t_0, x) \leqslant a(d(x, A)), a \in \mathcal{K}$;

(ii) $f \in C[\mathcal{J} \times R^k, R^k]$ *and the inequality* (2.2) *holds.*

Then the stability properties of the trivial solution of the scalar differential equation (2.3) *imply the corresponding stability criteria of the conditional invariant set* B *with respect to the system* (2.1).

Proof. We shall prove only the uniform stability of the conditionally invariant set. The proof for the other cases can be constructed on the basis of this with appropriate modifications.

Since $V(t, x)$ is positive definite with respect to the set B, there exists a function $b \in \mathcal{K}$ such that

$$V(t, x) \geqslant b(d(x, B)). \tag{4.1}$$

Let $0 < \varepsilon < \rho$ be given. Assume that the trivial solution of (2.3) is uniformly stable. Then, given $b(\varepsilon) > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $u_0 < \delta$ implies

$$u(t, t_0, u_0) < b(\varepsilon), \quad t \in \mathcal{J}. \tag{4.2}$$

Choose $u_0 = a(d(x_0, A))$ and $a(\delta_1) = \delta$. We claim that $x(t, t_0, x_0) = x(t) \in S(B, \varepsilon), t \in \mathcal{J}$ provided $x_0 \in S(A, \delta_1)$. Suppose that this is not true. Then, there exists a solution $x(t)$ with $d(x_0, A) < \delta_1$ and a $t_1 > t_0$ such that

$$d(x(t_1), B) = \varepsilon \text{ and } d(x(t), B) < \varepsilon, \quad t \in [t_0, t_1].$$

Thus we have, by (4.1),

$$V(t_1, x(t_1)) \geqslant b(\varepsilon). \tag{4.3}$$

For $t \in [t_0, t_1]$, $d(x(t), B) < \rho$ and hence the choice $u_0 = a(d(x_0, A))$ and the inequality (2.2) yields, as a consequence of theorem 2.1, the estimate

$$V(t, x(t)) \leqslant r(t, t_0, u_0), t \in [t_0, t_1] \tag{4.4}$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.3). At $t = t_1$, we obtain, using (4.2) and (4.4), the inequality

$$V(t_1, x(t_1)) \leqslant r(t_1, t_0, u_0) < b(\varepsilon)$$

which contradicts (4.3). This proves the uniform stability of the conditionally invariant set B.

Corollary 3.1. If $g(t, u) \equiv 0$ in theorem 4.1, the conditionally invariant set B is uniformly stable.

Corollary 3.2. If $g(t, u) = -C(u), C \in \mathcal{K}$, in theorem 4.1, the conditionally invariant set B is uniformly asymptotically stable.

Remark. It is clear that corollary 3.1 corresponds to Lyapunov's first theorem. Corresponding to Lyapunov's second theorem, we have the following two situations:

(i) $\quad D^+ V(t, x) \leqslant -C(d(x, A)), \quad C \in \mathcal{K}, \quad (t, x) \in \mathcal{J} \times S(B, \rho),$

(ii) $\quad D^+ V(t, x) \leqslant -C(d(x, B)), \quad C \in \mathcal{K}, \quad (t, x) \in \mathcal{J} \times S(B, \rho),$

instead of the inequality (2.2).

In case (i), it can be easily shown using the standard arguments [1] with appropriate changes, that the set B is uniformly asymptotically stable. However, since $V(t, x)$ is decrescent only at $t = t_0$, case (i) does not reduce to corollary 3.2 as is to be expected.

In the situation (ii), it is only possible to prove that

$$\lim_{t \to \infty} \inf d(x(t), B) = 0, \tag{4.5}$$

and the problem of finding additional restriction to guarantee uniform asymptotic stability of the set B is open. It is easy to observe that (4.5) is true even when (ii) is replaced by

$$D^+ V(t, x) \leqslant -(d(x, A_0)),$$

where A_0 is any set such that $A \subset A_0 \subset B$, because $d(x, A_0) \geqslant d(x, B)$.

5. Finally, let us consider the perturbed system

$$x' = f(t, x) + R(t, x), \quad x(t_0) = x_0, \tag{5.1}$$

and state a result which concludes the preservation of stability properties under constantly acting perturbations.

Theorem 5.1. *Assume that*

(i) *The conditional invariant set* B *is generalized exponentially stable with respect to the set* A *and the system* (2.1);

(ii) $\quad f \in C[\mathcal{J} \times R^k, R^k]$ *and for* $(t, x), \quad (t, y) \in \mathcal{J} \times R^k$,

$$\| f(t, x) - f(t, y) \| \leqslant L(t) \| x - y \|,$$

where $L \in C[\mathcal{J}, R_+]$;

(iii) $\quad R \in C[\mathcal{J} \times R^k, R^k]$ *and for* $(t, x) \in \mathcal{J} \times R^k$,

$$\| R(t, x) \| \leqslant w(t, d(x, B)),$$

where $w \in C[\mathcal{J} \times R_+, R_+], w(t, 0) = 0$ *and* $w(t, u)$ *is non-decreasing in* u.

Then, the stability criteria of the trivial solution of scalar differential equation

$$u' = -\rho'(t)\,u + K(t)\,w(t,u), \quad u(t) = u(t_0) = u_0 \geq 0,$$

imply the corresponding stability properties of the set B with respect to the perturbed system (5.1).

Proof: By virtue of theorem 3.2, there exists a function $V(t,x)$ such that

$$V(t_0, x) = d(x, A),$$

$$|V(t,x) - V(t,y)| \leq K(t)\,\|x - y\|,$$

$$D^+ V(t,x)_{(2.1)} \leq -\rho'(t)\,V(t,x).$$

Now, the standard computation shows that

$$D^+ V(t,x)_{(5.1)} \leq -\rho'(t)\,V(t,x) + K(t)\,w(t, V(t,x))$$

under the assumptions of theorem. It is readily seen that theorem 4.1 is applicable and the proof is complete.

Remark: It is interesting to prove a converse theorem parallel to Massera's theorem [1] in this set up. We are, however, unable to prove such a result.

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STABILITATE LIAPUNOV A UNOR MULŢIMI CONDIŢIONAL INVARIANTE

Rezumat

Se obţin condiţii necesare şi suficiente pentru criterii de stabilitate ale mulţimilor condiţional invariante şi de asemenea se discută un rezultat cu privire la perturbaţii. Metoda de lucru este paralelă cu aceea a teoriei lui Liapunov şi o conţine ca un caz particular. Funcţia Liapunov utilizată cere condiţii mai puţin restrictive privitoare la caracterul descrescător. De asemenea, în teorema 3.1 se demonstrează că, indiferent dacă sistemul este stabil sau nu, există funcţii Liapunov care au toate proprietăţile obişnuite cu excepţia caracterului de a fi pozitiv definite.

ON ASYMPTOTIC BEHAVIOUR OF FUNCTIONAL DIFFERENTIAL SYSTEMS

BY

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1. Introduction. In the last few years, there has been an increasing interest in the theory of differential equations with retarded arguments. These equations describe physical systems, the future state of which depends not only on the present state but also on its past history. Much of the recent literature has been devoted to the extension of known results for ordinary differential equations to differential equations with retarded arguments (functional differential equations).

We, in the present paper, wish to investigate this problem further. First, we study the asymptotic behaviour of the solutions of a functional differential system, through a new comparison principle. Using these results we study the asymptotic relationships between the solutions of a nonlinear perturbed functional differential system with those of the corresponding unperturbed differential system. Our results generalize recent results of Brauer and Wong [2] for ordinary differential systems. Next we study the concept of „asymptotic manifold of solutions", motivated by the work of Toroshelidze [6], for scalar ordinary differential equations, and introduced recently by Hallam and Heidel [3] and also studied by Ladas, Lakshmikantham and Leela [4]. Finally we apply one of our results to a second order differential-difference equation obtaining a generalization of a well-known result in Bellman [1, pp. 114-115] for ordinary differential equations.

2. Notation. Let $\mathcal{J}$ denote the half-line $0 \leq t < \infty$ and R^n the euclidean n space. Let $\|\cdot\|$ denote any convenient norm of a vector and the corresponding norm of a matrix. Given any $\tau \geq 0$, let $\mathcal{C}^n = C[[-\tau, 0], R^n]$ denote the space of continuous functions with domain $[-\tau, 0]$ and range in R^n. For any element $\Phi \in \mathcal{C}^n$, define the norm