RESULTS ON GENERALIZED HYPERGEOMETRIC AND JACOBI POLYNOMIALS

BY

MANILAL SHAH

In this paper the integral involving the product of Jacobi and two generalized hypergeometric polynomials are obtained in terms of Kampé de Fériet's double hypergeometric function. An expansion-formula for product of two hypergeometric polynomials in series of Jacobi polynomials has been established with the help of this integral and orthogonality-property for the Jacobi polynomials. A number of new, known and interesting results have also derived as particular cases of integral and expansion formulae.

1. **Integral.** The integral to be proved is

$$\int_0^1 x^{\xi}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x)\left\{{}_{p+1}F_q\left[\begin{matrix}-l,\ a_p;\ \mu x\\ b_q\end{matrix}\right]\right\}\left\{{}_{p+1}F_p\left[\begin{matrix}-m,\ A_p;\ \lambda x\\ B_q\end{matrix}\right]\right\}dx =$$

(1.1)

$$= \frac{\Gamma(\beta+n+1)(\alpha-\xi)_n\Gamma(\xi+1)}{n!\,\Gamma(\beta+\xi+n+2)} F\left[\begin{array}{c|c|c} 2 & \xi+1,\ 1-\alpha+\xi & \lambda \\ p+1 & -m,\ A_p;\ -l,\ a_p & \\ 2 & 1-\alpha+\xi-n,\ \beta+\xi+n+2 & \mu \\ q & B_q;\ b_q & \end{array}\right]$$

where $\mathrm{Re}(\beta) > -1$, $\mathrm{Re}(\xi) > -1$.

Proof. In the known result due to Shah [4], setting $\lambda = c = 1$, and $m = 0$, we have

$$\int_0^1 x^{\xi}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x)\left\{{}_{p+1}F_q\left[\begin{matrix}-l,\ a_p\\ b_q\end{matrix};\ \mu x\right]\right\}dx = \tag{1.2}$$

$$= \frac{\Gamma(\beta+n+1)\,\Gamma(\xi+1)(\alpha-\xi)_n}{n!\ \Gamma(\beta+\xi+n+2)}\ {}_{p+3}F_{q+2}\left|\begin{matrix}-l,\xi+1,\ 1-\alpha+\xi,\ a_p;\mu\\ 1-\alpha+\xi-n,\ \beta+\xi+n+2,\ b_q\end{matrix}\right|$$

where $\mathrm{Re}(\beta)>-1$, $\mathrm{Re}(\xi)>-1$.

In (1.1), expressing the polynomial $\left\{{}_{p+1}F_q\left[\begin{matrix}-m,\ A_p\\ B_q\end{matrix};\lambda x\right]\right\}$ in series, then changing the order of integration and summation which is easily seen to be justified due to the absolute convergence of the integral and summation involved in the process, we obtain

$$\sum_{r=0}^{\infty}\frac{(-m)_r(A_p)_r\lambda^r}{r!\ (B_q)_r}\int_0^1 x^{\xi+r}(1-x)^{\beta}P_n^{(\alpha,\beta)}(1-2x)\left\{{}_{p+1}F_q\left[\begin{matrix}-l,\ a_p\\ b_q\end{matrix};\ \mu x\right]\right\}dx. \tag{1.3}$$

Now evaluating the integral of (1.3) by using (1.2) etc., we get

$$\frac{\Gamma(\beta+n+1)\,(\alpha-\xi)_n\,\Gamma(\xi+1)}{n!\ \Gamma(\beta+\xi+n+2)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\xi+1)_{r+s}}{r!\ s!\ (1-\alpha+\xi-n)_{r+s}} \frac{(1-\alpha+\xi)_{r+s}}{(\beta+\xi+n+2)_{r+s}}\frac{(-m)_r(A_p)_r}{(B_q)_r}\frac{(-l)_s(a_p)_s}{(b_q)_s}\lambda^r\mu^s \tag{1.4}$$

which yields the expression on the right of (1.1) by virtue of known Kampé de Fériet double hypergeometric function of higher order [2].

2. Expansion. The expansion formula to be stablished is

$$x^{\xi}\left\{{}_{p+1}F_q\left[\begin{matrix}-l,\ a_p\\ b_q\end{matrix};\ \mu x\right]{}_{p+1}F_q\left[\begin{matrix}-m,\ A_p\\ B_q\end{matrix};\ \lambda x\right]\right\}= \tag{2.1}$$

$$=(\alpha+1)_{\xi}\sum_{u=0}^{\infty}\frac{(-\xi)_u(\alpha+\beta+2u+1)}{(\alpha+\beta+u+1)_{\xi+1}\,(\alpha+1)_u}\,F\left[\begin{array}{c|c|c}2 & \alpha+\xi+1,\ 1+\xi & \lambda\\ p+1 & -m,\ A_p;\ -l,\ a_p & \\ 2 & 1+\xi-u,\ \beta+\alpha+\xi+u+2 & \\ q & B_q;\ b_q & \mu\end{array}\right]$$

$$P_u^{(\alpha,\beta)}(1-2x).$$

Proof. Let

$$f(x)=x^{\xi}\left\{{}_{p+1}F_q\left(\begin{matrix}-l,\ a_p\\ b_q\end{matrix};\ \mu x\right){}_{p+1}F_q\left(\begin{matrix}-m,\ A_p\\ B_q\end{matrix};\ \lambda x\right)\right\}= \tag{2.2}$$

$$=\sum_{n=0}^{\infty}C_nP_n^{(\alpha,\beta)}(1-2x),\quad (0<x<1).$$

From (2.2) we obtain C_n in a purely formal manner. With that value for C_n, we have then assumed that the series on the right in (2.2) actually converges to $f(x)$, provided $f(x)$ is sufficiently well behaved [6, p. 176, § 100].

From (2.2) it follows formally that

$$\int_0^1 x^{\xi+\alpha}(1-x)^{\beta}P_n^{(\alpha,\beta)}(1-2x)\left\{{}_{p+1}F_q\left(\begin{matrix}-l,\ a_p\\ b_q\end{matrix};\ \mu x\right){}_{p+1}F_q\left(\begin{matrix}-m,\ A_p\\ B_q\end{matrix};\ \lambda x\right)\right\}dx= \tag{2.3}$$

$$=\sum_{u=0}^{\infty}C_u\int_0^1 x^{\alpha}(1-x)^{\beta}P_u^{(\alpha,\beta)}(1-2x)P_n^{(\alpha,\beta)}(1-2x)dx.$$

All the integrals on the right in (2.3) vanish except for the single term for which $u=n$. Therefore, making use of the orthogonality-property of the Jacobi polynomials [1, p. 285, (5)]:

$$\int_0^1 x^{\alpha}(1-x)^{\beta}[P_n^{(\alpha,\beta)}(1-2x)]^2\,dx=\frac{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\ (\alpha+\beta+2n+1)\ \Gamma(\alpha+\beta+n+1)},$$

$$\mathrm{Re}(\alpha)>-1,\qquad \mathrm{Re}(\beta)>-1,$$

on the right and using (1.1) on the left of (2.3) we obtain

$$C_n=\frac{(-\xi)_n(\alpha+1)_{\xi}(\alpha+\beta+2n+1)}{(\alpha+\beta+n+1)_{\xi+1}\,(\alpha+1)_n}\,F\left[\begin{array}{c|c|c}2 & \alpha+\xi+1,\ 1+\xi & \lambda\\ p+1 & -m,\ A_p;\ -l,\ a_p & \\ 2 & 1+\xi-n,\ \beta+\alpha+\xi+n+2 & \\ q & B_q;\ b_q & \mu\end{array}\right] \tag{2.4}$$

In view of (2.2) an I (2.4) we obtain the expansion-formula (2.1).

3. Particular cases of (1.1) and (2.1)

(i) Taking $a_1=l+\gamma+\delta+1,\quad b_1=1+\gamma,\quad b_2=1/2,$
$A_1=m+\rho+\sigma+1,\quad B_1=1+\rho,\quad B_2=1/2$

and multiplying both sides by $(1+\gamma)_l/l!\ (1+\rho)_m/m!$, we obtain

$$\int_0^1 x^{\xi}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x) f_l^{(\gamma,\delta)}\begin{pmatrix} a_2,\dots,a_p \\ b_3,\dots,b_q \end{pmatrix}; \mu x\Big) f_m^{(\rho,\sigma)}\begin{pmatrix} A_2,\dots,A_p \\ B_3,\dots,B_q \end{pmatrix}; \lambda x\Big) dx =$$

$$(3.1)\qquad = \frac{\Gamma(\beta+n+1)\,\Gamma(\xi+1)(\alpha+\xi)_n\,(1+\gamma)(1+\rho)_m}{l!\; m!\; n!\; \Gamma(\beta+\xi+n+2)}$$

$$F\left[\begin{array}{c|c|c} 2 & \xi+1,\ 1-\alpha+\xi & \lambda \\ p+1 & -m,\ m+\rho+\sigma+1,\ \ -l,\ l+\gamma+\delta+1, & \\ 2 & A_2,\dots,A_p;\ a_2,\dots,a_p\ \ 1-\alpha+\xi-n,\ \ \beta+\xi+n+2 & \mu \\ q & 1+\rho,\ 1/2,\ B_3,\dots,B_q;\ 1+\gamma,\ 1/2,\ b_3,\dots,b_q & \end{array}\right]$$

$$x^{\xi} f_l^{(\gamma,\delta)}\begin{pmatrix} a_2,\dots,a_p \\ b_3,\dots,b_q \end{pmatrix}; \mu x\Big) f_m^{(\rho,\sigma)}\begin{pmatrix} A_2,\dots,A_p \\ B_3,\dots,B_q \end{pmatrix}; \lambda x\Big) =$$

$$(3.2)\qquad = \frac{(\alpha+1)_{\xi}\,(1+\gamma)_l\,(1+\rho)_m}{l!\; m!} \sum_{u=0}^{\infty} \frac{(-\xi)_u\,(\alpha+\beta+2u+1)}{(\alpha+\beta+u+1)_{\xi+1}\,(1+\alpha)_u}$$

$$F\left[\begin{array}{c|c|c} 2 & \alpha+\xi+1,\ 1+\xi & \lambda \\ p+1 & -m,\ m+\rho+\sigma+1,\ A_2,\dots,A_p;\ \ -l,\ l+\gamma+\delta+1,\ \ a_2,\dots,a_p & \\ 2 & 1+\xi-u,\ B+\alpha+\xi+u+2 & \mu \\ q & 1+\rho,\ 1/2,\ B_3,\dots,B_q;\ 1+\gamma,\ 1/2,\ b_3,\dots,b_q & \end{array}\right]$$

$$P_u^{(\alpha,\beta)}(1-2x)$$

where $\mathrm{Re}(\beta) > -1$ and $\mathrm{Re}(\xi) > -1$ and

$$f_n^{(\alpha,\beta)}\begin{pmatrix} a_2,\dots,a_p \\ b_3,\dots,b_q \end{pmatrix}; x\Big) = \frac{(1+\alpha)_n}{n!}\, {}_{p+1}F_q\left[\begin{matrix} -n,\ n+\alpha+\beta+1,\ a_2,\dots,a_p \\ 1+\alpha,\ 1/2,\ b_3,\dots,b_q \end{matrix}; x\right]$$

is a generalized Sister Celine polynomial [3].

With $m=0$ we obtain the known results [3] and [5].

In (3.1) and (3.2), setting $p=q=3$, $a_2=1/2$, $a_3=\zeta$, $b_3=h$, $A_2=1/2$, $A_3=\eta$, $B_3=k$, we have

$$\int_0^1 x^{\xi}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x) H_l^{(\gamma,\delta)}(\zeta,\ h,\ \mu x) H_m^{(\rho,\sigma)}(\eta,\ k,\ \lambda x)\, dx =$$

$$(3.3)\qquad = \frac{\Gamma(\beta+n+1)\,\Gamma(\xi+1)(\alpha-\xi)_n\,(1+\gamma)_l\,(1+\rho)_m}{l!\; m!\; n!\; \Gamma(\beta+\xi+n+2)}$$

$$F\left[\begin{array}{c|c|c} 2 & \xi+1,\ 1-\alpha+\xi & \lambda \\ 3 & -m,\ m+\rho+\sigma+1,\ \eta;\ \ -l,\ l+\gamma+\delta+1,\ \zeta & \\ 2 & 1-\alpha+\xi-n,\ \beta+\xi+n+2 & \mu \\ 2 & 1+\rho,\ k;\ 1+\gamma,\ h & \end{array}\right]$$

$$x^{\xi} H_l^{(\gamma,\delta)}(\zeta,\ h,\ \mu x) H_m^{(\rho,\sigma)}(\eta,\ k,\ \lambda x) =$$

$$= \frac{(\alpha+1)_{\xi}\,(1+\gamma)_l\,(1+\rho)_m}{l!\; m!} \sum_{u=0}^{\infty} \frac{(-\xi)_u\,(\alpha+\beta+2u+1)}{(\alpha+\beta+u+1)_{\xi+1}\,(1+\alpha)_u}.$$

$$(3.4)\qquad F\left[\begin{array}{c|c|c} 2 & \alpha+\xi+1,\ 1+\xi & \lambda \\ 3 & -m,\ m+\rho+\sigma+1,\ \eta;\ \ -l,\ l+\gamma+\delta+1,\ \zeta & \\ 2 & 1+\xi-u,\ \beta+\alpha+\xi+u+2 & \mu \\ 2 & 1+\rho,\ k;\ 1+\gamma,\ h & \end{array}\right] P_u^{(\alpha,\beta)}(1-2x),$$

where $\mathrm{Re}(\beta) > -1$, $\mathrm{Re}(\xi) > -1$ and

$$H_n^{(\alpha,\beta)}(\rho,\ \sigma,\ x) = \frac{(1+\alpha)_n}{n!}\, {}_3F_2\left[\begin{matrix} -n,\ n+\alpha+\beta+1,\ \rho \\ 1+\alpha,\ \sigma \end{matrix}; x\right]$$

is the generalized Rice's polynomial.

Further substituting $\zeta=h$, $\eta=k$, $\mu=\lambda=1$, we obtain the results for Jacobi polynomials

$$\int_0^1 x^{\xi}(1-x)^{\beta} P_n^{(\alpha,\beta)}(1-2x)\, P_l^{(\gamma,\delta)}(1-2x)\, P_m^{(\rho,\sigma)}(1-2x)\, dx =$$

$$(3.5)\qquad = \frac{\Gamma(\beta+n+1)\,\Gamma(\xi+1)(\alpha-\xi)_n(1+\gamma)_l(1+\rho)_m}{l!\; m!\; n!\; \Gamma(\beta+\xi+n+2)}$$

$$F\left[\begin{array}{c|c|c} 2 & \xi+1,\ 1-\alpha+\xi & 1 \\ 2 & -m,\ m+\rho+\sigma+1;\ \ -l,\ l+\gamma+\delta+1 & \\ 2 & 1-\alpha+\xi-n,\ \beta+\xi+n+2 & 1 \\ 1 & 1+\rho;\ 1+\gamma & \end{array}\right]$$

$$x^{\xi} P_l^{(\gamma,\delta)}(1-2x)\, P_m^{(\rho,\sigma)}(1-2x) =$$

$$\frac{(\alpha+1)_\xi\,(1+\gamma)_l\,(1+\rho)_m}{l!\; m!} \sum_{u=0}^{\infty} \frac{(-\xi)_u\,(\alpha+\beta+2u+1)}{(\alpha+\beta+u+1)_{\xi+1}\,(1+\alpha)_u}$$

$$(3.6)\qquad F\left[\begin{array}{c|c|c} 2 & \alpha+\xi+1,\ 1+\xi & \\ 2 & -m,\ \ m+\rho+\sigma+1;\ \ -l,\ \ l+\gamma+\delta+1 & 1 \\ 2 & 1+\xi-u,\ \ \beta+\alpha+\xi+u+2 & \\ 1 & 1+\rho;\ \ 1+\gamma & 1 \end{array}\right] P_n^{(\alpha,\beta)}(1-2x)$$

where $\mathrm{Re}(\beta) > -1$ and $\mathrm{Re}(\xi) > -1$.

(ii) Setting $p=0$, $q=1$, $b_1 = 1+\rho$, $B_1 = 1+\sigma$ and multiplying both sides by $(1+\rho)_l/l!$, $(1+\sigma)_m/m!$ we obtain the relations involving the generalized Laguerre polynomials

$$\int_0^1 x^{\xi}(1-x)^{\beta}\, P_n^{(\alpha,\beta)}(1-2x)\, L_l^{(\rho)}(\mu x)\, L_m^{(\sigma)}(\lambda x)\, dx =$$

$$(3.7)\qquad = \frac{\Gamma(\beta+n+1)(\alpha-\xi)_n\,\Gamma(\xi+1)(1+\rho)_l\,(1+\sigma)_m}{l!\; m!\; n!}$$

$$F\left[\begin{array}{c|c|c} 2 & \xi+1,\ 1-\alpha+\xi & \\ 1 & -m;\ -l & \lambda \\ 2 & 1-\alpha+\xi-n,\ \ \beta+\xi+n+2 & \\ 1 & 1+\sigma;\ \ 1+\rho & \mu \end{array}\right].$$

$$x^{\xi} L_l^{(\rho)}(\mu x)\, L_m^{(\sigma)}(\lambda x) = \frac{(\alpha+1)_\xi(1+\rho)_l\,(1+\sigma)_m}{l!\;\; m!} \sum_{u=0}^{\infty} \frac{(-\xi)_u\,(\alpha+\beta+2u+1)}{(\alpha+\beta+u+1)_{\xi+1}\,(\alpha+1)_u}$$

$$(3.8)\qquad F\left[\begin{array}{c|c|c} 2 & \alpha+\xi+1,\ 1+\xi & \\ 1 & -m;\ -l & \lambda \\ 2 & 1+\xi-u,\ \ \beta+\alpha+\xi+u+2 & \\ 1 & 1+\sigma,\ \ 1+\rho & \mu \end{array}\right] P_n^{(\alpha,\beta)}(1-2x),$$

where $\mathrm{Re}(\beta) > -1$ and $\mathrm{Re}(\xi) > -1$.

(iii) Substituting $p=1$, $q=0$, $a_1 = n+a-1$, $A_1 = m+A-1$, $\mu = -1/b$ and $\lambda = -1/B$, we have

$$\int_0^1 x^{\xi}(1-x)^{\beta}\, P_n^{(\alpha,\beta)}(1-2x)\, \gamma_l(x,y,b)\, \gamma_m(x,A,B)\, dx =$$

$$(3.9)\qquad = \frac{\Gamma(\beta+n+1)(\alpha-\xi)_n\,\Gamma(1+\xi)}{n!\; \Gamma(\beta+\xi+n+2)}$$

$$F\left[\begin{array}{c|c|c} 2 & \xi+1,\ 1-\alpha+\xi & \\ 2 & -m,\ \ m+A-1;\ \ -l,\ \ l+a-1 & -1/B \\ 2 & 1-\alpha+\xi-n,\ \ \beta+\xi+n+2 & \\ 0 & \cdots;\cdots & -1/b \end{array}\right]$$

$$x^{\xi}\gamma_l(x,a,b)\,\gamma_m(x,A,B) = (\alpha+1)_\xi \sum_{u=0}^{\infty} \frac{(-\xi)_u\,(\alpha+\beta+2u+1)}{(\alpha+\beta+u+1)_{\xi+1}\,(1+\alpha)_u}$$

$$(3.10)\qquad F\left[\begin{array}{c|c|c} 2 & \alpha+\xi+1,\ \ 1+\xi & \\ 2 & -m,\ \ m+A-1;\ \ -l,\ \ l+a-1 & -1/B \\ 2 & 1+\xi-u,\ \ \beta+\alpha+\xi+u+2 & \\ 0 & \cdots;\cdots & -1/b \end{array}\right] P_n^{(\alpha,\beta)}(1-2x)$$

where $\mathrm{Re}(\beta) > -1$, $\mathrm{Re}(\xi) > -1$ and

$$\gamma_n(x,a,b) = {}_2F_0\left[\begin{array}{c} -n,\ \ n+a-1 \\ \cdots \end{array};\ -1/bx\right]$$ is the generalized Bessel polynomial.

Department of Mathematics,
P.M.B.G. Science College,
Sanyogitaganj, Indore (M.P.), India

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UNELE REZULTATE PRIVITOARE LA POLINOAME HIPERGEOMETRICE ŞI POLINOAME JACOBI GENERALIZATE

Rezumat

Se evaluează o integrală dintr-un produs de două funcţii hipergeometrice şi funcţii Jacobi, cu ajutorul funcţiei hipergeometrice duble a lui J. Kampé de Fériet. Aceasta este utilizată pentru a stabili o formulă de dezvoltare pentru un produs de polinoame hipergeometrice.

ON INFINITE SUMMATION FORMULAE INVOLVING KAMPÉ DE FÉRIET'S FUNCTION

BY

V. L. DESHPANDE

1. The object of this paper is to establish the summation formulae for the Kampé de Fériet function [1, p. 150] of two variables. While obtaining the results, the operational technique as well as the method of finite mathematical induction is utilised. The function being the generalization of Appell's functions [1, p. 14] of two variables, yields many interesting special cases. Few are mentioned in the paper.

In what follows, (a_p) denotes the sequence of p parameters $a_1, a_2, \ldots, a_p$ and $((a_p))_m$ has the interpretation $\prod_{i=1}^{m} (a_i)_m$ and so on.

Formulae required in the proof are [2, p. 309, (3.4) (3.5)].

These formulae are valid provided $p \leq q$, $r \leqslant s$, and $|x|$, $|y|$ and $|z|$ are so constrained that the two sides have a meaning; we need also formula [3, p. 762], where

$p \leqslant q$, $r \leqslant s$, $\operatorname{Re}(\mu) > |\operatorname{Re}(\nu)|$; $\operatorname{Re}(z) > 0$ for $p < q$, $r < s$ or $\operatorname{Re}(z) > \operatorname{Re}(x) + \operatorname{Re}(y)$; $p = q$, $r = s$.

Replacing in (3.4) of [2], p. 309 x by xt, y by yt, multiplying both sides by $(t)^{\mu-1} . e^{-zt} . k_\nu(zt)$, integrating w.r. to t between the limits 0 to ∞ and interchanging the order of integration and summation, we have,

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^n \int_0^\infty e^{-zt} . t^{\mu-1} . k_\nu(zt) . {}_{p+1}F_q \left[\begin{matrix} -n, (a_p), \\ (b_q), \end{matrix} xt \right] . {}_{r+1}F_s \left[\begin{matrix} \lambda + n, (\alpha_r), \\ (\beta_s) \end{matrix} yt \right] . dt$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n ((a_p))_n ((\alpha_r))_n}{n! ((b_q))_n ((\beta_s))_n} \left(\frac{-xyz}{(1-z)^2} \right)^n . \int_0^\infty e^{-zt} . t^{\mu+2n-1} . k_\nu(zt).$$