UNELE REZULTATE PRIVITOARE LA POLINOAME HIPERGEOMETRICE ŞI POLINOAME JACOBI GENERALIZATE

Rezumat

Se evaluează o integrală dintr-un produs de două funcţii hipergeometrice şi funcţii Jacobi, cu ajutorul funcţiei hipergeometrice duble a lui J. Kampé de Fériet. Aceasta este utilizată pentru a stabili o formulă de dezvoltare pentru un produs de polinoame hipergeometrice.

ON INFINITE SUMMATION FORMULAE INVOLVING KAMPÉ DE FÉRIET'S FUNCTION

BY

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1 The object of this paper is to establish the summation formulae for the Kampé de Fériet function [1, p. 150] of two variables. While obtaining the results, the operational technique as well as the method of finite mathematical induction is utilised. The function being the generalization of Appell's functions [1, p. 14] of two variables, yields many interesting special cases. Few are mentioned in the paper.

In what follows, (a_p) denotes the sequence of p parameters $a_1, a_2, \ldots, a_p$ and $((a_p))_m$ has the interpretation $\prod_{i=1}^{m} (a_i)_m$ and so on.

Formulae required in the proof are [2, p. 309, (3.4) (3.5].

These formulae are valid provided $p \leqslant q$, $r \leqslant s$, and $|x|$, $|y|$ and $|z|$ are so constrained that the two sides have a meaning; we need also formula [3, p. 762], where

$p \leqslant q$, $r \leqslant s$, $\mathrm{Re}(\mu) > |\mathrm{Re}(\nu)|$; $\mathrm{Re}(z) > 0$ for $p < q$, $r < s$ or
$\mathrm{Re}(z) > \mathrm{Re}(x) + \mathrm{Re}(y)$; $p = q$, $r = s$.

Replacing in (3.4) of [2], p. 309 x by xt, y by yt, multiplying both sides by $(t)^{\mu-1}.\ e^{-zt}.\ k_\nu(zt)$, integrating w.r. to t between the limits 0 to ∞ and interchanging the order of integration and summation, we have,

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^n \int_0^{\infty} e^{-zt}.\ t^{\mu-1}.\ k_\nu(zt).\ {}_{p+1}F_q \left| \begin{matrix} -n, (a_p), \\ (b_q), \end{matrix} xt \right| .\ {}_{r+1}F_s \left| \begin{matrix} \lambda+n, (\alpha_p), \\ (\beta_s) \end{matrix} yt \right| .dt$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n ((a_p))_n ((\alpha_p))_n}{n! ((b_q))_n ((\beta_s))_n} \left(\frac{-xyz}{(1-z)^2} \right)^n .\int_0^{\infty} e^{-zt}.\ t^{\mu+2n-1}.\ k_\nu(zt).$$

$$ {}_{p+1}F_{q}\left|\begin{matrix}\lambda+n,(a_p+n), & \dfrac{xzt}{z-1}\\ (b_q+n), & \end{matrix}\right| \cdot {}_{r+1}F_{s}\left|\begin{matrix}\lambda+n,(\alpha_p+n), & \dfrac{yt}{1-z}\\ (\beta_s+n), & \end{matrix}\right| dt. $$

Evaluating the integrals on both sides of [3, p. 762] and replacing x by $2xz$, y by $2yz$ and μ,ν by $\dfrac{u+w}{2}, \dfrac{u-w}{2}$, we arrive at the formula,

$$ \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\cdot F\left[\begin{array}{l|l} u,\ w:\ -n,\ (a_p);\ \lambda+n,\ (\alpha_p) & x\\ \dfrac{u+w+1}{2}:\ (b_q);\qquad\qquad (\beta_s) & y \end{array}\right]\cdot z^n $$

(2.1)

$$ =(1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n\,((a_p))_n\,((\alpha_p))_n\,(u)_{2n}\,(w)_{2n}}{n!\,((b_q))_n\,((\beta_s))_n\left(\dfrac{u+w+1}{2}\right)_{2n}}\left(\frac{-xyz}{(1-z)^2}\right)^n $$

$$ \cdot F\left[\begin{array}{l|l} u+2n,\ w+2n:\ \lambda+n,\ (a_p+n);\ \lambda+n,\ (\alpha_p+n) & \dfrac{xz}{x-1}\\ \dfrac{u+w+1}{2}+2n:\qquad (b_q+n);\qquad (\beta_s+n) & \dfrac{y}{1-z} \end{array}\right] $$

Proceeding on the same lines as above with the second, we obtain the formula,

$$ \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,F\left[\begin{array}{l|l} u,\ w\quad :\ \lambda+n,\ (a_p);\ \lambda+n,\ (\alpha_p) & x\\ \dfrac{u+w+1}{2}:\qquad (b_q);\qquad (\beta_s) & y \end{array}\right].z^n $$

(2.2)

$$ =(1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n\,((a_p))_n\,((\alpha_p))_n\,(u)_{2n}\,(w)_{2n}}{n!\,((b_q))_n\,((\beta_s))_n\left(\dfrac{u+w+1}{2}\right)_{2n}}\left(\frac{xyz}{(1-z)^2}\right)^n $$

$$ F\left[\begin{array}{l|l} u+2n,\ w+2n:\ \lambda+n,\ (a_p+n);\ \lambda+n,\ (\alpha_p+n) & \dfrac{x}{1-z}\\ \dfrac{u+w+1}{1}+2n:\qquad (b_q+n);\qquad (\beta_s+n) & \dfrac{y}{1-z} \end{array}\right]. $$

The results (2.1) and (2.2) are valid provided $p\leqslant q$, $r\leqslant s$, $\mathrm{Re}\,(u+w)>|\mathrm{Re}(u-w)|$ and $|x|$, $|y|$ and $|z|$ are so constrained that the two sides have a meaning.

We can imagine from (2.1) and (2.2) more general results as given below:

$$ \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,F\left[\begin{array}{l|l} u,w,\qquad (C_\rho):\ -n,\ (a_p):\ \lambda+n,\ (\alpha_p) & x\\ \dfrac{u+w+1}{2}\,(C'_\sigma):\qquad (b_q):\qquad (\beta_s) & y \end{array}\right]\cdot z^n $$

(2.3)

$$ =(1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n.\,((a_p))_n\,((\alpha_p))_n\,(u)_{2n}\,(w)_{2n}\,((C_\rho))_{2n}}{n!\,((b_q))_n\,((\beta_s))_n\left(\dfrac{u+w+1}{2}\right)_{2n}((C'_\sigma))_{2n}}\left(\frac{-xyz}{(1-z)^2}\right)^n. $$

$$ F\left[\begin{array}{l|l} u+2n,\quad w+2n,\ (C_\rho+2n):\lambda+n,\ (a_p+n);\ \lambda+n,\ (\alpha_p+n) & \dfrac{xz}{z-1}\\ \dfrac{u+w+1}{2}+2n,\ (C'_\sigma+2n):\qquad (b_q+n);\qquad (\beta_s+n) & \dfrac{y}{1-z} \end{array}\right] $$

and

$$ \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,F\left[\begin{array}{l|l} u,w,\qquad (C_\rho):\ \lambda+n,\ (a_p):\ \lambda+n,\ (\alpha_p) & x\\ \dfrac{u+w+1}{2},\ (C'_\sigma):\qquad (b_q):\qquad (\beta_s) & y \end{array}\right] z^n $$

(2.4)

$$ =(1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n\,((a_p))_n\,((\alpha_p))_n\,(u)_{2n}\,(w)_{2n}\,((C_\rho))_{2n}}{n!\,((b_q))_n\,((\beta_s))_n\left(\dfrac{u+w+1}{2}\right)_{2n}((C'_\sigma))_{2n}}\left(\frac{xyz}{(1-z)^2}\right)^n. $$

$$ .F\left[\begin{array}{l|l} u+2n,\quad w+2n,\ (C_\rho+2n):\ \lambda+n,\ (a_p+n);\ \lambda+n,\ (\alpha_p+n) & \dfrac{x}{1-z}\\ \dfrac{u+w+1}{2}+2n,\ (C'_\sigma+2n):\qquad (b_q+n);\qquad (\beta_s+n) & \dfrac{y}{1-z} \end{array}\right] $$

provided $p\leqslant q$, $r\leqslant s$, Re $(u+w)>|\mathrm{Re}(u-w)|$ and $|x|$, $|y|$ and $|z|$ are so constrained that the two sides have a meaning.

We observe that (2.3) is true for $\rho=\sigma=0$ because of (2.1). For the proof of (2.3) by the method of finite mathematical induction, let us assume it to be true for some values of ρ and σ. Replacing x by xt, y by yt in (2.3) and multiplying both sides by $(t)^{C_{\rho+1}-1}$ and taking their Laplace transform with respect to t, we find that ρ is replaced by $\rho+1$. Again in (2.3), substituting x by x/t, y by y/t, multiplying both sides by $(t)^{-C'_{\sigma+1}}$ and taking the inverse Laplace transform, we observe that σ is replaced by $\sigma+1$. Thus the induction on ρ and σ is complete. Hence (2.3) is proved. Similarly, starting with (2.2) and employing the Laplace transform technique the formula (2.4) can easily be established.

Particular cases: (i) Setting $p=q=r=s=0$, $\rho=1$, $\sigma=2$, $u=w+1$ and $C_1=w$ in (2.3) and (2.4), we obtain the results involving Appell's function F_4[1, p. 14].

(ii) In (2.3), substituting $p = r = 0$, $q = s = \rho = \sigma = 1$, $u = w + 1$ and $c'_1 = w$, we have a results involving Appel's function F_2[1, p. 14], with $\lambda = b_1$ and employing the formula [4, p. 105(3)], we have after little simplification, an interesting formula,

$$\sum_{n=0}^{\infty} \frac{(b_1)_n}{n!} F_2(c_1, -n, b_1, +n, b_1, \beta_1; x, y) . z^n = (1-z)^{c_1-b_1} . (1 + xz - \tag{2.5}$$
$$- y - z)^{-c_1} . H_3\left(c_1, \beta_1 - b_1; \beta_1; \frac{-xyz}{(1+xz-y-z)^2}, \frac{-y}{1+xz-y-z}\right)$$

where H_3 is the Horn's function [4, p. 225, (15)].

(iii) In (2.4), substituting $p = r = 0$, $q = s = \rho = \sigma = 1$, $c'_1 = w$, and $u = w + 1$, we have a result involving Appell's function F_2.

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ASUPRA UNOR FORMULE DE SUMARE CUPRINZÎND FUNCŢIILE LUI KAMPÉ DE FÉRIET

Rezumat

Obiectul articolului este stabilirea unor formule de sumare pentru funcţia lui Kampé de Fériet. Pentru obţinerea rezultatelor se utilizează tehnica operaţională şi metoda inducţiei finite. Unele din cazurile particulare sînt menţionate.

ON A RELATION INVOLVING HARDY'S TRANSFORM

BY

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1. Let $f(t)$ denote any function of a prescribed class of functions defined on a given interval (a, b). Further let $k(t, p)$ denote a definite function of t on that interval for each value of p, a parameter whose domain is prescribed; then an integral transform $T\{f(t); p\}$ of $f(t)$ is defined by means of the integral equation:

$$T\{f(t); p\} = p \int_0^{\infty} k(t, p) f(t)\, dt, \tag{1}$$

where the class of functions and the domain of parameters are so prescribed that the integral has sense. The function $k(t, p)$ is known as the kernel of the transform. Here we shall restrict our attention to the case in which the kernel $k(t, p)$ is a function of tp alone, $k(t, p) = k(p, t) = k(tp)$ as this case has the widest applications.

By taking a suitable Special Function for the kernel $k(t, p)$ in (1), various integral transforms have been defined and developed by many authors including Erdélyi [4], Meijer [9], Varma [15], Bora, Kalla & Saxena [2] and Gupta and Mittal [6].

By taking a kernel $t F_{\nu,\alpha}(pt)$, where

$$F_{\nu,\alpha}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k [(1/2)t]^{\nu+2\alpha+2k}}{\Gamma(\alpha+1+k)\,\Gamma(\alpha+1+\nu+k)}$$

Hardy [7, p. 62] has given a generalization of Hankel's transform