ON THE LINEAR CONNEXIONS ON A MANIFOLD ADMITTING AN ALMOST COSYMPLECTIC STRUCTURE

BY

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Let us consider on the $(2n+1)$-dimensional, C^∞-differentiable manifold M an almost cosymplectic structure, defined by the quadratic differential form Ω and the Pfaffian form ω, satisfying the condition:

$$\Omega^n \wedge \omega \neq 0.$$

We have determined, [6, 7], the general family of the almost cosymplectic and conformal cosymplectic connexions. In this short note we shall show that there exists a linear connexion defined on M so that the covariant derivatives of the tensor fields associated with the forms Ω and ω, are two prescribed tensor fields satisfying only a condition of skew-symmetry. Let us denote by A the tensor field of type $(0, 2)$, defined by:

$$A(X, Y) = \langle X \wedge Y; \Omega \rangle,$$

where X, Y are two arbitrary vector fields. Let us consider two tensor fields Q, q of type $(0, 3)$, $(0, 2)$, respectively, satisfying the condition:

$$Q(X, Y, Z) = -Q(X, Z, Y).$$

There exists a linear connexion defined on M, so that the covariant derivatives of A and ω, with respect to this connexion are Q and q.

To prove this result we need some results on the algebraic properties of the tensor fields A and ω, [6, 7]. There exists a vector field E and a tensor field $\tilde{A}$ of type $(2, 0)$, both defined on M, so that if X is an arbitrary vector field, then:

$$
(1)\qquad
\begin{aligned}
&A(E, X) = -A(X, E) = 0,\\
&\omega(E) = 1,\\
&\omega\widetilde{A} = -\omega A = 0,\\
&A_X \cdot \widetilde{A} = X - \omega(X)E,
\end{aligned}
$$

where A_X is the 1-form defined by: $(A_X)(Y) = A(X, Y)$ and the operations $\omega\widetilde{A}$, $\omega.\widetilde{A}$ are given in local coordinates by the following formulae:

$$
\begin{aligned}
&(\omega.\widetilde{A})^i = \omega_j \widetilde{A}^{ji},\\
&(\omega \cdot \widetilde{A})^i = \omega_j \widetilde{A}^{ij}; \quad i, j = 0, 1, 2, \ldots, 2n.
\end{aligned}
$$

We shall determine the general family of the linear connexions satisfying the conditions:

$$
(2)\qquad
\begin{aligned}
&(\nabla_X A)(Y, Z) = Q(X, Y, Z),\\
&(\nabla_X \omega)(Y) = q(X, Y),
\end{aligned}
$$

where by ∇ we have denoted the covariant derivative, and we shall make some remarks on the connexions of this type on the manifolds admitting an almost symplectic or a Riemannian structure.

Let $\dot{\nabla}$ be an, initially given, connexion and let us suppose that the connexion ∇ is defined by:

$$
(3)\qquad \nabla_X Y = \dot{\nabla}_X Y + P(X, Y),
$$

where the tensor field P of type $(1, 2)$, must satisfy the following system obtained from the conditions (2):

$$
(4)\qquad
\begin{aligned}
&A(P(X, Y), Z) + A(Y, P(X, Z)) = (\dot{\nabla}_X A)(Y, Z) - Q(X, Y, Z),\\
&\omega(P(X, Y)) = (\dot{\nabla}_X \omega)(Y) - q(X, Y).
\end{aligned}
$$

It can be easily verified that this system is equivalent to the following system:

$$
(5)\qquad
\begin{aligned}
&P(X, Y) - \omega(P(X, Y))E - A_Y P_X \widetilde{A} = (\dot{\nabla}_X A)_Y \cdot \widetilde{A} - Q_{XY} \cdot \widetilde{A},\\
&\omega(P(X, Y)) = (\dot{\nabla}_X \omega)(Y) - q(X, Y),
\end{aligned}
$$

where Q_{XY} is the 1-form defined by the relation $(Q_{XY})(Z) = Q(X, Y, Z)$, and P_X is the tensor field of type $(1, 1)$ defined by: $(P_X)(Y) = P(X, Y)$.

The system (5) is equivalent to the equation:

$$
(6)\qquad P(X, Y) - A_Y P_X \widetilde{A} = (\dot{\nabla}_X A)_Y \cdot \widetilde{A} - Q_{XY} \cdot \widetilde{A} + (\dot{\nabla}_X \omega)(Y)E - q(X, Y)E.
$$

Introducing the tensor fields Φ, Ψ, Θ, of type $(2, 2)$ defined by:

$$
(7)\qquad
\begin{aligned}
&\Psi = \frac{1}{2}(I \otimes I - A \otimes \widetilde{A}),\\
&\Phi = \frac{1}{2}(I \otimes I + A + \widetilde{A},\\
&\Theta = \frac{1}{2}(I \otimes \omega \otimes E + \omega \otimes E \otimes I - \omega \otimes E \otimes \omega \otimes E),
\end{aligned}
$$

the equation (6) can be written as follows:

$$
(8)\qquad \Psi_Y P_X = \frac{1}{2}(\dot{\nabla}_X A)_Y \cdot \widetilde{A} + \frac{1}{2}(\dot{\nabla}_X \omega)(Y)E - \frac{1}{2} Q_{XY} \cdot \widetilde{A} - \frac{1}{2} q(X, Y)E,
$$

where the operation ΨP is given in local coordinates by:

$$
(\Psi P)^i_{jk} = \Psi^{il}_{jh} P^h_{lk}; \quad i, j, k, l, h = 0, 1, \ldots, 2n.
$$

Considering the products $\Phi\Psi$ given in local coordinates by:

$$
(\Phi\Psi)^{kl}_{ij} = \Phi^{ml}_{ih} \Psi^{kh}_{mj},
$$

it is proved [6, 7] that:

$$
(9)\qquad
\begin{aligned}
&\Phi + \Psi = I \otimes I; \quad \Phi\Phi = \Phi - \frac{1}{2}\Theta; \quad \Psi\Psi = \Psi - \frac{1}{2}\Theta,\\
&\Phi\Psi = \Phi\Theta = \Psi\Phi = \Psi\Theta = \Theta\Phi = \Theta\Psi = \Theta\Theta = \frac{1}{2}\Theta.
\end{aligned}
$$

Then the equation (8) is equivalent to the following equation:

$$
(10)\qquad
\begin{aligned}
(\Psi + \Theta)_Y P_X = {}&\frac{1}{2}(\dot{\nabla}_X A)_{Y+\omega(Y)E} \cdot \widetilde{A} + (\dot{\nabla}_X \omega)(Y)E -\\
&- \frac{1}{2} Q_{X,\, Y+\omega(Y)E} \cdot \widetilde{A} - q(X, Y)E.
\end{aligned}
$$

One verifies readily that $\Psi + \Theta$, $\Phi - \Theta$ are projectors i.e.

$$
(11)\left\{
\begin{aligned}
&(\Psi + \Theta) + (\Phi - \Theta) = I \otimes I; \quad (\Psi + \Theta)(\Psi + \Theta) = \Psi + \Theta;\\
&(\Phi - \Theta)(\Phi - \Theta) = \Phi - \Theta; \quad (\Psi + \Theta)(\Phi - \Theta) = (\Phi - \Theta)(\Psi + \Theta) = 0.
\end{aligned}
\right.
$$

The equation (10) has a solution if and only if the operator $\Phi - \Theta$, applied to the right hand side of this equation gives zero. This condition can be verified by a straightforward calculation and then the general solution of the equation (10) is given by the formula:

$$P(X, Y) = \frac{1}{2} (\dot{\nabla}_X A)_{Y+\omega(Y)E} \cdot \tilde{A} +$$

$$(12) \qquad + (\dot{\nabla}_X \omega)(Y)E - \frac{1}{2} Q_{X,\, Y+\omega(Y)E} \cdot \tilde{A} - q(X, Y)E + (\Phi - \Theta)_Y S_X,$$

where S is an arbitrary tensor field of type $(1,2)$.

Theorem 1. *The general family of the connexions* ∇, *satisfying the conditions*; $(\nabla_X A)(Y, Z) = Q(X, Y, Z)$; $(\nabla_X \omega)(Y) = q(X, Y)$, *where* $Q \in \mathfrak{F}^0_3(M)$, $q \in \mathfrak{F}^0_2(M)$ *are given tensor fields with* $Q(X, Y, Z) = - Q(X, Z, Y)$, *is given by the formula:*

$$\nabla_X Y = \dot{\nabla}_X Y + \frac{1}{2} (\dot{\nabla}_X A)_{Y+\omega(Y)E} \cdot \tilde{A} +$$

$$+ (\dot{\nabla}_X \omega)(Y)E - \frac{1}{2} Q_{X,\, Y+\omega(Y)E} \cdot \tilde{A} - q(X, Y)E + (\Phi - \Theta)_Y S_X,$$

where $\dot{\nabla}$ *is an arbitrary initial connexion and* S *is a tensor field of type* $(1,2)$.

Particular cases: If $Q = q = 0$, we obtain the almost cosymplectic connexions and if $Q = \lambda \otimes A$, $q = \mu \otimes \omega$, we obtain the conformal almost cosimplectic connexions.

Remark 1. If we consider an almost symplectic structure on the $2n$-dimensional manifold M, then we can find always a connexion so that the covariant derivative of the fundamental tensor field is a given tensor field Q of type $(0,3)$, satisfying the condition $Q(X, Y, Z) = - Q(X, Z, Y)$.

Remark 2. Considering a Riemannian structure on the manifold M, we obtain a connexion so that the covariant derivative of the fundamental tensor field is a given tensor field Q of type $(0,3)$, satisfying the condition $Q(X, Y, Z) = Q(X, Z, Y)$.

Remark 3. In the case of the almost horsymplectic structure considered in [9], we can find a connexion ∇ such that $(\nabla_X A)(Y, Z) = Q(X, Y, Z)$, if and only if Q satisfies the following conditions:

$$Q(X, Y, Z) = - Q(X, Z, Y),$$

$$Q(X, vY, vZ) = 0,$$

where v is the projector on the vertical distribution of the given almost horsymplectic structure.

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ASUPRA CONEXIUNILOR LINIARE PE VARIETĂŢI ADMIŢÎND O STRUCTURĂ APROAPE COSIMPLECTICĂ

Rezumat

Se arată că pe o varietate diferenţiabilă de dimensiune impară, admiţînd o structură aproape cosimplectică, există o conexiune liniară astfel că derivatele covariante, în raport cu această conexiune, ale tensorilor ce definesc structura aproape cosimplectică sînt nişte tensori prescrişi, satisfăcînd doar o condiţie de antisimetrie. Sînt făcute cîteva observaţii privitoare la aceeaşi problemă pentru structurile aproape simplectice şi Riemanniene