Les résultats obtenus donnent:

Lemme 20. $R_i(V^*) = \{(u, z_1)/u \in R_i(V_i)\}$.

Ce résultat, son dual, et le lemme 15 nous donnent:

Lemme 21. *V^* a un moins une transversale de largeur 2.*

Lemme 22. *Si $u \in V$ est un élément d'une transversale de largeur $k(k = 2, 3)$ alors cela a lieu aussi pour l'élément (u, z_1), de V.*

Démonstration. Soit $u \cup a \cup b = 1$, $u \cap a \cap b = 0$ et la transversale (u, a, b) est irréductible. Alors les éléments (u, z_1), (a, z_2), (b, z_2) forment une transversale de largeur 3 de V^*, parce que $(u, z_1) \cup (a, z_2) \cup (b, z_2) = (u \cup a \cup b, z_2) = (1, z_2)$ et $(u, z_1) \cap (a, z_2) \cap (b, z_2) = (u \cap a \cap b, z_1) = (0, z_1)$. En outre on voit aisément que la transversale $((u, z_1), (u, z_2), (b, z_2))$ est irréductible dans V.

Théorème 7. *Un treillis fini et distributif V peut être plongé dans un treillis distributif $\overline{V}$ de telle manière que outre 0 et 1 chaque élément de $\overline{V}$ appartienne à une transversale de largeur au plus 3.*

Si V a n éléments alors $\overline{V}$ a au plus $4n$ éléments.

Démonstration. Nous distinguerons deux cas selon que, dans V, a) l'ensemble B de V est vide et b) le cas contraire.

a) Le lemme 15 nous assure que V a une transversale formée seulement de 2 éléments (a, b). Alors $((u, z_1), (a, z_2), (b, z_2))$ est une transversale irréductible de (u, z_1) et $\overline{V} = V^*$.

b) Si V a une transversale de largeur 2 alors on utilise le raisonnement de a). Dans le cas contraire nous aurons selon le lemme 21 une telle ransversale dans V^* et $\overline{V} = (V^*)^*$.

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O PROPRIETATE A LATICELOR FINITE ŞI DISTRIBUTIVE

Rezumat

Se generalizează pentru un reticul distributiv noţiunea de complement Se dau criterii pentru ca un element al reticulului să aibă un complement format din 1, 2 sau 3 elemente. Se dă o teoremă care generalizează o teoremă a lui H. Stone.

WEAK RING EXTENSIONS

BY

TUDORA LUCHIAN

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1. A weak ring $R^{\cdot}$ [1] is an additive abelian group R_+, in which we define a second composition law „$\cdot$" related to the addition by the weak distributive laws

$$a \cdot (b_1 + b_2) + a \cdot 0 = a \cdot b_1 + a \cdot b_2, \tag{1}$$

$$(b_1 + b_2) \cdot a + 0 \cdot a = b_1 \cdot a + b_2 \cdot a, \tag{2}$$

which imply

$$(3) \quad a \cdot 0 = a\varphi + c, \qquad (4) \quad 0 \cdot a = a\psi + c \quad \forall a \in R_+,$$

where φ, ψ are group endomorphisms of R_+ and c a fixed element of R_+

To an ordinary ring $R^{\times}$ (let denote by „$\times$" its multiplication, not necessarily associative), one can associate a weak ring $R^{\cdot}$, by choosing two endomorphisms φ, ψ of its additive group R_+ and defining a new multiplication „$\cdot$" by

$$a \cdot b = a \times b + a\varphi + b\psi + c, \quad \forall a, b \in R_+. \tag{5}$$

Conversely, to any weak ring $R^{\cdot}$ one can associate a unique ordinary ring $R^{\times}$ with the same additive group R_+ and the multiplication „$\times$" defined by

$$a \times b = a \cdot b - a \cdot 0 - 0 \cdot b + 0 \cdot 0. \tag{6}$$

If a weak ring $R^{\cdot}$ possesses an element d with the property

$$a \cdot d = d \cdot a = d, \quad \forall a \in R^{\cdot}, \tag{7}$$

we shall call $R^{\cdot}$ a *weak ring with* d; the necessary and sufficient conditions for the existence of d were given in [3]. For a weak ring with d, (5), (6) become respectively

$$(5_1) \qquad a \cdot b = (a - d) \times (b - d) + d,$$

$$(6_1) \qquad a \times b = (a + d) \cdot (b + d) - d.$$

A weak ring $R^{\cdot}$ with d is a *weak integral domain* if

$$a \cdot b = d \Rightarrow a = d \text{ or } b = d.$$

A weak ring with d, $R^{\cdot}$, is a weak field of species II if $R^{\cdot} - \{d\}$ is a multiplicative group. In any weak ring $R^{\cdot}$ we have defined [4], [5], the endomorphisms D_a, S_a of R_+ as follows

$$(8_1) \qquad rD_a = r \cdot a - 0 \cdot a, \qquad (8_2) \qquad rS_a = a \cdot r - a \cdot 0,$$

which imply

$$(9_1) \quad D_a = R_a + \varphi, \qquad (9_2) \quad S_a = L_a + \psi, \qquad D_0 = \varphi, \quad S_0 = \psi$$

with φ, ψ from (3), (4), whereas L_a, R_a are the left and right multiplications in the associated ordinary ring $R^{\times}$.

2. A first approach to the problem of weak ring extensions was given in [6], where weak polynomials over the ring $R^{\cdot}$ were constructed. We recall from [6] the fact that, giving first a more general definition, we found that — for a weak ring $R^{\cdot}$ with d, whose $R^{\times}$ has the identity e — a weak polynomial over $R^{\cdot}$ can be defined as

$$(10) \qquad (a_i)_n = f^{*}(x) = a_0 + a_1 D_x + a_2 D_{x^2} + \cdots + a_n D_{x^n},$$

where the *indeterminate* x is the weak polynomial

$$(11) \qquad x = (-e\varphi, e, 0, \cdots),$$

$D_x, \cdots, D_{x^n}$ are the endomorphisms defined by (8_1), while x^n is the n-th power in the sense of polynomial multiplication, given by the following formulas:

$$(12) \qquad (a_i)_n \cdot (b_i)_t = (a_0 \cdot b_0, c_1, \cdots, c_{n+t}, 0, \cdots)$$

$$\text{with } c_p = b_p S_{a_0} + a_p D_{b_0} + \sum_{\substack{s=1 \\ (p \geq 2)}}^{p-1} a_s \times b_{p-s}, \quad p = 1, 2, \cdots$$

$$(12_1) \qquad (a_i)_n \cdot (0) = (a_0 \cdot 0, a_1 \varphi, \cdots, a_n \varphi, 0, \cdots),$$

$$(12_2) \qquad (0) \cdot (a_i)_n = (0 \cdot a_0, a_1 \psi, \cdots, a_n \psi, 0, \cdots),$$

$$(12_3) \qquad (0) \cdot (0) = (c, 0, 0, \cdots).$$

The set of weak polynomials in the indeterminate x, (10) — with the addition defined as usually and the multiplication defined by (12) — (12_3) — is a weak ring $R^{\cdot}[x] \supset R^{\cdot}$. We shall call it *the weak polynomial ring of the indeterminate* x *over* $R^{\cdot}$.

The existence conditions of this weak ring hold, in particular, when $R^{\cdot}$ is associative (this implies the associativity of the associated ring $R^{\times}$, [4]).

Another aspect of the extension problem, the construction of the weak quotient field for a weak integral domain, was treated in [7].

In the present Note, we study the simple weak ring extensions.

3. Let $R^{\cdot}$ be a weak ring with d, $R^{\times}$ its associated ordinary ring with the identity e, $A^{\cdot}$ a weak subring of $R^{\cdot}$, $e \in A^{\cdot}$, and $v \in R^{\cdot}$ an element weakly permutable with every element of $A^{\cdot}$. Let us consider the weak subring of $R^{\cdot}$, $A^{\cdot}[v]$, obtained by the adjunction of v to $A^{\cdot}$, that is the intersection of all weak subrings of $R^{\cdot}$ which contain v and $A^{\cdot}$.

On the other hand, let us consider the set $A^{*}[v]$ of the weak polynomials of v over $A^{\cdot}$, $f^{*}(v)$.

$$(10') \qquad f^{*}(v) = a_0 + a_1 D_v + \cdots + a_n D_{v^n}, \quad a_i \in A^{\cdot}.$$

Defining the addition of $f^{*}(v)$, $g^{*}(v)$ by the addition of their coefficients and the multiplication by (12) — (12_3), the set $A^{*}[v]$ is a weak ring containing $A^{\cdot}$ and $v = d + eD_v$. Obviously, $A^{*}[v] \subseteq R^{\cdot}$, but, in general, $A^{*}[v]$ is not a weak subring of $R^{\cdot}$, because the multiplication defined above does not coincide with the one defined for $f^{*}(v)$, $g^{*}(v)$, considered as sums of products from $R^{\cdot}$ (that is writing D_{v^n} as in (8_1)). But, if $R^{\cdot}$ is an associative weak ring, the two multiplications mentioned above are coincident, and thus $A^{*}[v]$ is a weak subring of $R^{\cdot}$, containing $A^{\cdot}$ and v, hence

$$A^{*}[v] \supseteq A^{\cdot}[v].$$

Writing explicitly (10') (by (8_1)) one can observe that $A^{*}[v] \subseteq A^{\cdot}[v]$, hence these two rings are coincident. Moreover, if we recall that the associativity of $R^{\cdot}$ and the existence of $e \in R^{\times}$ imply the existence of $d = -e\varphi$ (a consequence of Remark 8 from [6]), we have proved:

Proposition 1. *If $R^{\cdot}$ is an associative weak ring such that $R^{\times}$ has the identity e, $A^{\cdot}$ is a weak subring of $R^{\cdot}$, $e \in A^{\cdot}$, and $v \in R^{\cdot}$ is weakly permutable with all the elements of $A^{\cdot}$, then the adjunction weak ring $A^{\cdot}[v]$ coincides with $A^{*}[v]$, the weak ring of the weak polynomials of v over $A^{\cdot}$*

$$A^{\cdot}[v] = A^{*}[v].$$

Under the same hypotheses as in Proposition 1, let $R^{\times}$ be the ordinary ring, associated to $R^{\cdot}$ by (6_1); then, [2], $A^{\cdot}$ is transformed in the ordinary subring $A^{\times} \subseteq R^{\times}$, $e \in A^{\times}$, while $A^{*}[v] = A^{\cdot}[v]$ is transformed in an ordinary subring A of $R^{\times}$, whose elements can be written as

$$(13) \qquad f^{*}(v) = a_0 + \sum_{k=1}^{n} a_k D_{v^k} = a_0 + \sum_{k=1}^{n} a_k \times (v - d)^k = f^{\times}(v - d).$$

This shows that $\bar{A}$ is the ordinary adjunction ring

$$\bar{A} = A^{\times}[v-d] \subset R^{\times}.$$

Conversely, from $A^{\times}[v-d]$ one can obtain $A^{*}[v] = A^{\cdot}[v]$, by using (5_1). Consequently, the following proposition holds:

Proposition 2. *Under the same hypothesis as in Proposition 1, the adjunction weak ring $A^{\cdot}[v]$ has the same additive group as the ordinary adjunction ring $A^{\times}[v-d]$ and it is obtained from this one by writing the polynomials $f^{\times}(v-d)$ as weak polynomials (13) and performing their multiplication by the rules (12) — (12_3); $A^{\times}[v-d]$ is the ordinary ring associated to $A^{\cdot}[v]$.*

4. Under the conditions of Proposition 1, the weak ring of the indeterminate x, $R^{\cdot}[x]$, and its weak subring $A^{\cdot}[x]$ do exist. Defining the mapping

$$(14) \qquad \chi : A^{\cdot}[x] \rightarrow A^{\cdot}[v], \quad [f^{*}(x)]\chi = f^{*}(v)$$

one can easily verify that χ is a weak ring epimorphism. Applying the weak ring homomorphism theorem ([2], Th. 3), we obtain the isomorphism

$$(15) \qquad A^{\cdot}[v] \overset{\xi^{\cdot}}{\simeq} \frac{A^{\cdot}[x]}{N_{\chi}^{\cdot}},$$

where $N_{\chi}^{\cdot} = \operatorname{Ker} \chi$, is a weak ideal

$$(16) \qquad N_{\chi}^{\cdot} = \{ q^{*}(x) \mid q^{*}(x) \in A^{\cdot}[x], \ q^{*}(v) = 0 \}.$$

The following two situations can arise:

I. $N_{\chi}^{\cdot} = (0)$, the zero weak ideal, then

$$(17) \qquad A^{\cdot}[v] \cong A^{\cdot}[x]$$

and the only weak polynomial $q^{*}(x)$ with $q^{*}(v) = 0$ is the zero weak polynomial, 0^{*}.

II. $N_{\chi}^{\cdot} \neq (0)$, then there exist weak polynomials $q^{*}(x) \neq 0^{*}$ so that $q^{*}(v) = 0$ and we have obviously

$$A^{\cdot}[v] \ni q_1^{*}(v) = q_2^{*}(v) \Longleftrightarrow q_1^{*}(x) - q_2^{*}(x) \in N_{\chi}^{\cdot} \subset A^{\cdot}[x].$$

We have thus proved:

Theorem 3. *If $R^{\cdot}$ is an associative weak ring such that $R^{\times}$ has the identity e, $A^{\cdot}$ is a weak subring of $R^{\cdot}$, $e \in A^{\cdot}$ and $v \in R^{\cdot}$ is weakly permutable with the elements of $A^{\cdot}$, then for the simple extension weak ring $A^{\cdot}[v] = A^{*}[v]$ we have*

I. *Either* (17) $A^{\cdot}[v] \overset{\xi^{\cdot}}{\cong} A^{\cdot}[x]$, *then v is called a weakly transcendental element over $A^{\cdot}$.*

II. *Or* (15) $A^{\cdot}[v] \overset{\xi^{\cdot}}{\cong} \dfrac{A^{\cdot}[x]}{N_{\chi}^{\cdot}}$,

where $N_{\chi}^{\cdot} = \operatorname{Ker} \chi$, χ from (14); then v is called weakly algebraic over $A^{\cdot}$.

To find the relation between the concepts of weak transcendency and algebraicity over $A^{\cdot}$ — on one hand — and the same concepts, in the ordinary sense, over $A^{\times}$ — on the other hand — let us observe that, from the validity of Newton's formula for $(v-d)^{k}$ in the associative noncommutative ordinary ring $R^{\times}$ (the weak permutability of v gives, in particular, $v \cdot 0 = 0 \cdot v \Rightarrow v \times d = d \times v$), it follows that Taylor's formula holds for a polynomial $\varphi(X) \in A^{\times}[X]$, $X = x - d$. Hence $\varphi(v)$ can be expanded as a polynomial of $v - d$ in $R^{\times}$. By using this, one can prove:

Proposition 4. *Under the conditions of Theorem 3, $v \in R^{\cdot}$ is weakly transcendental over $A^{\cdot}$ if and only if v is transcendental over the associated ordinary ring $A^{\times}$.*

Proposition 5. *Under the conditions of Theorem 3, $v \in R^{\cdot}$ is weakly algebraic over $A^{\cdot}$ if and only if v is algebraic over $A^{\times}$.*

5. Now, let $R^{\cdot} \supset A^{\cdot}$ be commutative weak fields, $v \in R^{\cdot}$, $v \neq d$; we shall consider the simple extension weak field II, $A^{\cdot}(v) \subset R^{\cdot}$. Obviously,

$$(18) \qquad A^{\cdot}[v] \text{ weak subring } \subset A^{\cdot}(v),$$

which shows that $A^{\cdot}[v]$ is a commutative weak integral domain. As it was seen above, there exists the weak ring epimorphism (14) which gives $\xi^{\cdot}$ from (15), in the present case an isomorphism of commutative weak integral domains.

In the case I, $N_{\chi}^{\cdot} = (0)$, we extend the isomorphism $\xi^{\cdot}$ to the isomorphism $\bar{\xi}^{\cdot}$ between the weak fields of fractions $Q_1^{\cdot}$, $A^{\cdot}(x)$ of the two weak integral domains ([7], Theorem 2). It is known that $Q_1^{\cdot}$ is the weak field II generated by the weak ring $A^{\cdot}[v]$, hence $Q_1^{\cdot}$ is the minimal weak subfield of $R^{\cdot}$ containing $A^{\cdot}[v]$ and consequently containing $A^{\cdot}$ and v; since the later is a characteristic property for the weak field $A^{\cdot}(v)$, we have

$$Q_1^{\cdot} = A^{\cdot}(v)$$

and thus we obtain

$$(19) \qquad A^{\cdot}(v) \cong A^{\cdot}(x),$$

where $A^{\cdot}(x)$ is the weak quotient field of the weak polynomials over $A^{\cdot}$.

In the case II, $N_{\chi}^{\cdot} \neq 0$, let us observe that the ordinary rings, associated to the weak rings $A^{\cdot}[x]$ and $A^{\cdot}[v]$, are respectively $A^{\times}[X] = A^{\times}[x-d]$ and $A^{\times}[v-d]$, (Proposition 2). It is known, ([7], § 4, Lemma 1), that the epimorphism χ, (14), induces an epimorphism $\chi^{\times}$ of the associated ordinary rings

$$(14^{\times}) \qquad \chi^{\times} : A^{\times}[x-d] \rightarrow A^{\times}[v-d],$$

with $[f^{\times}(x-d)]\chi^{\times} = f^{\times}(v-d)$ and, at the same time, the isomorphism $\xi^{\cdot}$ induces the ordinary ring isomorphism $\xi^{\times}$

$$(15^{\times}) \qquad A^{\times}[v-d] \overset{\xi^{\times}}{\cong} \frac{A^{\times}[x-d]}{N_{\chi^{\times}}^{\times}} = \frac{A^{\times}[X]}{N_{\chi^{\times}}^{\times}}$$

where $N^{\times}_{\chi^{\times}} = \operatorname{Ker} \chi^{\times}$ is the associated ordinary ideal of the weak ideal $N^{\cdot}_{\chi}$, namely

$$N^{\times}_{\chi^{\times}} = \{ q^{\times}(X) \mid q^{\times}(X) \in A^{\times}[X],\ q^{\times}(v-d) = 0 \}.$$

As $R^{\cdot} \supset A^{\cdot}$ are commutative weak fields II and hence $R^{\times} \supset A^{\times}$ are ordinary fields, $A^{\times}[X]$ is a principal ideal domain, $N^{\times}_{\chi^{\times}}$ is a principal ideal generated by an irreducible polynomial $\varphi(X) \in A^{\times}[X]$, $A^{\times}[X]/N^{\times}_{\chi^{\times}}$ is a field and so is $A^{\times}[v-d]$. It follows that $A^{\cdot}[v]$ is a weak field II, containing $A^{\cdot}$ and v, and this, together with (18), shows that $A^{\cdot}[v]$ is just the simple extension weak field

$$A^{\cdot}[v] = A^{\cdot}(v) \simeq \frac{A^{\cdot}[x]}{N_{\chi}}. \tag{20}$$

So we have proved:

Theorem 6. *If $R^{\cdot} \supset A^{\cdot}$ are weak fields II and $v \in R^{\cdot}$, $v \neq d$, then the simple extension weak field $A^{\cdot}(v)$ can be only in the two following cases:*

I. (19) $A^{\cdot}(v) \simeq A^{\cdot}(x)$ the weak quotient field of weak polynomials over $A^{\cdot}$ and is called weakly transcendent over $A^{\cdot}$;

II. (20) $$A^{\cdot}(v) = A^{\cdot}[v] \simeq \frac{A^{\cdot}[x]}{N_{\chi}}.$$

hence $A^{\cdot}(v)$ is the weak ring of the polynomials of v over $A^{\cdot}$ and N_{χ} the weak ideal given by (16); then v is called weakly algebraic over $A^{\cdot}$.

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EXTENSIUNI DE INELE SLABE

Rezumat

Se studiază extensiunile simple ale inelelor slabe asociative, apoi corpurile slabe de extensiune simplă ale corpurilor slabe de specia a II-a comutative.

A GENERALIZATION OF THE GENERALIZED COMMUTATIVE RINGS

BY

GEETA DEVI

1. Introduction: Belluce — Herstein — Sain [1] have defined a ring R to be a generalized commutative ring (written as g.c. ring) if given $a, b \in R$, there exist positive integers $m = m(a,b)$, $n = n(a,b)$, such that

$$(ab)^m = (ba)^n.$$

The aim of this paper is to generalize this property. We shall call a Multiplicative Semi-Group S to have G-property if given $a, b \in S$ there exist positive integers $m = m(a,b)$, $n = n(a,b)$ such that

$$a^n b^m = b^m a^n. \tag{G}$$

2. It follows from lemma [1] that if G is a Multiplicative Semi-Group in which for each $a, b \in G$

$$(ab)^{m(a,b)} = (ba)^{n(a,b)}$$

where $m(a,b)$ and $n(a,b)$ are positive integers, then G has G-property. We assume for convenience that the ring R has unity. We shall now prove the following

3. Theorem: *Let G be a Multiplicative Semi-Group. Let $a, b, c \in G$, there exist positive integers m, n, r and S depending on a and b such that*

$$c\,a^n b^m c^{-1} = b^n a^m \quad \textit{and} \quad a\,b\,c^r a^{-1} b^{-1} = c^s$$

then there exist positive integers λ and μ such that $a^{\mu} b^{\lambda} = b^{\lambda} a^{\mu}$.

Proof: If a and b are of finite order then the result is obvious. So let a and b be not of finite order. Then if $a^n b^m = b^n a^m$, for positive integers m and n, we must have $m = n$. We have by hypothesis