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CVASI-CONEXIUNI PE VARIETĂŢI DIFERENŢIABILE, II

Rezumat

În [4] au fost abordate o serie de probleme legate de teoria cvasi-conexiunilor pe varietăţi diferenţiabile. Prezenta Notă se referă tot la cvasi-conexiuni pe varietăţi diferenţiabile. §1 conţine o caracterizare geometrică a cvasi-conexiunilor fără curbură şi fără torsiune, iar în §2 se consideră unele probleme legate de perechi de cvasi-conexiuni ca: cvasi-conexiunea medie a două cvasi-conexiuni, algebra de deformare a două cvasi-conexiuni etc.

TENSOR INTEGRAL AND EXTERIOR DERIVATIVE OF DOUBLE TENSORIAL FORMS

BY

I. GOTTLIEB

Relying on A. Moór's works [7], [8], we defined [2], [3], [4] in the case of a n-dimensional parallelizable space, a multiple tensor integral on a p-dimensional domain ($p \leq n$), established Stokes-type formulae and with their help we defined the exterior derivative of exterior tensorial forms. In this work we shall generalize these results in the case of the space with linear connection.

Introduction. Let V_n be a n-dimensional space with linear connection having the connection coefficients Γ^{i}_{kl}, x and $x' \in V_n$. Let T_x and $T_{x'}$ be the tangent vectorial spaces in x, x' respectively and T^*_x and $T^*_{x'}$ be their duals. One calls bi-tensor [11], an element of the space $(\otimes_\alpha T_x) \otimes (\otimes_\beta T^*_x) \otimes (\otimes_\gamma T_{x'}) \otimes (\otimes_\delta T^*_{x'})$ defined over the space $V_n \times V_n$.

We shall admit the convention that the indices with prime, without prime respectively, would refer to the points x', x respectively. Thus, $T^{i}_{\cdot j'}$ for instance are the components of the bi-tensor $T \in T_x \otimes T^*_{x'}$.

With the change of coordinates

$$\bar{x}^i = \bar{x}^i(x^k), \tag{0.1}$$

the components of the bi-tensor T transform themselves according to the relation

$$\bar{T}^{i}_{\cdot k'} = \frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial x'^{l'}}{\partial \bar{x}'^{k'}} T^{j}_{\cdot l'}, \tag{0.2}$$

which may be generalized for several numbers of indices.

Furthermore we give the definition of the transport bi-tensor formulated by A. Lichnerowicz [6]. Let $\Omega \in V_n$ be a domain homeomorphic to an open ball of R^n with the property that for every pair of points $x, x' \in \Omega$, there is only one geodesic arc $l(x, x')$ joining these points and lying entirely in Ω. We may define a canonical isomorphism $\mu_{x,x'}$ of T_x on $T_{x'}$ with the help of the parallelizable transport along $l(x, x')$. This isomorphism is the identity for $x = x'$. According to this isomorphism, to every $v \in T_x$ of components v^i, there coresponds the vector $\mu_{x,x'} v \in T_{x'}$ of components $v^{k'}$, wich are linear fonctions of v^i, namely

$$v^{k'} = h^{k'}_i(x, x')v^i. \tag{0.3}$$

The quantities $h^{k'}_i$ are components of a bi-tensor $h \in T_{x'} \otimes T^*_x$, which is called *transport bi-tensor*. Naturally we have

$$\lim_{x' \to x} h^{k'}_i(x, x') = \delta^{k'}_i. \tag{0.4}$$

Let $k^j_{l'}(x, x')$ be the components of the bi-tensor k associated with $\mu^{-1}_{x,x'}$. Naturally, the bi-tensor k achieves the isomorphism of $T_{x'}$ on T_x, therefore

$$v^j = k^j_{l'}(x, x')v^{l'} \tag{0.5}$$

and it satisfies the relations

$$h^{l'}_i k^j_{l'} = \delta^j_i, \qquad h^{k'}_j k^j_{l'} = \delta^{k'}_{l'}. \tag{0.6}$$

Similarly we may define an isomorphism $\nu_{x,x'}$ of T^*_x on $T^*_{x'}$. Out of the invariance at the parallel transport of the scalar product $u_i v^i$, it follows that this isomorphism is achieved by the bi-tensor k such that

$$u_{l'} = k^j_{l'}(x, x')u_j, \tag{0.7}$$

and the isomorphism $\nu^{-1}_{x,x'}$ is achieved by the bi-tensor h; therefore

$$u_j = h^{l'}_j(x, x')u_{l'}. \tag{0.8}$$

The transport bi-tensor k satisfies also the relation

$$\lim_{x' \to x} k^j_{l'}(x, x') = \delta^j_{l'}. \tag{0.9}$$

Because of the relations (0.5) and (0.8), the conditions (0.4) and (0.9) are preserved even if we invert the tendency towards the limit, that is

$$\lim_{x \to x'} h^{k'}_i(x, x') = \delta^{k'}_i \quad \text{and} \quad \lim_{x \to x'} k^j_{l'}(x, x') = \delta^j_{l'}. \tag{0.10}$$

Let t^s, $t^{s'}$ be the components of the tangent vectors to $l(x, x')$ in x, x' respectively. Since $\mu_{x,x'} v$ was obtained from v by parallel transport, it follows from (0.3) that

$$t^{s'}\nabla_{s'}(h^{k'}_i v^i) = (t^{s'}\nabla_{s'} h^{k'}_i)v^i = 0,$$

where $\nabla_{s'}$ is the covariant derivative operator. This relation being true for an arbitrary vector v, we will have

$$t^{s'}\nabla_{s'} h^{k'}_i = 0. \tag{0.11}$$

Similarly we obtain from (0.5), (0.7) and (0.8) the relation

$$t^s \nabla_s k^j_{l'} = 0; \qquad t^{s'}\nabla_{s'} k^j_{l'} = 0; \qquad t^s \nabla_s h^{l'}_j = 0. \tag{0.12}$$

It follows from (0.11) that $\lim_{x' \to x}(t^{s'}\nabla_{s'} h^{k'}_i) = 0$. But in this expression $t^{s'}$ may tend to whatever vector of T_x. Consequently it follows from (0.11) and similarly from (0.12),

$$\lim_{x' \to x} \nabla_{s'} h^{k'}_i = 0, \qquad \lim_{x' \to x} \nabla_{s'} k^j_{l'} = 0 \tag{0.13}$$

and

$$\lim_{x \to x'} \nabla_s k^j_{l'} = 0, \qquad \lim_{x \to x'} \nabla_s h^{l'}_j = 0. \tag{0.14}$$

By means of the products of transport bi-tensors h and k on may obtain an isomorphism of the space $(\otimes^u_x T^*_x) \otimes (\otimes^v_x T_x)$ on the space $(\otimes^u_{x'} T^*_{x'}) \otimes (\otimes^v_{x'} T_{x'})$ and conversely. Certainly, for an u-times co- and v-times contravariant tensors T we have

$$T_{(l')}^{\cdot\,(k')} = k^{(i)}_{(l')} h^{(k')}_{(j)} T_{(i)}^{\cdot\,(j)} \tag{0.15}$$

respectively

$$T_{(i)}^{\cdot\,(j)} = h^{(l')}_{(i)} k^{(j)}_{(k')} T_{(l')}^{\cdot\,(k')}, \tag{0.16}$$

where

$$\begin{cases} T_{(i)}^{\cdot\,(j)} = T_{i_1 \dots i_u}^{\cdots\; j_1 \dots j_v} \\ k^{(i)}_{(l')} = k^{i_1}_{l'_1}(x, x')\, k^{i_2}_{l'_2}(x, x') \dots k^{i_u}_{l'_u}(x, x') \quad \text{etc.} \end{cases} \tag{0.17}$$

We call a *double tensorial form* [9] of order $p + q$, the expression

$$\begin{cases} \overset{p+q}{\omega}{}_{(i)\cdot(k')}^{\cdot(j)\cdot(l')\cdot} = T_{(i)\cdot(k')\cdot s_1 \dots s_p t'_1 \dots t'_q}^{\cdot(j)\cdot(l')\cdot} \cdot \\ \cdot(dx^{s_1} \wedge \dots \wedge dx^{s_p})(dx'^{t'_1} \wedge \dots \wedge dx'^{t'_q}), \end{cases} \tag{0.18}$$

where T is a bi-tensor. The differentials dx^s and $dx'^{t'}$ commute themselves. Thus,

$$\overset{p+q}{\omega} \wedge \overset{r+v}{\omega} = (-1)^{pr+qv}\, \overset{r+v}{\omega} \wedge \overset{p+q}{\omega}. \tag{0.19}$$

§ 1. **The bi-tensors H and K.** The quantities $\nabla_l h^{k'}_i(x, x')$ are the components of a bi-tensor $T \in T^*_x \otimes T^*_x \otimes T_{x'}$; therefore, using (0.6) we may write

$$\nabla_l h^{k'}_i(x, x') = T_{l\,i}{}^{k'}(x, x') - T_{l\,i}{}^{j'}(x, x')\, k^{t}_{j'}(x, x')\, h^{k'}_t(x, x'),$$

i.e.:

$$\nabla_l h^{k'}_i(x, x') = H_{l\,i}{}^{t}(x, x')\, h^{k'}_t(x, x'). \tag{1.1}$$

The bi-tensor H is a tensor of order three in x and a scalar in x'.

Similarly, the quantities $\nabla_{s'} k^l_{j'}(x, x')$ are the components of a bi-tensor $U \in T^*_{x'} \otimes T^*_{x'} \otimes T_x$; therefore

$$\nabla_{s'} k^l_{j'}(x, x') = U_{s'j'}{}^{l}(x, x') = U_{s'j'}{}^{h}(x, x')\, h^{t'}_h(x, x')\, k^l_{t'}(x, x')$$

and consequently

$$\nabla_{s'} k^l_{j'}(x, x') = K_{s'j'}{}^{t'}(x, x')\, k^l_{t'}(x, x'). \tag{1.2}$$

The bi-tensor K is a scalar in x and a tensor of order three in x'.

Now, let us apply the operator ∇_l to the relation (0.6.1). Taking (1.1) and (0.5) into account and the fact that $\nabla_l \delta^j_i = 0$,

$$\nabla_l k^j_{t'}(x, x') = -H_{l\,i}{}^{j}(x, x')\, k^i_{t'}(x, x'). \tag{1.3}$$

In the same way, from (0.6.2) and (1.2) it follows

$$\nabla_{s'} h^{k'}_i = -K_{s'\,t'}{}^{k'}\, h^{t'}_i. \tag{1.4}$$

From the relations (0.4), (0.9), (0.11) and (0.12), it results the following properties of bi-tensors H and K

$$\begin{cases} \lim\limits_{x\to x'} H_{l\,i}{}^{t}(x, x') = 0, & \lim\limits_{x'\to x} K_{s'j'}{}^{t'}(x, x') = 0, \\ t^l H_{l\,i}{}^{t} = 0, & t^{s'} K_{s'j'}{}^{t'} = 0. \end{cases} \tag{1.5}$$

Applying the operator ∇_j to the relation (1.1) we have

$$\nabla_j \nabla_l h^{k'}_i = (\nabla_j H_{l\,i}{}^{t})\, h^{k'}_t + H_{j\,l}{}^{s}\, H_{s\,i}{}^{t}\, h^{k'}_t,$$

respectively

$$\nabla_{[j} \nabla_{l]} h^{k'}_i = (\nabla_{[j} H_{l]i}{}^{t} + H_{[j\,|s|}{}^{t}\, H_{l]i}{}^{s})\, h^{k'}_t. \tag{1.6}$$

On the other hand ([10], chap. III, § 4)

$$\nabla_{[j} \nabla_{l]} h^{k'}_i = -\frac{1}{2} R_{j\,l\,i}{}^{t}\, h^{k'}_t - S_{j\,l}{}^{s}\, \nabla_s h^{k'}_i, \text{ with } S_{j\,l}{}^{s} = \Gamma^s_{[jl]},$$

or using again (1.1),

$$\nabla_{[j} \nabla_{l]} h^{k'}_i = \left(-\frac{1}{2} R_{j\,l\,i}{}^{t} - S_{j\,l}{}^{s}\, H_{s\,i}{}^{t}\right) h^{k'}_t. \tag{1.7}$$

Identifying (1.6) to (1.7) we obtain

$$\nabla_{[j} H_{l]i}{}^{t} = -\frac{1}{2} R_{j\,l\,i}{}^{t} - S_{j\,l}{}^{s}\, H_{s\,i}{}^{t} - H_{[j\,|s|}{}^{t}\, H_{l]i}{}^{s}. \tag{1.8}$$

Applying the operator $\nabla_{u'}$ to the relation (1.2) and using the same method we obtain

$$\nabla_{[u'} K_{s']j'}{}^{t'} = -\frac{1}{2} R_{u's'j'}{}^{t'} - S_{u's'}{}^{r'}\, K_{r'j'}{}^{t'} - K_{[s'|r'|}{}^{t'}\, K_{u']j'}{}^{r'}. \tag{1.9}$$

From (1.5), (1.8) and (1.9) it follows

$$\lim_{x'\to x} \nabla_{[j} H_{l]i}{}^{t} = -\frac{1}{2} R_{j\,l\,i}{}^{t}; \qquad \lim_{x\to x'} \nabla_{[u'} K_{s']j'}{}^{t'} = -\frac{1}{2} R_{u's'j'}{}^{t'}. \tag{1.10}$$

As $\nabla_k \nabla_{s'} k^l_{j'} = \nabla_{s'} \nabla_k k^l_{j'}$ and

$$\begin{cases} \nabla_k \nabla_{s'} k^l_{j'} = K_{s'j'}{}^{t'}{}_{,k}\, k^l_{t'} - K_{s'j'}{}^{t'}\, H_{k\,h}{}^{l}\, k^h_{t'}, \\ \nabla_{s'} \nabla_k k^l_{j'} = -H_{k\,h}{}^{l}{}_{,s'}\, k^h_{j'} - H_{k\,h}{}^{l}\, K_{s'j'}{}^{t'}\, k^h_{t'}, \end{cases}$$

it follows

$$K_{s'j'}{}^{t'}{}_{,k}\, k^l_{t'} = -H_{k\,h}{}^{l}{}_{,s'}\, k^h_{j'}, \qquad K_{s'j'}{}^{t'}{}_{,k}\, h^{j'}_h = -H_{k\,h}{}^{l}{}_{,s'}\, h^{t'}_l. \tag{1.11}$$

§ 2. **Tensor integral and Stokes-type formulae.** Let $(S_p) \subset \Omega$ be a p-dimensional domain and $x' \in \Omega$. We shall define *the tensor integral* of the bi-tensor $T_{(f')\cdot(i)\cdot\, j_1 \ldots j_p}^{\cdot(g')\cdot(k)}$ completely antisymmetric in the indices $j_1, \ldots, j_p$ *on the domain* (S_p) by means of the relation [4]

$$\begin{cases} U_{(f')\cdot(i')}^{\cdot(g')\cdot(k')}(x') = \mathbf{I}_{(S_p)}\, T_{(f')\cdot(i)\cdot\, j_1 \ldots j_p}^{\cdot(g')\cdot(k)}(x, x')\, dS^{j_1 \ldots j_p} = \\ = \int_{(S_p)} T_{(f')\cdot(i')\cdot\, j_1 \ldots j_p}^{\cdot(g')\cdot(k')}(x, x')\, dS^{j_1 \ldots j_p}, \end{cases} \tag{2.1}$$

where $(i) = (i')$, $(k) = (k')$ and

$$dS^{j_1 \dots j_p} = dx^{j_1} \wedge \dots \wedge dx^{j_p}. \tag{2.2}$$

Using (0.15), the tensor integral (2.1) can be written as

$$\left\{ \begin{aligned} & U_{(j')\cdot\ (i')}^{\cdot\ (g')\ \cdot\ (k)}(x') = \\ & = \int_{(S_p)} k_{(i')}^{(s)}(x, x')\, h_{(t)}^{(k')}(x, x')\, T_{(j')\cdot\ (s)\cdot j_1 \dots j_p}^{\cdot\ (g')\ \cdot\ (t)}\, dS^{j_1 \dots j_p}. \end{aligned} \right. \tag{2.3}$$

From (2.3) it follows that the quantities $U_{\dots}^{\dots}$ are the components of a tensor.

If $V_{j_1 \dots j_p}$ is a p-vector, we have ([10], chap. II, § 8)

$$\int_{(S_p)} V_{j_1 \dots j_p}\, dS^{j_1 \dots j_p} = \int_{(S_{p+1})} \partial_{[l}\, V_{j_1 \dots j_p]}\, dS^{l j_1 \dots j_p} \tag{2.4}$$

(S_p) being the boundary of (S_{p+1}). But $k_{(i')}^{(s)} h_{(t)}^{(k')} T_{(j')\cdot\ (s)\cdot j_1 \dots j_p}^{\cdot\ (g')\ \cdot\ (t)}$ is a p-vector in x. Thus, if (S_p) is a closed hypersurface, being the boundary of (S_{p+1}), we obtain from (2.3) and (2.4)

$$\left\{ \begin{aligned} & U_{(j')\cdot\ (i')}^{\cdot\ (g')\ \cdot\ (k')}(x') = \\ & = \int_{(S_{p+1})} \partial_{[l}\, \{k_{(i')}^{(s)} h_{(t)}^{(k')} T_{(j')\cdot\ (s)\cdot\ |j_1 \dots j_p]}^{\cdot\ (g')\ \cdot\ (t)}\}\, dS^{l j_1 \dots j_p}. \end{aligned} \right. \tag{2.5}$$

On the other hand ([10], chap. III, § 2)

$$\partial_{[l}\, V_{j_1 \dots j_p]} = \nabla_{[l}\, V_{j_1 \dots j_p]} + pS_{[l j_1}^{\cdot\ \cdot\ h}\, V_{|h|\, j_2 \dots j_p]},$$

therefore (2.5) becomes

$$\left\{ \begin{aligned} & U_{(j')\cdot\ (i')}^{\cdot\ (g')\ \cdot\ (k')}(x') = \int_{(S_{p+1})} \{(\nabla_{[l|}\, k_{(i')}^{(s)} h_{(t)}^{(k')})\, T_{(j')\cdot\ (s)\cdot\ |j_1 \dots j_p]}^{\cdot\ (g')\ \cdot\ (t)} + \\ & + k_{(i')}^{(s)} h_{(t)}^{(k)}\, \nabla_{[l|}\, T_{(j')\cdot\ (s)\cdot\ |j_1 \dots j_p]}^{\cdot\ (g')\ \cdot\ (t)}\}\, dS^{l j_1 \dots j_p}, \end{aligned} \right. \tag{2.6}$$

where ([10], chap. III, § 5)

$$\overset{p}{\nabla}_{[l|}\, T_{\dots\ |j_1 \dots j_p]}^{\dots} = \nabla_{[l}\, T_{\dots\ j_1 \dots j_p]}^{\dots} + pS_{[l j_1|}^{\cdot\ \cdot\ h}\, T_{\dots h |j_2 \dots j_p]}^{\dots}. \tag{2.7}$$

We can write explicitely the first term under the integral with the help of (1.1) and (1.3) namely,

$$(\nabla_l\, k_{i'}^s h_t^{k'})\, T_s^{\cdot\, t} = (H_{l\, i'}^{\cdot\ \cdot\ b}\, k_{i'}^s h_b^{k'}\, T_s^{\cdot\, t} - H_{l\, a}^{\cdot\ \cdot\ s}\, k_{i'}^a h_t^{k'}\, T_s^{\cdot\, t},$$

or

$$(\nabla_l\, k_{i'}^s h_t^{k'})\, T_s^{\cdot\, t} = (H_{l\, b}^{\cdot\ \cdot\ t}\, T_s^{\cdot\, b} - H_{l\, s}^{\cdot\ \cdot\ a}\, T_a^{\cdot\, t})\, k_{i'}^s h_t^{k'}. \tag{2.8}$$

Using (2.8), (2.6) becomes

$$U_{(j')\cdot\ (i')}^{\cdot\ (g')\ \cdot\ (k')} = \int_{(S_{p+1})} k_{(i')}^{(s)} h_{(t)}^{(k')}\, \overset{p}{\nabla}{}^{*}_{[l|}\, T_{(j')\cdot\ (s)\cdot\ |j_1 \dots j_p]}^{\cdot\ (g')\ \cdot\ (t)}\, dS^{l j_1 \dots j_p}, \tag{2.9}$$

where

$$\left\{ \begin{aligned} & \overset{p}{\nabla}{}^{*}_{[l|}\, T_{\dots (i)\cdot\ |j_1 \dots j_p]}^{\dots\ \cdot\ (k)} = \overset{p}{\nabla}_{[l}\, T_{\dots (i)\cdot\ j_1 \dots j_p]}^{\dots\ \cdot\ (k)} - \\ & - \sum_{\alpha=1}^{u} H_{[l|\, i_\alpha}^{\cdot\ \cdot\ a_\alpha}\, T_{\dots i_1 \dots a_\alpha \dots i_u\ |j_1 \dots j_p]}^{\dots\ \cdot\ (k)} + \\ & + \sum_{\beta=1}^{r} H_{[l|\, b_\beta}^{\cdot\ \cdot\ k_\beta}\, T_{\dots (i)\cdot\ |j_1 \dots j_p]}^{\dots\ \cdot\ k_1 \dots b_\beta \dots k_r}. \end{aligned} \right. \tag{2.10}$$

Taking into account the definition of the tensor integral, the relation (2.9) may be written as

$$\underset{(S_p)}{\mathbf{I}}\, T_{(j')\cdot\ (i)\cdot\ j_1 \dots j_p}^{\cdot\ (g')\ \cdot\ (k)}\, dS^{j_1 \dots j_p} = \underset{(S_{p+1})}{\mathbf{I}}\, \overset{p}{\nabla}{}^{*}_{[l|}\, T_{(j')\cdot\ (i)\cdot\ |j_1 \dots j_p]}^{\cdot\ (g')\ \cdot\ (k)}\, dS^{l j_1 \dots j_p}, \tag{2.11}$$

where (S_p) is the boundary of (S_{p+1}). *The relation (2.11) is the generalization of the Stokes-type formulae for the tensor integral.*

The tensor integral can be defined as well with respect to the second variable of the bi-tensor. All the results obtained may be transposed for this case too, but instead of the operator $\overset{p}{\nabla}{}^{*}$, the operator $\overset{p'}{\nabla}{}^{*}$ appears, in whose definition the tensor K appears instead of tensor H.

§ 3 **Double tensorial forms and exterior derivative.** We shall call *double tensorial forms of order* $p + q'$ [5], [9] the expression

$$\overset{p+q'}{\omega}{}_{(j')\cdot\ (i)}^{\cdot\ (g')\ \cdot\ (k)} = T_{(j')\cdot\ (i)\cdot\ j_1 \dots j_p\, h_1 \dots h_{q'}}^{\cdot\ (g')\ \cdot\ (k)}\, dS^{j_1 \dots j_p}\, dS^{h_1 \dots h_{q'}}, \tag{3.1}$$

where T is a bi-tensor whose components are fonctions of necessary class.

We may introduce two exterior derivative operators [9] for double tensorial forms which will allow to find simple expressions for Stokes-type formulae.

We can derive these operators to act upon double tensorial forms by the relations

$$d_x \overset{p+q'}{\omega}{}_{(f')\cdot(i)}^{\cdot(g')\cdot(k)} = \overset{p}{\nabla}{}^{*}_{[l|} T_{(f')\cdot(i)\cdot}^{\cdot(g')\cdot(k)}{}_{|j_1\ldots j_p]h'_1\ldots h'_{q'}} dS^{lj_1\ldots j_p} dS^{h'_1\ldots h'_{q'}}, \tag{3.2}$$

respectively

$$d_{x'} \overset{p+q'}{\omega}{}_{(f')\cdot(i)}^{\cdot(g')\cdot(k)} = \overset{q'}{\nabla}{}^{*}_{[l'|} T_{(f')\cdot(i)\cdot}^{\cdot(g')\cdot(k)}{}_{j_1\ldots j_p|h'_1\ldots h'_{q'}]} dS^{j_1\ldots j_p} dS^{l'h'_1\ldots h'_{q'}}. \tag{3.3}$$

Thus, the Stokes-type formulae become

$$\underset{(S_p)}{I}\,\omega = \underset{(S_{p+1})}{I}\, d_x\,\omega; \qquad \underset{(S'_{q'})}{I}\,\omega = \underset{(S'_{q'+1})}{I}\, d_{x'}\,\omega. \tag{3.4}$$

We may call the operator (3.2) ((3.3) respectively) an exterior derivative ([1], chap. IV, § 6) because it satisfies the following properties:

1) if $p=0$, $d_x \overset{0}{\omega}$ agrees with the definition of the absolute differential of $\overset{0}{\omega}$;

2) domain of $\overset{p}{\omega}$ = domain of $d_x \overset{p}{\omega}$;

3) d_x is R linear (R = real numbers);

4) $d_x(\overset{p}{\omega} \wedge \overset{q}{\omega}) = (d_x \overset{p}{\omega}) \wedge \overset{q}{\omega} + (-1)^p \overset{p}{\omega} \wedge d_x \overset{q}{\omega}$;

5) $d_x(d_x \overset{p}{\omega}) = 0$ for all $\overset{p}{\omega}$.

It follows from (2.10) that the property 1) is satisfied with coefficients $\Gamma^i_{kl} + H_{k\,i}^{\cdot\,\cdot\,i}$ of the connexion. The properties 2) and 3) are easy to verify. The relation 4) was verified for the case of parallelizable spaces [4] when $\overset{p}{\nabla}{}^{*}$ is reduced to $\overset{p}{\nabla}$. The difference between the two operators doesn't produce difficulties in deriving the property 4).

It remains to verify the property 5). By omiting the indices with prime it follows from (2.10) that

$$\begin{cases} \overset{p+1}{\nabla}{}^{*}_{[m} \overset{p}{\nabla}{}^{*}_{l} T_{i\cdot}^{\cdot k}{}_{|j_1\ldots j_p]} = \\ \overset{p+1}{\nabla}_{[m} \overset{p}{\nabla}_{l|} T_{i\cdot}^{\cdot k}{}_{j_1\ldots j_p]} - \overset{p+1}{\nabla}_{[m} (T_{a\cdot}^{\cdot k}{}_{j_1\ldots j_p} H_{l]i}^{\cdot\,\cdot a}) + \overset{p+1}{\nabla}_{[m|} (T_{i\cdot}^{\cdot b}{}_{|j_1\ldots j_p} H_{l|b}^{\cdot\,\cdot k} - \\ - H_{[m|i}^{\cdot\,\cdot c}\{\overset{p}{\nabla}_{|l|} T_{c\cdot}^{\cdot k}{}_{j_1\ldots j_p]} - H_{l|c}^{\cdot\,\cdot a} T_{a\cdot}^{\cdot k}{}_{j_1\ldots j_p]} + H_{l\,b}^{\cdot\,\cdot k} T_{c\cdot}^{\cdot b}{}_{j_1\ldots j_p]}\} + \\ + H_{[m\,d}^{\cdot\,\cdot k}\{\overset{p}{\nabla}_{l|} T_{i\cdot}^{\cdot d}{}_{|j_1\ldots j_p]} - H_{l|i}^{\cdot\,\cdot a} T_{a\cdot}^{\cdot d}{}_{j_1\ldots j_p]} + H_{l|b}^{\cdot\,\cdot d} T_{i\cdot}^{\cdot b}{}_{|j_1\ldots j_p]}\}. \end{cases}$$

Using the relations ([10], chap. III, § 5)

$$\overset{p+1}{\nabla}_{[m} \overset{p}{\nabla}_{l|} T_{i}^{\cdot k}{}_{|j_1\ldots j_p]} = \frac{1}{2} R_{[ml|b}^{\cdot\,\cdot\,\cdot k} T_{i}^{\cdot b}{}_{j_1\ldots j_p]} - \frac{1}{2} R_{[ml\,i}^{\cdot\,\cdot\,\cdot a} T_{a\cdot}^{\cdot k}{}_{|j_1\ldots j_p]}$$

and

$$\overset{p+q}{\nabla}_{[l|} (T_{i\cdot}^{\cdot k}{}_{j_1\ldots j_p} U_{s\cdot}^{\cdot t}{}_{|h_1\ldots h_p]}) = (\overset{p}{\nabla}_{[l|} T_{i\cdot}^{\cdot k}{}_{|j_1\ldots j_p|}) U_{s\cdot}^{\cdot t}{}_{|h_1\ldots h_p]} + \\ + T_{i\cdot}^{\cdot k}{}_{[j_1\ldots j_p} \overset{p}{\nabla}_{l|} U_{s\cdot}^{\cdot t}{}_{|h_1\ldots h_q]}$$

we obtain

$$\begin{cases} \overset{p+1}{\nabla}{}^{*}_{[m} \overset{p}{\nabla}{}^{*}_{l|} T_{i\cdot}^{\cdot k}{}_{|j_1\ldots j_p]} = \\ = \left\{\frac{1}{2} R_{[ml|b}^{\cdot\,\cdot\,\cdot k} + \overset{1}{\nabla}_{[m} H_{l|b}^{\cdot\,\cdot k} + H_{[m|d}^{\cdot\,\cdot k} H_{l|b}^{\cdot\,\cdot d}\right\} T_{i\cdot}^{\cdot b}{}_{|j_1\ldots j_p]} - \\ - \left\{\frac{1}{2} R_{[ml|i}^{\cdot\,\cdot\,\cdot a} + \overset{1}{\nabla}_{[m} H_{l|i}^{\cdot\,\cdot a} + H_{[m|c}^{\cdot\,\cdot a} H_{l|i}^{\cdot\,\cdot c}\right\} T_{a\cdot}^{\cdot k}{}_{|j_1\ldots j_p]}. \end{cases}$$

But

$$\overset{1}{\nabla}_{[m} H_{l]b}^{\cdot\,\cdot k} = \nabla_{[m} H_{l]b}^{\cdot\,\cdot k} + S_{ml}^{\cdot\,\cdot s} H_{sb}^{\cdot\,\cdot k}$$

and thus, because of (1.8), the expressions in the brackets will vanish, and it follows

$$d_x (d_x \overset{p}{\omega}) = \overset{p+1}{\nabla}{}^{*}_{[m} \overset{p}{\nabla}{}^{*}_{l|} T_{i\cdot}^{\cdot k}{}_{|j_1\ldots j_p]} dS^{mlj_1\ldots j_p} = 0,$$

which was to show.

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TENSOR—INTEGRALA ŞI DIFERENŢIALA EXTERIOARĂ A FORMELOR DUBLE TENSORIALE

Rezumat

Utilizînd bitensorii de transport [11], [6], pentru care se deduc unele proprietăţi noi în §1, se defineşte tensor-integrala bitensorilor prin relaţia (2. 1). Pentru tensor—integrala astfel definită formulele de tip Stokes au forma (2. 11). Formula (2. 11) conduce la definiţia (3.2), respectiv (3.3), a diferenţialei exterioare a formelor duble tensoriale (3.1). Se arată că această operaţie satisface axiomele diferenţialei exterioare.

ON FUNCTIONS OF NON-HOMOGENEOUS MARKOV CHAINS

BY

JACK HACHIGIAN

Introduction: Functions of Markov chains have been considered by many authors [see bibliography] and a variety of results have been obtained This paper contains some analogous results in the case where the under lying Markov chain is non-stationary.

Statement of Problem and Results. Let X_n, $n = 0,1,2,\dots$ be a denumerable state Markov chain having transition matrix $P(n, n+1)$ with any initial distribution $p = (p_i > 0)$. Consider now a new process $Y_n = f(X_n)$ where f is a given function on the state space of X_n. The function f is many-one and the states i of X_n on which f assumes the same value are collected and considered as a single state of Y_n. The states of Y_n are denoted by S_a, $a = 1, 2, \dots$ [1].

Proposition: *Let $X_n\,[X(t)]$ $n = 0, 1, 2, \dots\,[0 \leqslant t < \infty]$ be a denumerable state Markov chain having transition matrix $P(n, n+1)\,[P(t,s), t < s]$ with any initial distribution $p = (p_i > 0)$. Let $Y_n = f(X_n)\,[Y(t) = f(X(t))]$. Then a sufficient condition that $Y_n\,[Y(t)]$ to be Markovian is given by:*

$$\sum_{j \in S_b} p_{ij}(n, n+1) = P[X_{n+1} \in S_b \mid X_n = i] = C^{(n,\,n+1)}_{S_a, S_b} \tag{1}$$

$$\left[\sum_{j \in S_b} p_{ij}(t,s) = P[X(s) \in S_b \mid X(t) = i] = C^{(t,s)}_{S_a, S_b} \right]$$

for fixed b, $b = 1, 2, \dots$ and all $i \in S_a$, and all $a = 1, 2, \dots$

Proof: We must show that $P[Y_0 = S_{a_0}, Y_1 = S_{a_1}, \dots, Y_n = S_{a_n}] =$

$P[Y_0 = S_{a_0}]\,P[Y_1 = S_{a_1} \mid Y_0 = S_{a_0}] \dots P[Y_n = S_{a_n} \mid Y_{n-1} = S_{a_{n-1}}]$.