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TENSOR—INTEGRALA ŞI DIFERENŢIALA EXTERIOARĂ A FORMELOR DUBLE TENSORIALE

Rezumat

Utilizînd bitensorii de transport [11], [6], pentru care se deduc unele proprietăţi noi în §1, se defineşte tensor-integrala bitensorilor prin relaţia (2.1). Pentru tensor—integrala astfel definită formulele de tip Stokes au forma (2.11). Formula (2.11) conduce la definiţia (3.2), respectiv (3.3), a diferenţialei exterioare a formelor duble tensoriale (3.1). Se arată că această operaţie satisface axiomele diferenţialei exterioare.

ON FUNCTIONS OF NON-HOMOGENEOUS MARKOV CHAINS

BY

JACK HACHIGIAN

Introduction: Functions of Markov chains have been considered by many authors [see bibliography] and a variety of results have been obtained This paper contains some analogous results in the case where the under lying Markov chain is non-stationary.

Statement of Problem and Results. Let X_n, $n = 0,1,2,\dots$ be a denumerable state Markov chain having transition matrix $P(n, n+1)$ with any initial distribution $p = (p_i > 0)$. Consider now a new process $Y_n = f(X_n)$ where f is a given function on the state space of X_n. The function f is many-one and the states i of X_n on which f assumes the same value are collected and considered as a single state of Y_n. The states of Y_n are denoted by S_a, $a = 1,2,\dots$ [1].

Proposition: *Let X_n $[X(t)]$ $n = 0,1,2,\dots$ $[0 \leqslant t < \infty]$ be a denumerable state Markov chain having transition matrix $P(n, n+1)$ $[P(t,s), t < s]$ with any initial distribution $p = (p_i > 0)$. Let $Y_n = f(X_n)$ $[Y(t) = f(X(t))]$. Then a sufficient condition that Y_n $[Y(t)]$ to be Markovian is given by:*

$$\sum_{j \in S_b} p_{ij}(n, n+1) = P[X_{n+1} \in S_b \mid X_n = i] = C^{(n,\,n+1)}_{S_a,\,S_b} \tag{1}$$

$$\left[\sum_{j \in S_b} p_{ij}(t,s) = P[X(s) \in S_b \mid X(t) = i] = C^{(t,\,s)}_{S_a,\,S_b} \right]$$

for fixed b, $b = 1,2,\dots$ and all $i \in S_a$, and all $a = 1,2,\dots$

Proof: We must show that $P[Y_0 = S_{a_0}, Y_1 = S_{a_1}, \dots, Y_n = S_{a_n}] =$

$P[Y_0 = S_{a_0}]\,P[Y_1 = S_{a_1} \mid Y_0 = S_{a_0}] \dots P[Y_n = S_{a_n} \mid Y_{n-1} = S_{a_{n-1}}]$.

The left hand side is easily seen to be

$$P[Y_0 = S_{a_0}]\, C^{(0,1)}_{S_{a_0} S_{a_1}} \ldots C^{(n-1,n)}_{S_{a_{n-1}} S_{a_n}}.$$

The right hand side can be seen to be the same after observing that

$$P[Y_k = S_{a_k} \mid Y_{k-1} = S_{a_{k-1}}] = C^{(k-1,k)}_{a_{k-1} a_k} \text{ for } k = 1, 2, \ldots, n.$$

The proof for the continuous parameter case is analogous to that of the discrete case.

Theorem. *Let $X_n, n = 1, 2, \ldots$ be a Markov chain with transition mechanism $P(n, n+1) = [p_{ij}(n, n+1);\ i, j = 1, 2, \ldots]$. Let $w = (w_i > 0)$ be any initial distribution at any time. Let f be a function which collapses only one set of states of the X_n chain. Then $Y_n = f(X_n)$ is Markovian if and only if*

(2) $$\sum_{j \in S} p_{kj}(n-1, n)\, p_{ju}(n, n+1) = p_{kS}(n-1, n)\, C^{(n-1,n)}_u$$

(3) *for all k, and all $u \notin S$, or $p_{iS}(n-1, n) \equiv 0$ for all $i \notin S$,*

for each time n, where S is the collapsed set of states.

Proof: Let us first consider the necessity of the conditions. If $p_{kS}(n-1, n) = 0$ for all k then (2) is automatically satisfied. Suppose then there exists k such $p_{kS}(n-1, n) \neq 0$. Since we have supposed Y_n to be Markovian then it must satisfy

(4) $$P[Y_{n-1} = i,\ Y_n = S,\ Y_{n+1} = u] = \frac{P[Y_{n-1} = i, Y_n = S]\, P[Y_n = S, Y_{n+1} = u]}{P[Y_n = S]}$$

for all paths $i \to S \to u$. Consider the case where $i \notin S$ and $u \notin S$, and compute (4) to obtain

$$\sum_{j \in S} w_i\, p_{ij}(n-1, n)\, p_{ju}(n, n+1) = \frac{w_i p_{iS}(n-1, n) \sum_k \sum_{j \in S} w_k p_{kj}(n-1, n)\, p_{ju}(n, n+1)}{\sum_k w_k p_{kS}(n-1, n)},$$

or

(5) $$\sum_{j \in S} p_{ij}(n-1, n)\, p_{ju}(n, n+1) = \frac{p_{iS}(n-1, n) \sum_k \sum_{j \in S} w_k p_{kj}(n-1, n)\, p_{ju}(n, n+1)}{\sum_k w_k p_{kS}(n-1, n)}$$

for all $i \notin S$ and $u \notin S$. If k is such that $p_{kS}(n-1, n) \neq 0$ and we let $w_k \to 1$, and $w_v \to 0$ for $v \neq k$ we see that (5) becomes

(6) $$\sum_{j \in S} p_{ij}(n-1, n)\, p_{ju}(n, n+1) = \frac{p_{iS}(n-1, n) \sum_{j \in S} p_{kj}(n-1, n)\, p_{ju}(n, n+1)}{p_{kS}(n-1, n)}.$$

If $p_{kS}(n-1, n) = 0$, (6) is clearly satisfied so that (6) holds for all k and $i \notin S$. But relation (6) is symmetric in i and k and therefore the roles of i and k may be interchanged. Hence (6) holds for all k, all i, and all $u \notin S$. But (6) implies condition (2) in the statement of the theorem. The case where $u \in S$ is subsumed in the previous one if we observe that summing both sides of (6) with respect to $u \in S$ the same relationship holds. We must consider the possibility that $p_{iS}(n-1, n) = 0$ for all $i \notin S$. But this is condition (3).

Consider now the sufficiency of conditions (2) and (3). If condition (3) is true for all $n = 1, 2, 3, \ldots$ then it is clearly sufficient. In these case where condition (2) holds for all $n = 1, 2, \ldots$ we must show Y_n to have the Markovian property. We need only consider the validity of

(7) $$P[X_n = i,\ X_{n+1} \in S, \ldots, X_{n+m+1} = u] = P[X_n = i]\, P[X_{n+1} \in S \mid X_n = i] \ldots P[X_{n+m+1} = u \mid X_{n+m} \in S]$$

for all i and any $u \notin S$ since any other path can be broken down into such paths. But (7) has the equivalent form

(8) $$P[X_{n+1} \in S, \ldots X_{n+m+1} = u \mid X_n = i] = P[X_{n+1} \in S \mid X_n = i] \ldots P[X_{n+m+1} = u \mid X_{n+m} \in S]$$

since we have assumed $w = (w_i > 0)$. The left side of (8) is given by

$$\sum_{\substack{i_r \in S \\ r = 1, \ldots, m}} p_{i\, i_1}(n, n+1) \ldots p_{i_m u}(n+m, n+m+1) =$$

$$\sum_{\substack{i_j \in S \\ j=1,\ldots,m-2}} p_{i,i_1}(n, n+1) \ldots$$

$$\ldots \sum_{i_{m-1} \in S} p_{i_{m-2}, i_{m-1}}(n+m-2, n+1)\, p_{i_{m-1}, S}(n+m-1, n+m)\, C_u^{(n+m-1,\, n+m)} =$$

$$\vdots$$

$$= \sum_{i_1 \in S}^{\vdots} p_{i,i_1}(n, n+1)\, C_S^{(n+1,\, n+2)} \ldots C_S^{(n+m-2,\, n+m-1)}\, C_u^{(n+m-1,\, n+m)}.$$

We observe however that

$$C_u^{(m-1,\, m)} = P[X_{m+1} = u \mid X_m \in S] \text{ for } u \notin S,$$

since

$$P[X_{m+1} = u \mid X_m \in S] =$$

$$\sum_i \sum_{k \in S} w_i\, p_{ik}(m-1, m) p_{ku}(m, m+1) \Big/ \sum_i w_i\, p_{iS}(m-1, m)$$

$$\frac{\sum_i w_i\, p_{iS}(m-1, m)\, C_u^{(m-1,\, m)}}{\sum_i w_i\, p_{iS}(m-1, m)} = C_u^{(m-1,\, m)}.$$

A similar computation yields C_S

$$C_S^{(m-1,\, m)} = P[X_{m+1} \in S \mid X_m \in S].$$

These last remarks imply the validity of (8) when condition (2) holds for all n. To conclude the proof of the theorem we must consider the situation where conditions (2) and (3) alternate in some way. That is, at time t condition (2) may be valid but at $t+1$ or any other time τ, condition (3) may be true. We must consider all paths of length m, for any finite m. There are two possible situations where we may have difficulty: a) If at time t the path leaves S and enters some state $j \in S$ and at this time condition (2) is valid, and b) the process is in state S and at this time (3) holds. For in case a) if the system is in any state $j \in S$ and at this time (3) is true, then the sufficiency is trivial as an identity in zero. However, if the path is in $j \in S$ and (2) holds, the sufficiency is still valid as pointed out in the computation of the paths in (7). In the second case, i. e. b), the fact that the X_n process (operating in the background, so to speak) is Markovian allows the sufficiency to go through. For it (the X_n process) has the governing probabilistic mechanism in this case. This concludes the proof.

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FUNCŢII DE LANŢURI MARKOV NEOMOGENE

Rezumat

Se generalizează, pentru cazul nestanţionar, rezultatele obţinute de diverşi autori (vezi bibliografia) pentru lanţuri Markov nestaţionare.