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ASUPRA TEOREMEI DE RECIPROCITATE ÎN TEORIA DINAMICĂ A ELASTICITĂŢII LINIARE

Rezumat

Se consideră ecuaţiile elasticităţii liniare în cazul dinamic pentru medii omogene şi anizotrope. Folosind metoda dată în [2] este stabilită teorema de reciprocitate fără a apela la transformata Laplace, extinzînd teorema dată în [1].

ON A BASIC THEOREM IN MINIMAL COST FLOWS

BY

IRINEL DRAGAN

Let $N = (V, A)$ be a directed network with m vertices and n arcs; denote these elements by $v_i \in V$, $(i = 1, \cdots, m)$, and $a_l \in A$, $(l = 1, \cdots, n)$, respectively. Suppose given a capacity n-vector $K \geqslant 0$, an unitar transportation costs n-vector $C \geqslant 0$ and a demand m-vector d. Then, we can state the minimal cost flow problem as: minimize

(1) $$C(X) = C'X,$$

subject to

(2) $$BX = d,$$

and

(3) $$0 \leqslant X \leqslant K,$$

where B is the incidence matrix of the network. Of course, this is the simple case of proportional costs.

In the following we shall suppose that our network has only one source v_1 and one sink v_m, in consequence the demand vector has the form $d = (v, 0, \cdots, 0, -v)$. A vector X satisfying (2), (3), with such a vector d, will be called as habitually a flow with value v. We shall suppose also integer capacities, costs and demands.

Let us denote by $C(\mu)$ the unitar cost of a cycle μ, i. e.

(4) $$C(\mu) = C'\mu,$$

where μ represents the corresponding vector cycle. A cycle μ will be called admissible cycle with respect to the flow X, when there exists at least a positive integer ε such that $\bar{X} = X + \varepsilon\mu$ be also a flow in the network. Now, we can state the central theorem in minimal cost flows:

(T) *A flow $\hat{X}$ is minimal if and only if $C(\mu) \geq 0$ for every cycle μ admissible with respect to this flow* (see [2], [5]). A proof is given in [6], where more general cost functions are considered.

As T. C. Hu remarks in his book [5], this theorem and other near results justify several algorithms for solving the minimal cost flow problem. In the present paper we shall consider such a result, which justifies an algorithm given by R. G. Busacker and P. J. Gowen (see [1], [2]). Essentially, this algorithm solves the minimal cost flow problem by solving successive shortest chain problems; the result which justifies the algorithm uses the concept of reduced cost of a chain.

We define the reduced cost of a chain ν, with respect to a flow X, by the formula

(5) $$C(\nu) = \begin{cases} C'\nu & \text{if } \nu \text{ is admissible,} \\ \infty & \text{otherwise.} \end{cases}$$

Of course, the number $C(\nu)$ represents the cost of sending a supplemental unitar flow along the chain, when we can send this flow.

Now, we shall give a new proof for the following theorem: *If $\hat{X}$ is a minimum cost flow with value q and $\hat{\nu}$ is a minimum reduced cost chain from v_1 to v_m, then $\hat{Y} = \hat{X} + \hat{\nu}$ will be a minimum cost flow with value $q+1$.*

Two different proofs are given in [2] and [4]. Our proof is very similar to that given by V. V. Menon for the theorem (T).

In our proof the following previous result will be used:

T_1: *If X is a flow with value q and Y is a flow with value $q+1$, then there exists a chain ν and a collection of cycles $\mu_1, \cdots, \mu_r$ such that the equality:*

(6) $$Y = X + \nu + \varepsilon_1 \cdot \mu_1 + \cdots + \varepsilon_r \cdot \mu_r$$

be true for some positive integers $\varepsilon_1, \cdots, \varepsilon_r$; moreover every cycle μ_s, $(1 \leq s \leq r)$ will be an admissible cycle with respect to X.

Proof: We can prove constructively this result, just as this was done for the theorem (T), (see [6], Th. 3). The crucial point is: we can suppose that every cycle μ_s, $(1 \leq s \leq r)$, has no common arcs with the chain ν; in truth, if a cycle μ_s has at least a common arc with ν, we can replace the chain ν by the chain $\nu + \varepsilon_s \mu_s$. The constructive proof shows every cycle μ_s, $(1 \leq s \leq r)$, being admissible with respect to the flow $X + \nu$, (see [3], Th. 4). Then, μ_s will be also admissible with respect to the flow X, because ν and μ_s have no common arcs.

T_2: *If $\hat{X}$ is a minimum cost flow with value q and $\hat{\nu}$ is a minimum reduced cost chain from v_1 to v_m, then $\hat{Y} = \hat{X} + \hat{\nu}$ will be a minimum cost flow with value $q+1$.*

Proof: Clearly, $\hat{Y}$ is a flow with value $q+1$ and

(7) $$C'\hat{Y} \leq C'(\hat{X} + \nu),$$

for each admissible chain ν. If $\hat{Y}$ is not an optimal flow and we denote by Y a minimum cost flow, we have

(8) $$C'Y < C'\hat{Y}.$$

According T_1 we can write

(9) $$Y = \hat{X} + \nu + \varepsilon_1 \cdot \mu_1 + \cdots + \varepsilon_r \cdot \mu_r,$$

where $\varepsilon_s > 0$, $(1 \leq s \leq r)$, and μ_s $(1 \leq s \leq r)$, are admissible cycles with respect to $\hat{X}$. As $\hat{X}$ is an optimal flow, we have for these admissible cycles

(10) $$C(\mu_s) = C'\mu_s \geq 0, \ (1 \leq s \leq r),$$

according to theorem (T). So, by using (9) and (7) we get

(11) $$C'Y = C'(\hat{X} + \nu) + \sum_{s=1}^{s=r} \varepsilon_s (C'\mu_s) \geq C'(\hat{X} + \nu) \geq C'\hat{Y},$$

which contradicts (8). Hence, $\hat{Y}$ is an optimal flow.

Now it is clear that a minimum cost flow with value v could be reached by constructing minimum cost flows with values $1, \cdots, v-1, v$ successively; in every step q we have an available minimum cost flow with value $q-1 < v$ and we are looking for a minimum reduced cost chain with respect to this flow. If such a chain doesn't exist we can't reach a flow with value v, because the maximal flow has the value $q-1$; otherwise, we get a minimal cost flow with value q, by sending a supplemental unitar flow along the chain. If $q = v$ the problem is solved, otherwise we have to pass to the next step. This is the algorithm given by R. G. Busacker and P. J. Gowen (see [1]).

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O TEOREMĂ DE BAZĂ ASUPRA FLUXURILOR DE COST MINIM

Rezumat

Se dă o nouă demonstraţie pentru teorema ce stabileşte legătura dintre problema fluxului de cost minim şi problema drumului minim.

НЕСКОЛЬКО ЗАМЕЧАНИЙ ОБ АЛГОРИТМАХ ОТСЕЧЕНИЯ (РЕШЕНИЕ ЦЕЛОЧИСЛЕННОЙ ЗАДАЧИ ЛИНЕЙНОГО ПРОГРАММИРОВАНИЯ)

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В работе описан алгоритм перебора вершин многогранника, заданного при помощи линейных ограничений, кратко изложен способ нахождения общего решения в целых числах системы линейных алгебраических уравнений с целыми коэффициентами, а в заключение приведены результаты вычислительного эксперимента для 100 задач, иллюстрирующие работу алгоритмов отсечения при решении ими нескольких классов целочисленных задач линейного программирования.

п. 1. *Алгоритм отсечения для перебора вершин многогранника.* В этом алгоритме производится перебор вершин многогранника, не удовлетворяющих заданным условиям, причем процесс начинается от оптимального решения задачи линейного программирования. Если заданными условиями являются условия целочисленности, то рассматриваемый алгоритм может быть использован для анализа специального класса целочисленных задач. В основу алгоритма перебора положены лексикографический двойственный симплекс — метод и идея отсечения.

Класс задач, для которых многогранное множество условий является целочисленным (т.е. все вершины этого многогранного множества — целочисленные векторы), представляет собой весьма узкий класс задач математического программирования.

Рассмотрим один специальный класс целочисленных задач. Под частично целочисленным многограным множеством будем понимать такое многогранное множество, для которого все точки, удовлетворяющие ограничениям целочисленности, являются только вершинами рассматриваемого множества.