where $N_{\chi^\times} = \operatorname{Ker} \chi^\times$ is the associated ordinary ideal of the weak ideal $N_\chi^\cdot$, namely

$$N_{\chi^\times} = \{ q^\times (X) \mid q^\times (X) \in A^\times [X],\ q^\times (v - d) = 0 \}.$$

As $R^\cdot \supset A^\cdot$ are commutative weak fields II and hence $R^\times \supset A^\times$ are ordinary fields, $A^\times [X]$ is a principal ideal domain, $N_{\chi^\times}$ is a principal ideal generated by an irreducible polynomial $\varphi (X) \in A^\times [X]$, $A^\times [X]/N_{\chi^\times}$ is a field and so is $A^\times [v - d]$. It follows that $A^\cdot [v]$ is a weak field II, containing $A^\cdot$ and v, and this, together with (18), shows that $A^\cdot [v]$ is just the simple extension weak field

$$A^\cdot [v] = A^\cdot (v) \simeq \frac{A^\cdot [x]}{N_\chi^\cdot}. \tag{20}$$

So we have proved:

Theorem 6. *If $R^\cdot \supset A^\cdot$ are weak fields II and $v \in R^\cdot$, $v \neq d$, then the simple extension weak field $A^\cdot (v)$ can be only in the two following cases:*

I. (19) $A^\cdot (v) \simeq A^\cdot (x)$ *the weak quotient field of weak polynomials over $A^\cdot$ and is called weakly transcendent over $A^\cdot$;*

II. (20)
$$A^\cdot (v) = A^\cdot [v] \simeq \frac{A^\cdot [x]}{N_\chi^\cdot}.$$

hence $A^\cdot (v)$ is the weak ring of the polynomials of v over $A^\cdot$ and $N_\chi^\cdot$ the weak ideal given by (16); then v is called weakly algebraic over $A^\cdot$.

BIBLIOGRAPHY

1. Climescu Al. — *O nouă clasă de inele slabe.* Bul. Inst. polit. Iaşi, T. X (XIV), (1964), p. 29.
2. Luchian T. — *Homomorfisme în inele slabe.* An. şt. Univ. Iaşi, Matematică, T. XIII (1967), p. 17.
3. Luchian T. — *Asupra noţiunii de domeniu de integritate în inele slabe.* An şt. Univ. Iaşi, Matematică, T. XIII (1967), p. 257.
4. Luchian T. — *Asupra unor transformări ale inelelor slabe.* An. şt. Univ. Iaşi, Matematică, T. XIV (1968), p. 245.
5. Luchian T. — *Asupra unei clase de endomorfisme asociate inelelor slabe.* An. şt. Univ. Iaşi, Matematică, T. XV (1969), p. 271.
6. Luchian T. — *Polynomials over weak rings.* An. şt. Univ. Iaşi, Matematică, T. XVI (1970), p. 1.
7. Luchian T. — *The weak fields of fractions of a weak integral domain.* An. şt. Univ. Iaşi T. XVII (1971), p. 245.

EXTENSIUNI DE INELE SLABE

Rezumat

Se studiază extensiunile simple ale inelelor slabe asociative, apoi corpurile slabe de extensiune simplă ale corpurilor slabe de specia a II-a comutative.

A GENERALIZATION OF THE GENERALIZED COMMUTATIVE RINGS

BY

GEETA DEVI

1. **Introduction**: Belluce — Herstein — Sain [1] have defined a ring R to be a generalized commutative ring (written as g.c. ring) if given $a, b \in R$, there exist positive integers $m = m(a, b)$, $n = n(a, b)$, such that

$$(a\,b)^m = (b\,a)^n.$$

The aim of this paper is to generalize this property. We shall call a Multiplicative Semi-Group S to have G-property if given $a, b \in S$ there exist positive integers $m = m(a, b)$, $n = n(a, b)$ such that

$$a^n b^m = b^m a^n. \tag{G}$$

2. It follows from lemma [1] that if G is a Multiplicative Semi-Group in which for each $a,\ b \in G$

$$(a\,b)^{m(a,b)} = (b\,a)^{n(a,b)}$$

where $m(a, b)$ and $n(a, b)$ are positive integers, then G has G-property. We assume for convenience that the ring R has unity. We shall now prove the following

3. **Theorem**: *Let G be a Multiplicative Semi-Group. Let $a, b, c \in G$, there exist positive integers m, n, r and S depending on a and b such that*

$$c\,a^n b^m c^{-1} = b^n a^m \quad \text{and} \quad a\,b\,c^r a^{-1} b^{-1} = c^s$$

then there exist positive integers λ and μ such that $a^\mu b^\lambda = b^\lambda a^\mu$.

Proof: If a and b are of finite order then the result is obvious. So let a and b be not of finite order. Then if $a^n b^m = b^n a^m$, for positive integers m and n, we must have $m = n$. We have by hypothesis

$$c\, a^n\, b^m\, c^{-1} = b^n\, a^m\, .$$

By induction we get $c^r a^{n^r} b^{m^r} c^{-r} = b^{n^r} a^{m^r}$ for all positive integers r. We write for convenience

(1) $$c^r\, a^n\, b^m\, c^{-r} = b^n\, a^m\, .$$

Consider the collection of all ordered pairs (m, n), satisfying (1). Let $x(r$ and $y(r)$ be the smallest positive integers among m's and n's respectively i.e. $x(r) = \min\, \{m \mid c^r a^n b^m c^{-r} = b^n a^m$ for some $n\}$ and similarly $y(r) = \min\, \{m \mid c^r a^n b^m c^{-r} = b^n a^m$ for some $n\}$. We claim $c^r\, a^{y(r)}\, b^{x(r)}\, c^{-r} = b^{y(r)}\, a^{x(r)}$.

Let x, y be such that

(2) $$c^r\, a^{y(r)}\, b^x\, c^{-r} = b^{y(r)}\, a^x\, ,$$

(3) $$c^r\, a^y\, b^{x(r)}\, c^{-r} = b^y\, a^{x(r)}\, .$$

Multiplying (2) by $y, x\,(r)$ and (3) by $y(r), x$, we have

(4) $$c^r\, a^{y(r)\cdot y}\, b^{x\cdot x(r)} = b^{y(r)x(r)}\, a^{xy},$$

(5) $$c^r\, a^{y\cdot y(r)}\, b^{x(r)\cdot x} = b^{yx}\, a^{x(r)y(r)}.$$

This would make right hand sides equal. Thus the right hand $b^{y(r)x(r)}\, a^{xy}$ and $b^{yx}\, a^{x(r)y(r)}$ are also equal. Thus by our remark in the beginning, this implies $y \cdot x = y(x)\, x(r)$. Since $x(r) \leqslant x$ and $y(r) \leqslant y$ we must have $x(r) = x$, $y(r) = y$.

Therefore we have

(6) $$c^r\, a^{y(r)}\, b^{x(r)}\, c^{-r} = b^{y(r)}\, a^{x(r)} \quad \textit{for each positive integer } r.$$

Suppose we have also

(7) $$c^r\, a^\mu\, b^\lambda\, c^{-r} = b^\mu\, a^\lambda\, .$$

(4) and (5) give: $a^{\mu x(r)}\, b^{\lambda y(r)} = b^{\mu x(r)}\, a^{\lambda y(r)}$. This means

$$a^{\mu x(r)} = a^{\lambda y(r)} \text{ and } b^{\lambda y(r)} = b^{\mu x(r)} \text{ i.e. } \mu\, x(r) = \lambda\, y(r).$$

So we obtain that for a given positive integer r, if the relation (7) is true, then the ratio λ/μ is constant and equals $x(r)/y(r)$.

Now, if s is another positive integer,

$$c^{r+s}\, a^{y(r)y(s)}\, b^{x(r)x(s)}\, c^{-r-s} = (c^{r+s}\, a^{y(r)}\, b^{x(r)}\, c^{-r-s})^{y(s)x(s)} =$$

$$= (c^s\, c^r\, a^{y(r)}\, b^{x(r)}\, c^{-r-s})^{y(s)x(s)} = (c^s\, b^{y(r)}\, a^{x(r)}\, c^{-s})^{y(s)x(s)} =$$

$$= (c^s\, b^{x(s)}\, a^{y(s)}\, c^{-s})^{y(r)x(r)} = b^{y(s)y(r)}\, a^{x(s)x(r)}.$$

Therefore, by remark just made:

$$\frac{x(r)\; x(s)}{y(r)\; y(s)} = \frac{x(r+s)}{y(r+s)}\, .$$

If we set $f(r) = x(r)/y(r)$, then $f(r+s) = f(r)\, f(s)$. This gives $f(r) = [f(1)]^r$ so that if we can prove $g(r) = 1$, for some r, then $f(r) = 1$ for each r.

In particular we would have for $r = 1$, the relation $a^{y(1)}\, b^{x(1)} = b^{x(1)}\, a^{y(1)}$, so we now proceed to show that $f(r) = 1$ for some r. So far we have not used our second hypothesis

(8) $$a\, b\, c^r\, a^{-1}\, b^{-1} = c^s\, .$$

(Note that r and s are some fixed positive integers satisfying the relation).

We conjugate (6) by ab and rewrite it as

$$a\, b\, c^r\, a^{-1}\, b^{-1}\, a^{y(r)}\, b^{x(r)}\, a\, b\, c^{-r}\, a^{-1}\, b^{-1} = b^{y(r)}\, a^{x(r)},$$

(9) $$c^s\, a^{y(r)}\, b^{x(r)}\, c^{-s} = b^{y(s)}\, a^{x(s)}\, .$$

Using (8)

$$c^r\, a^{y(r)}\, b^{x(r)}\, c^{-r} = b^{y(r)}\, a^{x(r)}\, .$$

Hence we obtain integers λ and μ such that,

(10) $$c^\lambda\, a^{y(r)}\, b^{x(r)}\, c^{-\lambda} = b^{y(r)}\, a^{x(r)}$$

and also

(11) $$c^\mu\, a^{y(r)}\, b^{x(r)}\, c^{-\mu} = b^{y(r)}\, a^{x(r)}\, .$$

Raising both sides of (10) by $x(\lambda)$ and both sides of (11) by $x(\mu)$ we have:

(12) $$\begin{aligned} c_\lambda\, b^{x(r)x(\lambda)}\, a^{y(r)x(\lambda)}\, c^{-\lambda} &= b^{y(r)x(\lambda)}\, a^{x(r)x(\lambda)}, \\ c^\mu\, b^{x(r)x(\mu)}\, a^{y(r)x(\mu)}\, c^{-\mu} &= b^{x(r)x(\mu)}\, a^{x(r)x(\mu)}. \end{aligned}$$

The left hand side of (10) is $b^{x(r)y(\lambda)}\, a^{y(r)y(\lambda)}$, i. e. $b^{y(\lambda)x(r)}\, a^{y(\lambda)y(r)}$; and $b^{y(\mu)x(r)}\, a^{y(\mu)y(r)}$.

Thus $b^{y(\lambda)x(r)}\, a^{y(\lambda)y(r)} = b^{x(r)x(\lambda)}\, a^{y(r)x(\lambda)}$ and similarly

$$b^{y(\mu)x(r)}\, a^{y(\mu)y(r)} = b^{x(r)x(\mu)}\, a^{y(r)x(\mu)}.$$

But this implies that $x(\lambda) = y(\lambda)$ and $x(\mu) = y(\mu)$. Hence $f(\mu) = 1$, $f(\lambda) = 1$. This completes the proof.

My thanks are due to Dr. H.M. Srivastava for his kind guidance and help in preparing this paper.

REFERENCES

1. Belluce L.P., Herstein I.N. and Sain S.K. — *Generalized commutative rings.* Nagoya Math. J. 27(1966) 1—5.
2. Herstein I.N. — *Two remarks on the commutativity of rings.* Canadian J. Math 7 (1955), 411—412

Department of Mathematics,
Lucknow University,
Lucknow (INDIA)

O GENERALIZARE A INELELOR COMUTATIVE GENERALIZATE

Rezumat

Autorul numeşte un semigrup multiplicativ cu proprietatea G o mulţime de elemente S, cu proprietatea că pentru o pereche $a,b \in S$ există $m = m(a,b)$ şi $n = n(a,b)$, aşa ca relaţia (G) să fie satisfăcută. Articolul studiază proprietăţile acestei structuri algebrice şi relaţiile ei cu alte structuri.

ASUPRA G-ALGEBRELOR ATAŞATE UNEI ALGEBRE DATE

DE

V. TAMAS

Încercînd să găsim clase de algebre neasociative care să generalizeze noţiunea de algebră Lie, în [5] am introdus noţiunea de G-algebră.

Dacă A este o algebră peste corpul $\mathfrak{K}$, cu înmulţirea ab pentru $a, b \in A$, vom numi G-algebră structura care se obţine din A, luînd pe lîngă structura de spaţiu liniar înmulţirea $a \ast b$ definită prin:

$$a \ast b = h(a,b)\, ab + k(a,b)\, ba, \tag{1}$$

unde $h(a,b)$ şi $k(a,b)$ sînt două funcţii de două argumente vectoriale cu valori în $\mathfrak{K}$ şi o vom nota cu $A_{h,k}$.

Clasa acestor algebre include clasa algebrelor Lie care se obţin pentru $h(a,b) = 1$, $k(a,b) = -1$; clasa algebrelor Jordan speciale care se obţin pentru $h(a,b) = 1/2$, $k(a,b) = 1/2$; clasa algebrelor cvasi-asociative obţinute pentru $h(a,b) = \lambda$ constant şi $k(a,b) = 1 - \lambda$, în cazul în care se pleacă de la algebra A asociativă.

Pornindu-se de la alte algebre se obţin noi clase de algebre neasociative. Astfel, dacă A este alternativă, atunci G-algebra corespunzătoare este o algebră Malţev

În [5] şi [6] am studiat unele proprietăţi ale acestor algebre şi ale unor structuri ternare pe care le-am numit sisteme triple generalizate. În Nota de faţă dorim să punem în evidenţă cîteva proprietăţi pe care le-am putea numi ereditare, care se transmit de la A la G-algebra corespunzătoare $A_{h,k}$.

Se ştie că dată algebra asociativă A peste corpul $\mathfrak{K}$ introducînd comutatorul, determinăm o algebră unică A^-, algebra Lie ataşată, iar dacă introducem simetrizatorul determinăm algebra Jordan A^+ ataşată. Această determinare este univocă întrucît reciproc nu mai este posibil să determinăm algebra asociativă pornind de la A^- sau A^+.