ON DISTORTION THEOREMS IN CLASSICAL DOMAINS OF SEVERAL COMPLEX VARIABLES *)

BY

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1. **Introduction**. The extension of Ahlfors' proof [1] of the classical Schwarz-Pick distortion theorem for the unit disc

$$C_1^1 = \{z : |z| < 1\} \tag{1.1}$$

to the n-dimensional unit ball

$$C_1^n = \{z : \|z\| < 1\} \tag{1.2}$$

of the complex (unitary) space $\mathbf{C}^n$ with the Euclidean norm $\|z\|$ ($\|z\|^2 = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$), requires the existence of a real solution $u = \{u(z, \bar{z}), z \in C_1^n\}$ (shorter: $\{u(z, \bar{z}), z \in C_1^n\}$ of the partial differential equation

$$\det[u_{z\bar{z}}] = \det[u_{z_i\bar{z}_k}]_n = e^{(n+1)u} \tag{1.3}$$

with boundary values $+\infty$ and $[u_{z\bar{z}}] > 0$ ($[u_{z\bar{z}}] > 0 \Leftrightarrow u_{z\bar{z}}\, dz d\bar{z}$ positive definite!). This has been shown first in [7] and shortly thereafter completed and extended in [8] and [9]. Generalizations of various kinds (Kernel functions of homogeneous domains) using the same method as in [7] has

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**) Senior Foreign Scientist Fellow of the National Science Foundation been given by J. Mitchell and K. Hahn [12], [13], and more general theorems involving non-Kählerian metrics were proved by S. S. Chern [6] and S. Kobayashi [11].

Since the definitions of H. Cartan's [5] and S. Bergman's polynomials [2], [3] resp. sequences of functions, and the Bergman Kernel (references in [4]) require considerable technique, it should be of interest to attack the problem, at least for the classical homogeneous domains, by methods of the theory of partial differential equations leading to metrics with constant (negative) Ricci-curvature (references in [7]). The unique solution of this equation with boundary values $+\infty$ and $[u_{z\bar{z}}] > 0$ is strongly connected with the Bergman Kernel. In this paper, the chosen domain shall be the Lie-sphere given by (2.4). The technique is based on a general determinant identity, which shall be proved in **2.** The reader will find further applications of this identity (on Hua's algebraic indentities) in an additional note.

2. Definitions and preparatory theorems. A matrix with real or complex entries ω_{ik} with p rows and q columns will be denoted in general by Ω_{pq}, $[\omega_{ik}]_{pq}$, or Ω. If $p = q$, then Ω_{pq} will be denoted by Ω_q, $[\omega_{ik}]_q$, or $[\omega_{ik}]$. The transposed matrix of Ω shall be denoted by Ω'_{pq}, $[\omega'_{ik}]_{pq}$, or Ω'. If the ω_{ik} are complex numbers we set $\bar{\Omega} = [\bar{\omega}_{ik}]_{pq}$ and $\Omega' = [\omega_{ik}]'_{pq} = [\omega'_{ik}]_{qp}$ $(\omega'_{ik} = \omega_{ki})$.

Definition. *Let D be a bounded region of the complex space $\mathbf{C}^n$ $(n \geq 1)$ with points $z = (z_1, \ldots, z_n)$ and let $f: D \to D$ be a mapping from D into D given by a system $f = (f_1, \ldots, f_n)$ of functions $f_1, \ldots, f_n$ holomorphic in D. Then f is called an inner mapping of D (references in [4]).*

Definition. *Let $n = pq(p, q$ integers $> 0)$. Then the four classical regions (references in [10]) $\mathfrak{R}_{\mathrm{I}}$, $\mathfrak{R}_{\mathrm{II}}$, $\mathfrak{R}_{\mathrm{III}}$, and $\mathfrak{R}_{\mathrm{IV}}$ will be defined as follows:*

$$\mathfrak{R}_{\mathrm{I}} = \{z : E_p - Z\bar{Z}' > 0, \quad Z = [z_{ik}]_{pq}, \quad E_p = [\delta_{ik}]_p\} \tag{2.1}$$

$$\mathfrak{R}_{\mathrm{II}} = \{z : E_q - Z\bar{Z}' > 0, \quad Z = [z_{ik}]_q, \quad Z' = Z\} \tag{2.2}$$

$$\mathfrak{R}_{\mathrm{III}} = \{z : E_q + Z\bar{Z}' > 0, \quad Z = [z_{ik}]_q, \quad Z' = -Z\} \tag{2.3}$$

$$\mathfrak{R}_{\mathrm{IV}} = \{z : 1 - 2\|z\|^2 + |zz'|^2 > 0, \quad |zz'| < 1\} \tag{2.4}$$

where $z = (z_1, \ldots, z_n)$ *and* $zz' = z_1^2 + \cdots + z_n^2$.

The region $\mathfrak{R}_{\mathrm{IV}}$ is called the Lie-sphere (more references in [10]). The following classical proposition will be stated without proof (references in [8]):

Lemma 1. *Let A_{pq}, B_{pq} be Hermitian matrices, $A_{pq} \geqq 0$ and $B_{pq} \geqq 0$. Then*

$$A_{qq} \geqq B_{qq} \Rightarrow 0 \leqq \det B_{qq} \leqq \det A_{qq}. \tag{2.5}$$

The following important lemma is for the development of sections 3 and 4:

Lemma 2. *Let $A_{pq} = [\alpha_{ik}]_{pq}$, $B_{pq} = [\beta_{ik}]_{pq}$, $C_q = [\gamma_{ik}]_q$ be arbitrary real or complex matrices. Then*

$$\det[\gamma_{ik} + \textstyle\sum_{\mu=1}^{p} \lambda_\mu \alpha_{\mu i} \beta_{\mu k}]_q = \det \begin{bmatrix} E_p & -\Lambda_p B_{pq} \\ A'_{pq} & C_q \end{bmatrix} \tag{2.6}$$

where

$$\Lambda_p = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_p \end{bmatrix}.$$

Clearly,

$$[\gamma_{ik} + \textstyle\sum_{\mu=1}^{p} \lambda_\mu \alpha_{\mu i} \beta_{\mu k}]_q = C_q + A'_{pq} \Lambda_p B_{pq}.$$

Proof of Lemma 2. We set $(p = 1!)$

$$X(\lambda_1) = \det[\gamma_{ik} + \lambda_1 \alpha_{1i} \beta_{1k}]_q$$

and

$$Y(\lambda_1) = \det \begin{bmatrix} 1 & -\lambda_1\beta_{11} \cdots -\lambda_1\beta_{1q} \\ \alpha_{11} & \\ \vdots & [\gamma_{ik}]_q \\ \alpha_{1q} & \end{bmatrix}.$$

Then we obtain

$$X''(\lambda_1) = Y''(\lambda_1) = 0$$

and therefore [7]

$$X(\lambda_1) = X(0) + \lambda_1 X'(0), \quad Y(\lambda_1) = Y(0) + \lambda_1 Y'(0).$$

On the other hand,

$$X(0) = Y(0) = \det[\gamma_{ik}]_q$$

and

$$X'(0) = -\det \begin{bmatrix} 0 & \beta_{11} \cdots \beta_{1q} \\ \alpha_{11} & \\ \vdots & [\gamma_{ik}]_q \\ \alpha_{1q} & \end{bmatrix} = Y'(0).$$

Hence (2.6) holds for $p=1$. Assume now (2.6) holds for some $p, p>1$. We set

$$\widetilde{\gamma}_{ik}=\gamma_{ik}+\lambda_p\alpha_{pi}\beta_{pk}$$

and

$$X(\lambda_p)=\det\left[\widetilde{\gamma}_{ik}+\sum_{\mu=1}^{p-1}\lambda_\mu\alpha_{\mu i}\beta_{\mu k}\right]_q.$$

Then by applying (2.6) we get

$$\widetilde{X}(\lambda_p)=\det\begin{vmatrix} E_{p-1} & -\Lambda_{p-1}B_{p-1q}\\ A'_{p-1q} & [\widetilde{\gamma}_{ik}]_q\end{vmatrix}.$$

Now

$$\Lambda_{p-1}B_{p-1q}=\begin{vmatrix}\lambda_1\beta_{11}\cdots\lambda_1\beta_{1q}\\ \cdot\quad\cdots\quad\cdot\\ \lambda_{p-1}\beta_{p-1 1}\cdots\lambda_{p-1}\beta_{p-1q}\end{vmatrix}$$

and therefore $X''(\lambda_p)$, as a function of λ_p, must be identically zero. Hence

$$\widetilde{X}(\lambda_p)=\widetilde{X}(0)+\lambda_p\widetilde{X}'(0)\,. \tag{2.7}$$

On the other hand, $X'(0)$ has the form

$$\alpha_{p1}P_1+\alpha_{p2}P_2+\cdots+\alpha_{pq}P_q$$

where P_ν means the determinant obtained from

$$\det\begin{bmatrix} E_p & -\Lambda_{p-1}B_{p-1q}\\ A'_{p-1q} & C_q\end{bmatrix}$$

if we replace the $p+\nu-1$ row by $0\cdots0\ \beta_{p1}\cdots\beta_{pq}$. This proves that (2.7) equals the right side of (2.6).

3. Special cases. By (2.6) we have for

$$X(\lambda)=X(\lambda_1,\dots,\lambda_p)=\det\left[\gamma_{ik}+\sum_{\mu=1}^{p}\lambda_\mu\alpha_{\mu i}\beta_{\mu k}\right]_q \tag{3.1}$$

the expression

$$\det[\gamma_{ik}]_q+\sum_{\mu=1}^{p}\Big\{\sum_{1\le\sigma_1<\dots<\sigma_\mu}\Phi_{\sigma_1\cdots\sigma_\mu}\lambda_{\sigma_1}\cdots\lambda_{\sigma_\mu}\Big\}$$

with coefficients $\Phi_{\sigma_1\cdots\sigma_\mu}$ not depending on $\lambda_1,\dots,\lambda_p$. Now setting

$$X^A_{\sigma_1\cdots\sigma_\mu}=\begin{vmatrix}\alpha_{\sigma_1 1}\cdots\alpha_{\sigma_1 q}\\ \cdots\\ \alpha_{\sigma_\mu 1}\cdots\alpha_{\sigma_\mu q}\end{vmatrix}$$

and

$$X^B_{\sigma_1\cdots\sigma_\mu}=\begin{vmatrix}\beta_{\sigma_1 1}\cdots\beta_{\sigma_1 q}\\ \cdots\\ \beta_{\sigma_\mu 1}\cdots\beta_{\sigma_\mu q}\end{vmatrix},$$

we obtain

$$\left.\frac{\partial^\mu X}{\partial\lambda_{\sigma_1}\cdots\partial\lambda_{\sigma_\mu}}\right|_{\lambda=0}=\det\begin{vmatrix}O_\mu & -X^B_{\sigma_1\cdots\sigma_\mu}\\ (X^A_{\sigma_1\cdots\sigma_\mu})' & C_q\end{vmatrix}$$

where O_μ denotes the zero matrix of order μ. Consequently,

$$\begin{aligned}&\det\left[\gamma_{ik}+\textstyle\sum_{\mu=1}^{p}\lambda_\mu\alpha_{\mu i}\beta_{\mu k}\right]=\\ &=\det[\gamma_{ik}]_q+\sum_{\mu=1}^{p}\Big\{\sum_{1\le\sigma_1<\dots<\sigma_\mu}\det\begin{vmatrix}O_\mu & -X^B_{\sigma_1\cdots\sigma_\mu}\\ (X^A_{\sigma_1\cdots\sigma_\mu})' & C_q\end{vmatrix}\lambda_{\sigma_1}\cdots\lambda_{\sigma_\mu}\Big\}.\end{aligned} \tag{3.2}$$

Supposing $[\gamma_{ik}]_q=[\delta_{ik}]_q$, setting $a_i=[\alpha_{ik}]_{1q}, b_i=[\beta_{ik}]_{1q}$ $(i=1,\dots,q)$, and introducing the inner products

$$\begin{aligned}&(a_{\sigma_1}\wedge\cdots\wedge a_{\sigma_\mu}, b_{\sigma_1}\wedge\cdots\wedge b_{\sigma_\mu})=\\ &=\det X^A_{\sigma_1\cdots\sigma_\mu}\det X^B_{\sigma_1\cdots\sigma_\mu}\end{aligned}$$

of the multi-vectors $a_{\sigma_1}\wedge\cdots\wedge a_{\sigma_\mu}$ and $b_{\sigma_1}\wedge\cdots\wedge b_{\sigma_\mu}$, we get

$$\begin{aligned}&\det\left[\delta_{ik}+\textstyle\sum_{\mu=1}^{p}\lambda_\mu\alpha_{\mu i}\beta_{\mu k}\right]_q=\\ &=1+\sum_{\mu=1}^{p}\Big\{\sum_{1\le\sigma_1<\dots<\sigma_\mu}(a_{\sigma_1}\wedge\cdots\wedge a_{\sigma_\mu}, b_{\sigma_1}\wedge\cdots\wedge b_{\sigma_\mu})\lambda_{\sigma_1}\cdots\lambda_{\sigma_\mu}\Big\}.\end{aligned} \tag{3.3}$$

The following identity is important for later developments:

Lemma 3. *Suppose that $\xi_1=(\xi_{1i},\dots,\xi_{1q})$ and $\xi_2=(\xi_{2i},\dots,\xi_{2q})$ are complex vectors. Then*

$$\begin{aligned}&\det[\delta_{ik}+\lambda_1\xi_{1i}\bar{\xi}_{1k}+\lambda_2\xi_{2i}\bar{\xi}_{2k}]=\\ &=1+\lambda_1\|\xi_1\|^2+\lambda_2\|\xi_2\|^2+\lambda_1\lambda_2\|\xi_1\wedge\xi_2\|^2.\end{aligned} \tag{3.4}$$

Proof. Write ξ_{1i},ξ_{2i} for α_{1i},α_{2i} and set $\beta_{1i}=\bar{\alpha}_{1i},\beta_{2i}=\bar{\alpha}_{2i}$.

4. Partial differential equations. We define $V = \{V(z)\}_{z \in L_1^n}$ in (Lie-sphere)

(4.1) $$L_1^n = \{z : 1 - 2\|z\|^2 + |zz'|^2 > 0, \ |zz'| < 1\}$$

by

(4.2) $$V = V(z) = (1 - 2\|z\|^2 + |zz'|^2)^{-1}$$

and we prove the following theorem:

Theorem 1. *The function V satisfies the partial differential equation*

(4.3) $$\det[(\log V)_{z\bar z}] = 2^n V^n.$$

Proof. Write ψ for $\log V$ and A for $zz' = z_1^2 + \cdots + z_n^2$. Then

$$\psi_{z_i \bar z_k} = 2V\{\delta_{ik} - 2z_i \bar z_k + 2V(\bar z_i - z_i \bar A)(z_k - \bar z_k A)\}$$

and consequently, using (3.4)

$$\det[\psi_{z_i \bar z_k}]_n = 2^n V\{1 - 2\|z\|^2 + 2V\|\bar z - zA\|^2 - 4V\|z \wedge (\bar z - zA)\|^2\}.$$

Now $z \wedge (\bar z - zA) = z \wedge \bar z$ and therefore

$$\|z \wedge (\bar z - zA)\|^2 = \|z \wedge \bar z\|^2 = \|z\|^4 - |zz'|^2 = \|z\|^4 - A\bar A.$$

On the other hand,

$$(1 - 2\|z\|^2)V^{-1} = (1 - 4\|z\|^2 + 4\|z\|^4) + (1 - 2\|z\|^2)A\bar A.$$

and

$$\|\bar z - zA\|^2 = \|z\|^2 - 2A\bar A + \|z\|^2 A\bar A.$$

Thus

$$(1 - 2\|z\|^2)V^{-1} + 2\|\bar z - zA\|^2 - 4\|z \wedge \bar z\|^2 =$$
$$= 1 - 2\|z\|^2 + |zz'|^2 = V^{-1}.$$

It is easy to see, that ψ is a strong plurisubharmonic function in L_1^n. We mark with m (as index) expressions working with the vector $[z_1, \cdots, z_m]$ and compute $[\psi_{z_i \bar z_k}]_m$ using (3.4). Then

$$2^{-m}V^{-(m+1)}\det[\psi_{z_i \bar z_k}]_m = T_m =$$
$$= (1 - 2\|z\|_m^2)V^{-1} + 2\|\bar z - zA\|_m^2 - 4\|z \wedge \bar z\|_m^2 =$$
$$= (1 - 2\|z\|_m^2)V^{-1} + 2\|\bar z - zA\|_m^2 - 4\|z\|_m^4 + 4A_m \bar A_m =$$
$$= V^{-1} - 2\|z\|_m^2(1 - 2\|z\|^2 + A\bar A) +$$
$$+ 2(\|z\|_m^2 + \|z\|_m^2 A\bar A - A_m \bar A - \bar A_m A) - 4\|z\|_m^4 + 4A_m \bar A_m.$$

Hence

$$I_m = V^{-1} + 4\|z\|_m^2(\|z\|^2 - \|z\|_m^2) - 2A_m(\bar A - \bar A_m) - 2\bar A_m(A - A_m).$$

Setting

$$\|z\|^2 = \|z\|_m^2 + d_m^2,$$

we obtain

$$|A_m(\bar A - \bar A_m)| = |\bar A_m(A - A_m)| \leqslant \|z\|_m^2 d_m^2,$$

and therefore $I_m \geqslant V^{-1} > 0$. This shows that $[\psi_{z\bar z}] > 0 \ (z \in L_1^n)$ holds. It still remains to show that $\log V$ is the only strong plurisubharmonic solution of the partial differential equation

(4.4) $$\det[(\log V)_{z\bar z}]_n = 2^n V^n$$

on L_1^n with boundary values $+\infty$. This shall be done in the next section.

5. Inner mappings. Distortion theorems. Using the identity

$$|zz'|^2 = \|z\|^4 - \|z \wedge \bar z\|^2$$

we can put (4.1) in the form [10]

$$L_1^n = \{z : 1 - \|z\|^2 > \|z \wedge \bar z\|\}$$

from which follows that $z \in L_1^n \Rightarrow \alpha z \in L_1^n (\forall \alpha, 0 < \alpha < 1)$.

Let $\omega = (\omega_1, \cdots, \omega_n)$ be an inner mapping of L_1^n having the property (property A): ω is holomorphic in $\widehat{L}_1^n$ (Hull of L_1^n).

We prove the theorem:

Theorem 2. *For every inner mapping $\omega : L_1^n \to L_1^n$, the following inequality holds*

(5.2) $$|\mathfrak{J}(\omega)|^2 V(\omega)^n \leqslant V(z)^n$$

where

(5.3) $$\mathfrak{J}(\omega) = \det\left|\frac{\partial \omega_i}{\partial z_k}\right|_n$$

is the Jacobian of the system $\omega_1, \cdots, \omega_n$ with respect to the variables $z_1, \cdots, z_n$.

Proof. Suppose that ω has the property A and let

$$\psi = \psi(z) = \log\left\{|\mathfrak{J}(\omega)|^2 \frac{V(\omega)^n}{V(z)^n}\right\}.$$

Then ψ has boundary values $-\infty$ and therefore has a maximum at a point L^n_1, say z. Hence

$$\psi_{z\bar{z}}\, dz\, d\bar{z} \leq 0$$

and consequently

$$(\log V(\omega))_{z\bar{z}}\, dz\, d\bar{z} \leqslant (\log V(z))_{z\bar{z}}\, dz\, d\bar{z}$$

at the point z. Thus (Lemma 1 !)

$$\det[(\log V(\omega))_{z\bar{z}}]_n = |\mathfrak{J}(\omega)|^2 \det[(\log V(\omega))_{\omega\bar{\omega}}]_n =$$
$$= 2^n |\mathfrak{J}(\omega)|^2 V(\omega)^n \leqslant \det[(\log V(z))_{z\bar{z}}]_n = 2^n V(z)^n$$

which contradicts the assumption $\psi > 0$ at the point z. Therefore (5.2) is valid in L^n_1.

The inequality (5.2) holds without assuming A. To prove this define $L^n_\delta (0 < \delta < +\infty)$ by

(5.4) $$L^n_\delta = \{z : 1 - 2\delta \| z \|^2 + \delta^2 |zz'|^2 > 0,\ \delta |zz'| < 1\}$$

and set

$$v_\delta = (1 - 2\delta \| z \|^2 + \delta^2 |zz'|^2)^{-1} \quad (z \in L^n_\delta).$$

Then

$$\det[(\log v_\delta)_{z\bar{z}}]_n = 2^n \delta^n v^n_\delta .$$

Choosing $1 < \delta < +\infty$, we can now prove by the same method (ω has the property A in L^n_δ) that

$$|\mathfrak{J}(\omega)|^2 V(\omega)^n \leq \delta^n v^n_\delta \quad (z \in L^n_\delta)$$

and ($\delta \to 1$!) the general validity of (5.2).

Theorem 3. *The partial differential equation*

(5.5) $$\det[(\log u)_{z\bar{z}}]_n = 2^n u^n$$

has exactly one strong plurisubharmonic solution in L^n_1 with the boundary values $+\infty$.

Proof. Assume (5.5) has a second solution $u \neq V$. We consider the function

$$\{\log u - \log v_\delta - \log \delta\}_{z \in L^n_\delta} (1 < \delta < +\infty)$$

with the boundary values $-\infty$, and prove first by means of theorem 2 that

$$u \leq \delta v_\delta \ (1 < \delta < +\infty,\ z \in L^n_\delta)$$

holds. On the other hand (again boundary values $-\infty$!)

$$\{\log v_\delta + \log \delta - \log u\}_{z \in L^n_1} \quad (0 < \delta < 1)$$

has a maximum in L^n_1 and hence by means of theorem 2

$$\delta v_\delta \leq u \quad (z \in L^n_1,\ 0 < \delta < 1).$$

Consequently ($\delta \to 1$!) $u = V$.

6. **Remarks**. The developments of sections 4 and 5 can be repeated for the simpler case of the unit sphere C^n_1 of $\mathbf{C}^n$ and lead to the partial differential equation

$$\det[(\log u)_{z\bar{z}}]_n = u^{n+1}$$

which has the strong plurisubharmonic solution $u = (1 - \| z \|^2)^{-1}$.

The proof is based on the identity [7,8].

$$\det[\delta_{ik} + \lambda \alpha_i \beta_k]_n = 1 - \lambda \det \begin{vmatrix} 0 & \beta_1 \cdots \beta_n \\ \alpha_1 & \\ \alpha_n & [\delta_{ik}]_n \end{vmatrix} .$$

The extension of the method to the general case $\mathfrak{R}_{\mathrm{I}}$ and to the cases $\mathfrak{R}_{\mathrm{II}}$ and $\mathfrak{R}_{\mathrm{III}}$ requires more technique and will be discussed on another occasion.

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REFERENCES

1. Ahlfors, L. V. : 1938 — *An extension of Schwarz's lemma*, 1938. Trans. Amer. Soc. 43. 359—364.
2. Bergman, S. : 1947 — *Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques*. Mem. Sci. Mathem. 106. Gauthier-Villars, Paris.
3. Bergman, S. : 1953 — *Kernel Functions and extended classes in the theory of functions of complex variables*. Colloque sur les fonctions de plusieurs variables (Bruxelles 1953) pp. 135—157, Masson, Paris.
4. Bochner, S. and Martin, W. T. : 1948 — *Several Complex Variables*. Princeton, Princeton Univ. Press.
5. Cartan, H. : 1931 — *Les fonctions de deux variables complexes et le problème de la représentation analytique*. Journ. d. Mathem. Pures et Appl. Ser. 9, 1—114.
6. Chern, S. S. : 1966 — *On Holomorphic Mapping of Hermitian Manifolds of the Same Dimension*. Proc. of Symposia in Pure Mathem. Vol. XI. Amer. Math. Soc. Providence, Rhode Island. 157—170.

7. D i n g h a s, A. : 1965 — *Ein n-dimensionales Analogon des Schwarz-Pickschen Satzes für holomorphe Abbildungen der komplexen Einheitskugel in eine Kähler-Mannigfaltigkeit.* Festschrift zur Gedächtnisfeier für Karl Weierstrass, 1815—1965. Westdeutscher Verlag, Cologne. 494—499.
8. D i n g h a s, A. : 1967 — *Über das Schwarzsche Lemma und verwandte Sätze.* Israel J. Math. 5. 157—169.
9. D i n g h a s, A. : 1968 — *Verzerrungssätze bei holomorphen Abbildungen von Hauptbereichen automorpher Gruppen mehrerer komplexer Veränderlicher in eine Kähler-Mannigfaltigkeit.* Sitzungsber der Heidelberg. Akad. der Wiss. Math.-naturw. Klasse. Springer-Verlag, Heidelberg.
10. H u a, L. K. : 1963 — *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains.* Transl. of Math. Monographs. Amer. Math. Soc. Providence, Rhode Island (p. 5 and 48).
11. K o b a y a s h i, S. : 1966 — *Volume Elements, Holomorphic Mappings and Schwarz's Lemma.* Proc. of Symposia in pure Mathem. Vol. XI. Amer. Math. Soc. Providence, Rhode Island. 253—260.
12. M i t c h e l l, J. and K. H a h n : 1967 — *Generalization of Schwarz-Pick lemma to invariant volume in a Kähler-mannifold.* Bull. Amer. Math. Soc. 73. 668—670.
13. M i t c h e l l, J. and K. H a h n : 1967 — *Generalization of Schwarz-Pick lemma to invariant volume in a Kähler-mannifold.* Trans. Amer. Math. Soc. 128. 221—231.

ASUPRA UNOR TEOREME DE DISTORSIUNE ÎN DOMENII CLASICE DE MAI MULTE VARIABILE COMPLEXE

Rezumat

Extensiunea demonstraţiei lui Ahlfors pentru teorema clasică de distorsiune a lui Schwarz-Pick pentru discul unitate la sfera unitate n-dimensională, necesită existenţa unei soluţii reale a ecuaţiei (1.3), cu anumite condiţii la frontieră. În Nota prezentă problema este atacată cel puţin pentru anumite domenii, prin metode ale teoriei ecuaţiilor cu derivate parţiale care conduc la metrici cu curburi Ricci constante. Unica soluţie a acestei ecuaţii cu valori la frontieră $+\infty$ şi $[u_{i\bar{j}}] > 0$ este strîns legată de nucleul lui Bergman. Domeniul ales este sfera Lie dată de (2.4). Tehnica este bazată pe o identitate generală între determinanţi, demonstrată în § 2.

ON HUA'S ALGEBRAIC IDENTITIES *)

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1. Introduction. The identity [1]

$$\det[\gamma_{ik} - \textstyle\sum_{\mu=1}^{p} \lambda_\mu \alpha_{\mu i} \beta_{\mu k}]_q = \det \begin{vmatrix} E_p & \Lambda_p B_{pq} \\ A'_{pq} & C_q \end{vmatrix} \tag{1.1}$$

for the real or complex matrices

$$A_{pq} = [\alpha_{ik}]_{pq},\ B_{pq} = [\beta_{ik}]_{pq},\ C_{qq} = C_q = [\gamma_{ik}]_q,$$

$E_p = [\delta_{ik}]_p$ (δ_{ik} = Kroneckers symbols)
and

$$\Lambda_p = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_p \end{bmatrix}$$

leads to an easy proof of two classical identities of L. K. H u a.

Lemma 1. *We set for a fixed μ, $1 \leq \mu \leq p$ and for $\sigma_1, \ldots, \sigma_\mu \in \{1, \ldots, p\}$*

$$X^{\sigma_1 \cdots \sigma_\mu}[\gamma_{ik}]_q = \begin{vmatrix} O_\mu & [\beta_{\sigma_j k}]_{\mu q} \\ [\alpha_{\sigma_j k}]'_{\mu q} & [\gamma_{ik}]_q \end{vmatrix}. \tag{1.2}$$

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**) Senior Foreign Scientist Fellow of the National Science Foundation.