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ASUPRA UNOR TEOREME DE DISTORSIUNE ÎN DOMENII CLASICE DE MAI MULTE VARIABILE COMPLEXE

Rezumat

Extensiunea demonstrației lui Ahlfors pentru teorema clasică de distorsiune a lui Schwarz-Pick pentru discul unitate la sfera unitate n-dimensională, necesită existența unei soluții reale a ecuației (1.3), cu anumite condiții la frontieră. În Nota prezentă problema este atacată cel puțin pentru anumite domenii, prin metode ale teoriei ecuațiilor cu derivate parțiale care conduc la metrici cu curburi Ricci constante. Unica soluție a acestei ecuații cu valori la frontieră $+\infty$ și $[u_{z\bar z}] > 0$ este strîns legată de nucleul lui Bergmann. Domeniul ales este sfera Lie dată de (2.4). Tehnica este bazată pe o identitate generală între determinanți, demonstrată în § 2.

ON HUA'S ALGEBRAIC IDENTITIES *)

BY

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1. Introduction. The identity [1]

$$\det[\gamma_{ik} - \sum_{\mu=1}^{p} \lambda_\mu \alpha_{\mu i} \beta_{\mu k}]_q = \det \begin{vmatrix} E_p & \Lambda_p B_{pq} \\ A'_{pq} & C_q \end{vmatrix} \tag{1.1}$$

for the real or complex matrices

$$A_{pq} = [\alpha_{ik}]_{pq}, \ B_{pq} = [\beta_{ik}]_{pq}, \ C_{qq} = C_q = [\gamma_{ik}]_q,$$

$E_p = [\delta_{ik}]_p$ (δ_{ik} = Kroneckers symbols)
and

$$\Lambda_p = \begin{vmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_p \end{vmatrix}$$

leads to an easy proof of two classical identities of L. K. Hua.

Lemma 1. *We set for a fixed* μ, $1 \leq \mu \leq p$ *and for* $\sigma_1, \dots, \sigma_\mu \in \{1, \dots, p\}$

$$X^{\sigma_1 \cdots \sigma_\mu}[\gamma_{ik}]_q = \begin{vmatrix} O_\mu & [\beta_{\sigma_i k}]_{\mu q} \\ [\alpha_{\sigma_i k}]'_{\mu q} & [\gamma_{ik}]_q \end{vmatrix}. \tag{1.2}$$

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Then

$$\det[\gamma_{ik} - \sum_{\mu=1}^{p} \lambda_\mu \alpha_{\mu i} \beta_{\mu k}]_q =$$

(1.3) $$\det[\gamma_{ik}]_q + \sum_{\mu=1}^{p} \{\sum_{1 \leq \sigma_1 < \dots < \sigma_\mu} \lambda_{\sigma_1} \cdots \lambda_{\sigma_\mu} \det X^{\sigma_1 \cdots \sigma_\mu}[\gamma_{ik}]\}.$$

Proof. Denote by $\Phi(\lambda)$ the right hand side of (1.2) and use the Taylor formula

$$\Phi(\lambda) = \Phi(0) + \sum_{\mu=1}^{q} \frac{1}{\mu!}\left(\lambda_1 \frac{\partial}{\partial h_1} + \dots + \lambda_p \frac{\partial}{\partial h_p}\right)^\mu \Phi(h) \Big|_{h=0}.$$

Now

$$\Lambda_\mu \beta_{\mu\nu} = \begin{vmatrix} \lambda_1 \beta_{11} \cdots \lambda_1 \beta_{1q} \\ \cdot \quad \cdots \quad \cdot \\ \lambda_p \beta_{p1} \cdots \lambda_p \beta_{pq} \end{vmatrix}$$

and therefore

$$\frac{\partial^2}{\partial h_\mu^2} \Phi(h) = 0 \quad (\mu = 1, \dots, p).$$

This leads to the expansion

$$\Phi(\lambda) = \Phi(0) + \sum_{\mu=1}^{p} \left(\sum_{1 \leq \sigma_1 < \dots < \sigma_\mu} \lambda_{\sigma_1} \cdots \lambda_{\sigma_\mu} \frac{\partial^\mu}{\partial h_{\sigma_1} \cdots \partial h_{\sigma_\mu}}\right) \Phi(h) \Big|_{h=0}$$

and, remarking that

$$\frac{\partial^\mu}{\partial h_{\sigma_1} \cdots \partial h_{\sigma_\mu}} \Phi(h) \Big|_{h=0}$$

is obtained from the determinant $\det X^{\sigma_1 \cdots \sigma_\mu}[\gamma_{ik}]$ by substituting the k-th row ($k = 1, \dots$) of $\Phi(1)$ by

$$0 \cdots 0 \quad \beta_{k1} \cdots \beta_{kq} \quad (k \in \{\sigma_1, \cdots, \sigma_\mu\})$$

respectively

$$\delta_{k1} \dots \delta_{kp} \; 0 \dots 0 \quad (k \notin \{\sigma_1, \dots, \sigma_\mu\}),$$

to (1.3).

2. Two theorems of Hua. The identity (1.3) can be used for proving the following two theorems of L. K. Hua [2] (algebraic machinery).

Theorem 1. (*L. K. Hua*) *Let the power series*

(2.1) $$f(z) = a_0 + a_1 z + \cdots$$

converge for $|z| < \rho$. *Then for* $|x_i y_k| < \rho$, $i, k = 1, \dots, q$, *we have*

(2.2) $$\det[f(x_i y_k)]_q = \sum_{0 \leq \alpha_1 < \dots < \alpha_q} a_{\alpha_1} \dots a_{\alpha_q} \det[x_i^{\alpha_k}]_q \det[y_i^{\alpha_k}]_q .$$

Proof. Suppose $p > q$ and set

$$f_p(x_i y_k) = a_0 + a_1(x_i y_k) + \dots + a_p(x_i y_k)^p.$$

Then choosing in (1.3)

$$\gamma_{ik} = a_0 (i, k = 1, \dots, q), \quad \lambda_i = -a_i \quad (i = 1, \dots, p)$$

and

$$\alpha_{\mu i} = x_i^\mu, \quad \beta_{\mu k} = y_k^\mu \quad (\mu = 1, \dots, p;\; i, k = 1, \dots, q).$$

we get

$$\det[f_p(x_i y_k)]_q =$$

(2.3) $$= \det[a_0]_q + \sum_{\mu=1}^{q} (-1)^\mu \{\sum_{1 \leq \sigma_1 < \dots < \sigma_\mu} a_{\sigma_1} \dots a_{\sigma_\mu} \det X^{\sigma_1 \cdots \sigma_\mu}[a_0]_q\}$$

where the $X^{\sigma_1 \cdots \sigma_\mu}[a_0]_q$ are the corresponding matrices to (1.2). Now it can be confirmed without difficulty that all determinants $X^{\sigma_1 \cdots \sigma_\mu}[a_0]_q$ with a $\mu \neq q-1, q$ (use Laplacian expansion rule!) identically vanish. That gives the identity

$$(-1)^{q-1} \det[f_p(x_i y_k)]_q =$$

$$= \sum_{1 \leq \sigma_1 < \dots < \sigma_{q-1}} a_{\sigma_1} \dots a_{\sigma_{q-1}} \det X^{\sigma_1 \cdots \sigma_{q-1}}[a_0]_q - \sum_{1 \leq \sigma_1 < \dots < \sigma_q} a_{\sigma_1} \dots a_{\sigma_q} \det X^{\sigma_1 \cdots \sigma_q}[a_0]_q .$$

Now (Laplacian expansion!)

$$(-1)^q \det X^{\sigma_1 \cdots \sigma_q}[a_0]_q = \det[x_i^{\sigma_k}]_q \det[y_i^{\sigma_k}]_q =$$

$$= \det \begin{vmatrix} x_1^{\sigma_1} \dots x_q^{\sigma_1} \\ \cdot \quad \dots \quad \cdot \\ x_1^{\sigma_q} \dots x_q^{\sigma_q} \end{vmatrix} \det \begin{vmatrix} y_1^{\sigma_1} \dots y_q^{\sigma_1} \\ \cdot \quad \dots \quad \cdot \\ y_1^{\sigma_q} \dots y_q^{\sigma_q} \end{vmatrix}$$

and (similarly Laplacian expansion)

$$\det X^{\sigma_1 \cdots \sigma_{q-1}}[a_0]_q =$$

$$= (-1)^{q-1} a_0 \det \begin{vmatrix} 1 & \dots & 1 \\ x_1^{\sigma_1} & \dots & x_q^{\sigma_1} \\ \cdot & \dots & \cdot \\ x_1^{\sigma_{q-1}} & \dots & x_q^{\sigma_{q-1}} \end{vmatrix} \det \begin{vmatrix} 1 & \dots & 1 \\ y_1^{\sigma_1} & \dots & y_q^{\sigma_1} \\ \cdot & \dots & \cdot \\ y_1^{\sigma_{q-1}} & \dots & y_q^{\sigma_{q-1}} \end{vmatrix} .$$

This proves (2.3) and $p \to +\infty$ leads to (2.2).

More generally Hua proved:

Theorem 2. *Let the power series*

$$f_i(z) = a_0^{(i)} + a_1^{(i)} z + \cdots \quad (i = 1,\dots, q)$$

converge for $|z| < \rho$. *Then for* $|z_1|, \dots, |z_q| < \rho$ *the following identity holds:*

$$\det [f_i(z_k)]_q = \sum_{0 \leqslant \alpha_1 < \dots < \alpha_q} \det [a_h^{(i)}]_q \det [z_i^{\alpha_k}]_q. \tag{2.4}$$

Proof. We consider again the segments

$$f_p^{(i)}(z) = \sum_{\mu=0}^{p} a_\mu^{(i)} z^\mu \quad (p \geqslant q)$$

and chose

$$\alpha_{\mu i} = a_\mu^{(i)}, \quad \beta_{\mu k} = z_k^\mu \quad (i, k = 1, \dots, q)$$

$$\gamma_{ik} = a_0^{(i)} \quad (i, k = 1, \dots, q) \quad \lambda_1 = \cdots = \lambda_p = -1.$$

Then we get for $q \geqslant 1$

$$\det [f_p^{(i)}(z_k)]_q =$$
$$= \sum_{\mu=1}^{p} (-1)^\mu \{ \sum_{1 \leqslant \sigma_1 < \dots < \sigma_\mu} \det X^{\sigma_1 \cdots \sigma_\mu} [a_0^{(i)}]_q \}.$$

It can be proved again that all the determinants $X^{\sigma_1 \cdots \sigma_\mu} [a_0^{(i)}]_q$ with $\mu \neq q - 1, q$ must vanish identically (they are linear combinations of determinants with at least two equal columns).

Therefore

$$(-1)^{q-1} \det [f_i^p(z_k)]_q =$$
$$= \sum_{1 \leqslant \sigma_1 < \dots < \sigma_{q-1}} \det X^{\sigma_1 \cdots \sigma_{q-1}} [a_0^{(i)}]_q - \sum_{1 \leqslant \sigma_1 < \dots < \sigma_q} \det X^{\sigma_1 \cdots \sigma_q} [a_0^{(i)}]_q =$$
$$= (-1)^{q-1} \sum_{0 \leqslant \sigma_1 < \dots < \sigma_q} \det [a_{\sigma_i}^{(k)}]_q \det [z_k^{\sigma_i}]_q.$$

Without entering in details it should be remarked that the identity (1.1) can also be used to prove by a more general way a classical identity given by Wirtinger in 1936 [3].

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ASUPRA IDENTITĂŢILOR ALGEBRICE ALE LUI HUA

Rezumat

Folosind identitatea (1.1) din 1, se demonstrează două teoreme ale lui Hua: 1) Dacă seriile (2.1) converg pentru $|z| < \rho$, atunci pentru $|x_i, y_k| < \rho$ $(i, k = 1, 2, \dots, q)$ este valabilă identitatea (2.2) şi 2). În aceeaşi ipoteză este valabilă identitatea (2.4).