invariante à gauche sur $GL(\mathbf{E})$ définit un sous-groupe de Lie $H \subset GL(\mathbf{E})$ ayant θ comme forme banachique invariante à gauche, si et seulement si. l'équation (9) est complètement intégrable.

La démonstration résulte du théorème 6 et du corollaire 1.

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ASUPRA GRUPURILOR LIE-BANACH

Rezumat

Prin metoda formelor diferenţiale banachice, dezvoltată în [1], se obţin primele două teoreme ale lui Lie pentru grupuri Lie-Banach. Cu ajutorul lor se obţine teorema fundamentală relativ la subgrupurile unui grup Lie-Banach.

ON SOME HOMOTOPY LIFTING AND EXTENSION PROPERTIES FOR SEQUENCES OF CONTINUOUS MAPS

BY

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1. The main aim of this paper is to define the concept of fH-sequence by analogy with that one of f-sequence [1] and to give necessary and sufficient conditions for a sequence $X_1 \xrightarrow{p_1} X_2 \xrightarrow{p_2} X_3$ to be a fH-sequence.

The covering sequences are introduced and as application we prove that the connected locally path connected spaces are the objects of a category with covering projections as morphisms.

The dual notion, is also considered, namely the f'-sequence, which extends the notion of cofibration. A necessary and sufficient condition that a sequence be f'-sequence is given and a theorem of duality is proved.

2. A sequence of topological spaces and continuous maps

$$X_1 \xrightarrow{p_1} X_2 \xrightarrow{p_2} X_3 \tag{1}$$

is called to be a fH-sequence, if for every topological space Y, every continuous map $f : Y \to X_1$ and every homotopy, $F : Y \times I \to X_2$ for $p_1 \circ f$ there exists a homotopy $G : Y \times I \to X_1$ for f, such that $p_2 p_1 G = p_2 F$.

Remarks 1. The map $p : X \to Y$ is a Hurewicz fiber space if and only if the sequence $X \xrightarrow{p} Y \xrightarrow{1_Y} Y$ and $X \xrightarrow{1_X} X \xrightarrow{p} Y$ are fH-sequences.

2. If p_1 or $p_2 \circ p_1$ in (1) is a Hurewicz fiber space then (1) is a fH-sequence.

A sequence

$$X_1 \xrightarrow{p_1} X_2 \xrightarrow{p_2} X_3 \to \ldots\ldots X_i \xrightarrow{p_i} X_{i+1} \xrightarrow{p_{i+1}} X_{i+2} \to \ldots\ldots \tag{2}$$

is called a fH-sequence if $X_i \xrightarrow{p_i} X_{i+1} \xrightarrow{p_{i+1}} X_{i+2}$ is fH-sequence for $i \geqslant 1$.

The following proposition can be proved in an obvious way:

Proposition 1. *If (2) is a fH-sequence, Y a topological space, $f : Y \to X_j$ a continuous map and $F : Y \times I \to X_i$, $(i > j)$, a homotopy for $p_{i-1} \cdots p_j f$ then there exists $G : Y \times I \to X_j$ a homotopy for f such that $p_{i+1} \cdots p_j G = p_{i+1} F$.*

Let (1) be a sequence of continuous maps. We consider th e space $\bar{X}_2 = \{ (x_1, \omega) \in X_1 \times X_2^I / \omega(0) = p_1(x_1)\}$. A continuous map $\lambda : \bar{X}_2 \to X_1^I$ is called a *lifting function* for (1) if

a) $\lambda(x_1, \omega)(0) = x_1$ and

b) $p_2 p_1 \lambda (x_1, \omega)(t) = p_2 \omega(t)$.

The proof of the following theorem is analogous as for Hurewicz fiber spaces ([2], p. 123).

Theorem 1. *The sequence (1) is a fH-sequence if and only if there exists a lifting function for (1).*

Let (1) be a sequence of continuous maps and $f : Y \to X_2$ a continuous map. A map $g : Y \to X_1$ such that $p_2 p_1 g = p_2 f$ is called a *lifting of maping f, with respect to the sequence (1).*

Generally, given the sequence (2), a continuous map $f : Y \to X_i$ and $j < i$, then a continuous map $g : Y \to X_j$ is called a *lifting at X_j of f with respect to the sequence (2).* if $p_{i+1} \cdots p_j g = p_{i+1} f$.

Lemma 1. *If (2) is a fH-sequence, then for any path $\omega \in X_i^I$ with $\omega(0) = x_i$ and $x_j \in X_j (j < i)$ such that $p_{i-1} \cdots p_j(x_j) = x_i$, there exists a lifting ω' of ω at X_j with respect to (2) such that $\omega'(0) = x_j$.* The proof is obvious in view of proposition 1.

We say that the *fH*-sequence (2) has a unique path lifting if for every two paths $\omega, \omega' \in X_i^I$ with $\omega(0) = \omega'(0)$ and $p_{i+1} \cdots p_j \omega = p_{i+1} \cdots p_j \omega'$ $(j < i)$, it follows $\omega = \omega'$ and if $p_{i+1} \cdots p_j \omega \simeq p_{i+1} \cdots p_j \omega'$ it follows $\omega \simeq \omega'$.

Remarks 1. If $p_2 \circ p_1$ is a Hurewicz fiber space which has a unique lifting path, then (1) is a *fH*-sequence which has a unique lifting path

2. If p_1 is a Hurewicz fiber space with unique path lifting we don't know if the sequence (1) is a *fH*-sequence with a unique path lifting.

The folowing lemmas are obvious:

Lemma 2. *Let (2) be a fH-sequence. $f : Y \to X_i$ a continuous map and Y a contractible space. If $f(Y) \subset p_{i-1} \cdots p_j(X_j)$ then there exists a lifting at X_j of f with respect to (2).*

Lemma 3. *Let $(X_1, x_1) \xrightarrow{p_1} (X_2, x_2) \xrightarrow{p_2} (X_3, x_3)$ be a fH-sequence with unique path lifting. If y_0 is a strong deformation retract of Y then every map $f : (Y, y_0) \to (X_2, x_2)$ has a lifting $(Y, y_0) \to (X_1, x_1)$ with respect to the fH-sequence given above.*

Lemma 4. *If (1) is a fH-sequence with unique path lifting, then a necessary and sufficient condition for $f : (Y, y_0) \to (X_2, x_2)$—where Y is connected and locally path connected — to have a lifting $(Y, y_0) \to (X_1, x_1)$ with respect to (1) is that:*

$$p_{2*} f_* \pi (Y, y_0) \simeq p_{2*} p_{1*} \pi(X_1, x_1).$$

Let now be the diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{p_1} & X_2 & \xrightarrow{p_2} & X_3 \\ \varphi_1 \downarrow & & \downarrow \varphi_2 & & \downarrow \varphi_3 \\ Y_1 & \xrightarrow{p_1'} & Y_2 & \xrightarrow{p_2'} & Y_3 \end{array}$$

with commutative squares. We remember, the first square (for instance) is a $u.i'$ square ([3], p. 33) if he is exact, that is, from $p'(y_1) = \varphi_2(x_2)$, it results that there exists $x_1 \in X_1$ such that $\varphi_1(x_1) = y_1$, $p_1(x_1) = x_2$ and, moreover, X_1 is homeomorphic with the subspace $\{ (x_2, y_1) / \varphi_2(x_2) = p_1' (y_1) \}$ and $X_1 \xrightarrow{\Delta} X_1 \times X_1 \xrightarrow{p_1 \times \varphi_1} X_2 \times Y_1$ is the inclusion. We can prove.

Proposition 2. *If in (3) the squares are u. i' and the second sequence is a fH-sequence, then the first one is a fH-sequence.*

Proof. We use the theorem 1. Let be

$$\bar{Y}_2 = \{ (y_1, \omega) \in Y_1 \times Y_2^I / \omega(0) = p_1'(y_1)\} \quad \text{and}$$
$$\bar{X}_2 = \{ (x_1, \omega) \in X_1 \times X_2^I / \omega(0) = p_1(x_1)\}.$$

By assumptions of the theorem there exists $\lambda' : \bar{Y}_2 \to Y_1^I$ which is a lifting function with respect to sequence (2). We define a continuous map $\lambda : \bar{X}_2 \to X_1^I$. Let, for this aim $(x_1, \omega) \in \bar{X}_2$. Then $(\varphi_1(x_1), \varphi_2 \omega) \in \bar{Y}_2$. We can consider $\lambda'(\varphi_1(x_1), \varphi_2\omega)(t) \in Y_1$. We have $\lambda'(\varphi_1(x_1), \varphi_2\omega)(0) = \varphi_1(x_1)$ and $p_2' p_1' \lambda' (\varphi_1(x_1), \varphi_2\omega)(t) = p_2'\varphi_2\omega (t) = \varphi_3 p_2 \omega(t)$. Let $\alpha : \{ (y_2, x_3) / p_2'(y_2) = \varphi_3(x_3)\} \to X_2$ be the homeomorphism which exists since the second square is $u.i'$. We denote $h(x_1, \omega)(t) = \alpha(p_1'\lambda'(\varphi_1(x_1), \varphi_2\omega)(t), p_2\omega(t)) \in X_2$. Then $\varphi_2(h(x_1, \omega)(t)) = p_1'\lambda'(\varphi_1(x_1), \varphi_2\omega)(t)$ and $p_2(h(x_1, \omega)(t)) = p_2\omega(t)$. We have $\varphi_2(h(x_1, \omega)(0) = p_1'\varphi_1(x_1)$. If we consider the homeomorphism $\beta : \{ (y_1, x_2) / p_1'(y_1) = \varphi_2(x_2) \} \to X_1$, let $\lambda : \bar{X}_2 \to \bar{X}_1$ be the map defined by $\lambda(x_1, \omega)(t) = \beta(h(x_1, \omega)(t), \lambda'(\varphi_1(x_1), \varphi_2\omega)(t)$. Therefore $\varphi_1\lambda (x_1, \omega)(t) = \lambda'(\varphi_1(x_1), \varphi_2\omega)(t)$ and also $p_1\lambda(x_1, \omega)(t) = h(x_1, \omega)(t)$. Hence $p_2 p_1 \lambda(x_1, \omega)(t) = p_2 h(x_1, \omega)(t) = p_2 \omega(t)$. Moreover $\varphi_2(h(x_1, \omega)(0)) = p_1'\varphi_1 (x_1) = \varphi_2 p_1(x_1)$ and $p_2 h(x_1, \omega)(0) = p_2\omega(0) = p_2 p_1(x_1)$. Whence it follows $h(x_1, \omega)(0) = p_1(x_1)$. We have $\varphi_1\lambda(x_1, \omega)(0) = \varphi_1(x_1)$, $p_1\lambda(x_1, \omega)(0) = h(x_1, \omega)(0) = p_1(x_1)$, therefore $\lambda(x_1, \omega)(0) = x_1$. Then λ is a lifting function for the first sequence and by theorem 1 it follows that it is a *fH*-sequence.

Corollary 1. *If the diagram*

$$(4) \qquad \begin{array}{ccccccccccc} X_1 & \xrightarrow{p_1} & X_2 & \xrightarrow{p_2} & X_3 \to & \cdots & X_i & \xrightarrow{p_i} & X_{i+1} & \xrightarrow{p_{i+1}} & X_{i+2} \cdots \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_i & & \downarrow \varphi_{i+1} & & \downarrow \varphi_{i+2} \\ Y_1 & \xrightarrow{p_1'} & Y_2 & \xrightarrow{p_2'} & Y_3 \to & \cdots & Y_i & \xrightarrow{p_i'} & Y_{i+1} & \xrightarrow{p_{0+1}'} & Y_{i+2} \to \cdots \end{array}$$

contains u.i'-squares and the second sequence is a fH-sequence, the first one is a fH-sequence.

Corollary 2. *If in (3) the squares are u. i', the second sequence is a fH-sequence with unique path lifting and φ_1 is a Hurewicz fiber space with unique path lifting, then the first one is a fH-sequence with unique path lifting.*

Generally, if in (4) the squares are $u.\ i'$, the second sequence is a *fH*-sequence with unique path lifting, while the maps φ_i are Hurewicz fiber spaces with unique path lifting, then the first is a *fH*-sequence with unique path lifting.

3. Now we suppose that (1) is a *fH*-sequence with unique path lif- ting and that X_1, X_2, X_3 are connected locally path connected spaces. Under these conditions we say that the sequence (1) is a *covering sequence* for X_2 if there exists an open covering $\mathfrak{U}$ of X_2 such that

$$p_{2*}\pi(\mathfrak{U}, p_1(x_1)) \subset p_{2*} p_{1*}\pi(X_1, x_1). \tag{5}$$

Remarks 1. If in (1), the spaces $X_i (i = 1, 2, 3)$ are connected locally path connected spaces, p_1 is a covering map and p_2 is a Hurewicz fiber map with unique path lifting, then (1) is a covering sequence for X_2. In fact, under conditions given above, (1) is a *fH*-sequence with unique path lif- ting and there exists, ([2], p. 108) an open covering $\mathfrak{U}$ of X_2 such that $\pi\mathfrak{U}, (p_1(x_1)) \subset p_{1*}\pi(X_1, x_1)$, whence it follows condition (5). Then if p_1 and p_2 are covering projections and $X_i (i = 1, 2, 3)$ are connected lo- cally path connected spaces, then (1) is a covering sequence for X_2.

2. If $p_2 \circ p_1$ is a covering projection and X_i are connected locally path connected spaces, then (1) is a *fH*-sequence with unique path lifting and there exists a covering $\mathfrak{V}$ of X_3 such that $\pi(\mathfrak{V}, \cdot) \subset p_{2*}p_{1*}\pi(X_1, x_1 \cdot)$. Let $\mathfrak{U} = \{ p_2^{-1}(V)/V \in \mathfrak{V}\}$: we have $p_{2*}\pi(\mathfrak{U}, p_1(x_1) \subset \pi(\mathfrak{V}, p_2 p_1(x_1))$ and therefore $p_{2*}\ \pi(\mathfrak{U}, p_1(x_1)) \subset p_{2*}p_{1*}\pi(X_1, x_1)$, hence (1) is a covering sequence.

3. If (1) is a *fH*-sequence with unique path lifting, X_1 and X_3 are connected locally path connected spaces and X_2 is a connected locally path conected and a semilocally 1-connected space, then (1) is a covering sequence for X_2.

In fact, since X_2 is semilocally 1-connected, there exists a covering U of X_2 such that $\pi(\mathfrak{U}, x_2) = 0$, [2].

Proposition 3. *If (1) is a fH-sequence with unique path lifting, then the sequence*

$$\Omega(X_1, x_1) \xrightarrow{\Omega p_1} \Omega(X_2, x_2) \xrightarrow{\Omega p_2} \Omega(X_3, x_3) \tag{6}$$

is a *fH*-sequence with unique path lifting.

The proof follows directly by definition or by theorem 1.

Corollary. If (1) is a *fH*-sequence with unique path lifting and X_i are connected locally path connected spaces, then (6) is a covering sequence for $\Omega(X_2, x_2)$.

One can use the proposition 3 and that $\pi(\Omega(X_2, x_2)) = 0$, whence $\mathfrak{U} = \{ \Omega(X_2, x_2), \varnothing\}$ satisfies (5).

Proposition 4. *If in (3) the diagrams are u. i' squares, X_i, and Y_i are connected locally path connected spaces, φ_1 is a homeomorphism, $\varphi_{3*} : \pi(X_3, \cdot) \to \pi(Y_3, \cdot)$ is a monomorphism and the second sequence is a covering sequence, then the first one is a covering sequence.*

Proof. Let $\mathfrak{U}'$ be a covering from Y_2 which satisfies (5) and let $\mathfrak{U} = \{ U = \varphi_2^{-1}(U')/U' \in \mathfrak{U}' \}$. Let $[\omega * \omega' * \omega^{-1}] \in \pi(\mathfrak{U})$. Then

$$[\varphi_2 (\omega * \omega' * \omega^{-1})] \in \pi(\mathfrak{U}', \cdot)$$

We have $p_2'\varphi_2(\omega * \omega' * \omega^{-1}) \simeq p_2'p_1'\bar{\bar{\omega}}$, where $[\bar{\bar{\omega}}] \in \pi(Y_1, y_1) = \pi(X_1, x_1)$. It follows

$$\varphi_3 p_2(\omega * \omega' * \omega^{-1}) \simeq p_2'p_1'\varphi_1(\varphi_1^{-1}\bar{\omega}) \text{ rel } \dot{I}, \text{ therefore :}$$

$$\varphi_3 p_2 (\omega * \omega' * \omega^{-1}) \simeq p_2'\varphi_2 p_1(\varphi_1^{-1}\bar{\omega}) \text{ rel } \dot{I}, \text{ hence}$$

$$\varphi_3 p_2 (\omega * \omega' * \omega^{-1}) \simeq \varphi_3 p_2 p_1(\varphi_1^{-1}\bar{\omega}) \text{ rel } \dot{I} \text{ whence}$$

$$p_2(\omega * \omega' * \omega^{-1}) \simeq p_2 p_1 (\varphi_1^{-1}\bar{\omega}) \text{ rel } \dot{I} \text{ that is}$$

$$p_{2*}[\omega * \omega' * \omega^{-1}] \in p_{2*}p_{1*}\pi(X_1, x_1) \text{ therefore}$$

$$p_{2*}\pi(\mathfrak{U}, p_1(x_1)) \subset p_{2*}p_{1*}\pi(X_1, x_1).$$

Since in view of Corollary 2 of Proposition 2 it follows that the first se- quence is with unique path lifting, it follows that this one is a covering sequence.

We give now another characterization of the open sets U of cove- ring which satisfies (5).

Let (1) be a covering sequence. Let $\mathfrak{U}$ a covering which satisfies (5) and we suppose that there exists $U \in \mathfrak{U}$ such that : U is connected $p_1(x_1) \in U$, therefore $p_1^{-1}(U) \neq \varnothing$ and $p_2(U)$ is an open set in X_3. Let be the diagram

$$\begin{array}{ccccc} (X_1, x) & \xrightarrow{p_1} & (X_2, p_1(x)) & \xrightarrow{p_2} & (X_3, (p_2 p_1(x)) \\ & & \Big\uparrow{\scriptstyle i} & & \\ & & (U, p_1(x)) & & \end{array}$$

for $x \in (p_2 \circ p_1)^{-1} (p_2p_1(x_1))$. Because $p_{2*}i_*\pi(U, p_1(x)) \subset p_{2*}\ p_{1*}\ \pi(X_1, x)$ it follows by lemma 4, ((1) being with unique path lifting, it has this property with respect to any connected path space) that there exists a unique map $\tilde{i}_x : (U, p_1(x)) \to (X_1, x)$ such that $p_2 p_1\tilde{i}_x = p_2 i$ for $x \in (p_2 \circ p_1)^{-1} (p_2p_1(x_1))$. Since $p_1^{-1}(U) \neq \varnothing$ it follows that $(p_2 \circ p_1)^{-1} (p_2(U)) \neq \varnothing$. We prove that $(p_2 \circ p_1)^{-1} (p_2(U)) = \bigcup \tilde{i}_x(U)$, for: $x \in (p_2 \circ p_1)^{-1} (p_2p_1(x_1))$. Let $\bar{x}_1 \in (p_2 \circ p_1)^{-1} (p_2(U))$, ω_2 a path in U joining $p_1(\bar{x}_1)$ and $p_1(x_1)$ and ω_1 be a lif- ting of this path with $\omega_1(0) = \bar{x}_1$ and $p_2p_1\omega_1 = p_2\omega$. Then $p_2p_1\omega_1(1) = p_2$

$\omega_2(1)=p_2p_1(x_1)$, therefore $\omega_1(1) \in (p_2 \circ p_1)^{-1}(p_2p_1(x_1))$. Since (1) is a fH-se-
quence with unique path lifting and the sets $i''_x(U)$ are path connected,
it follows that these sets are disjoint. It follows that the path ω_1 is con-
tained into $i''_{\omega_1(1)}(U)$. Therefore $x_1 \in i''_{\omega_1(1)}(U)$ and hence $(p_2 \circ p_1)^{-1}(p_2(U)) \subset$
$\subset \bigcup i''_x(U)$ for $x \in (p_2 \circ p_1)^{-1}(p_2p_1(x_1))$. The inverse is obvious from $p_2p_1i''_x =$
$= p_2 i$.

Since $(p_2 \circ p_1)^{-1}(p_2(U))$ is open in X_1 which is locally path connected
and $i''_x(U)$ are connected and disjoint it follows that $i''_x(U)$ are open sets
in X_1. Moreover, since $p_2 \circ p_1/i''_x(U)$ is continuous bijection and $i''_x(U)$ and
$p_2(U)$ are path connected it follows that $i''_x(U)$ is applied homeomorphi-
cally onto $p_2(U)$ by $p_2 \circ p_1$. Therefore we have:

Theorem 2. *If (1) is a covering sequence for X_2, $\mathfrak{U}$ fullfils (5), $U \in \mathfrak{U}$ is connected, $p_2(U)$ is open and $p_1^{-1}(U) \neq \emptyset$, then $p_2(U)$ is evenly covered by p_2p_1.*

Corollary. *The connected locally path connected spaces are the objects of a category with covering projections as morphisms.*

In fact, if $p_1 : X_1 \to X_2$ and $p_2 : X_2 \to X_3$ are covering maps, then
there exists a covering $\mathfrak{U}$ of X_3, containing connected open maps and such
that $\pi(\mathfrak{U}, p_1(x_1)) \subset p_{1*}\pi(X_1, x_1)$. The map p_2 which is a covering map,
is a locally homeomorphism, therefore an open map and hence $p_2(U)$ is an
open set for every $U \in \mathfrak{U}$. Moreover, since p_1 is surjection, $p_1^{-1}(U) \neq \emptyset$.
By theorem 2, $p_2(U)$ is evenly covered by p_2p_1. Moreover, $\{p_2(U)/U \in \mathfrak{U}\}$
is a covering of X_3 because p_2 is surjection and therefore p_2p_1 is a cove-
ring map. The other axioms in definition of a category are obvious satisfied.

4. Let

$$(1') \qquad X_1 \xleftarrow{q_1} X_2 \xleftarrow{q_2} X$$

be a sequence of topological spaces and continuous maps.

We say that the sequence (1′) is a *f'-sequence* if for any topological
space Y, any continuous map $g : X_1 \to Y$ and any homotopy $G : X_2 \times I \to$
$\to Y$ for gq_1, there exists a homotopy $F : X_1 \times I \to Y$ for g, such that
$F(q_1 \times 1)(q_2 \times 1) = G(q_2 \times 1)$.

A sequence

$$(2') \qquad X_1 \xleftarrow{q_1} X_2 \xleftarrow{q_2} X_3 \leftarrow \cdots \leftarrow X_i \xleftarrow{q_i} X_{i+1} \xleftarrow{q_{i+1}} X_{i+2} \leftarrow \cdots$$

is called to be a f'-sequence if every sequence

$$X_i \xleftarrow{q_i} X_{i+1} \xleftarrow{q_{i+1}} X_{i+2}$$

is a f'-sequence.

Remarks. 1. If in (1′) q_1 or $q_1 \circ q_2$ is cofibration then (1′) is a f'-se-
quence.

2. If (2′) is a f'-sequence then for every topological space Y, every
continuous map $g : X_i \to Y$ and every homotopy $G : X_j \times I \to Y$, $(j > i)$
for $gq_i, \ldots\ldots q_{j-1}$, there exists a homotopy $F : X_i \times I \to y$ for g, such that
$F(q_i \times 1) \ldots\ldots (q_j \times 1) = G(q_j \times 1)$.

Let $A \subset Y$ and $f : X \to A$ be a continuous map. We say that A
is a *f-retract* of Y if there exists $r : Y \to A$ such that $rif = f$. A is retract
of Y iff A is 1_A-retract of Y.

Proposition 5. *Let $X \xrightarrow{f} A \xrightarrow{i} Y$ be a f-sequence such that A is a weak retract of Y. Then A is a f-retract of Y.*

Proof. Let $r : Y \to A$ with $ri \simeq 1_A$ and let $G : A \times I \to A$ with $G(a,0)=a$.
There exists $F : Y \times I \to A$ with $F(y,0) = r(y)$ and $F(i \times 1)(f \times 1) = G(f \times$
$\times 1)$. Let $r' : Y \to A$ with $r'(y) = F(y,1)$. We have $F(i \times 1)(f \times 1)(x,1) =$
$= G(f \times 1)(x,1)$ and therefore $F(if(x),1) = f(x)$ hence $r'if(x) = f(x)$.

We give now a necessary and sufficient condition for a sequence (1′)
be a f'-sequence. Let be the sequence

$$(8) \qquad X_1 \xrightarrow{q_1} X_2 \xrightarrow{q_2} X_3.$$

We consider the space $\bar{X}_3$ obtained from the summ $(X_2 \times I) \vee (X_3 \times 0)$
by identification $(x_2, 0) \in X_2 \times I$ with $(q_2(x_2), 0) \in X_3 \times 0$. We denote the
elements of this space by $[x_2, t]$ and $[x_3, 0]$. Therefore $[x_2, 0] = [q_2(x_2), 0]$. We
define the maps:
$\bar{i} : \bar{X}_3 \to X_3 \times I$ and $q'_1 : X_1 \times I \to \bar{X}_3$ by $\bar{i}[x_2, t] = (q_2(x_2), t)$, $\bar{i}[x_3, 0] = (x_3, 0)$
and $q'_1(x_1, t) = [q_1(x_1), t]$ respectivelly. We say that a continuous map
$\rho : X_3 \times I \to \bar{X}_3$ is a *retracting function for the sequence (8) if*

$$(9) \qquad \rho i q'_1 = q'_1.$$

Theorem 3. *A necessary and sufficient condition for the sequence (8) be a f'-sequence is that there exists for it a retracting function.*

Proof. We suppose that (8) is a f'-sequence. Let be the diagram

$$\begin{tikzcd}
X_1 \arrow[rr, "q_1"] & & X_2 \arrow[rr, "q_2"] \arrow[dd] & & X_3 \arrow[dl, "g"'] \arrow[dd, dashed] \\
& & & \bar{X}_3 & \\
X_1 \times I \arrow[rr, "q_1 \times 1"] & & X_2 \times I \arrow[rr, "q_2 \times 1"] \arrow[ur, "G"] & & X_3 \times I
\end{tikzcd}$$

where $g(x_3) = [x_3, 0]$ and $G(x_2, t) = [x_2, t]$. Then we have $G(x_2, 0) =$
$= [x_2, 0] = [q_2(x_2), 0] = gq_2(x_2)$ and therefore there exists $\rho : X_3 \times I \to$
$\to \bar{X}_3$ with $\rho(x_3, 0) = g(x_3) = [x_3, 0]$ and $\rho(q_2 \times 1)(q_1 \times 1) = G(q_1 \times 1)$ therefore
$\rho i q'_1 = q'_1$. Conversely, let be the diagram

$$\begin{tikzcd} X_1 \arrow[r, "q_1"] & X_2 \arrow[r, "q_2"] \arrow[dd] & X_3 \arrow[dd] \arrow[dl, "g"] \\ & Y & \\ X_1\times I \arrow[r, "q_1\times 1"] & X_2\times I \arrow[r, "q_2\times 1"] \arrow[u, "G"] & X_3\times I \end{tikzcd}$$

with $G(x_2, 0) = gq_2(x_2)$. Let $\overline{G} : \overline{X}_3 \to Y$ with $\overline{G}\,[x_2, t] = G\ (x_2, t)$ and $\overline{G}[x_3, t] = g(x_3)$. We define $F : X_3 \times I \to Y$ with $F = \overline{G} \circ \rho$. We have $F(x_3, 0) = \overline{G}, \rho(x_3, 0) = \overline{G}[x_3, 0] = g(x_3)$ and $F(q_2 \times 1)(q_1 \times 1)\ (x, t) = F(q_2, q_1(x_1), t) = G\rho(q_2\, q_1(x_1), t) = G(q_1(x_1), t) = G(q_1 \times 1)(x_1, t)$ therefore (8) is a f'-sequence. An analogous proof as for the theorem 2 may be used for :

Proposition 6. *If in the diagram*

$$(9)\qquad \begin{tikzcd} X_1 & X_2 \arrow[l, "q_1"'] & X_3 \arrow[l, "q_2"'] & \cdots \arrow[l] & X_i \arrow[l] & X_{i+1} \arrow[l, "q_i"'] & X_{i+2} \arrow[l, "q_{i+1}"'] & \cdots \arrow[l] \\ Y_1 \arrow[u, "\psi_1"] & Y_2 \arrow[l, "q'_1"] \arrow[u, "\psi_2"] & Y_3 \arrow[l, "q'_2"] \arrow[u, "\psi_3"] & \cdots \arrow[l] & Y_i \arrow[l] \arrow[u, "\psi_i"] & Y_{i+1} \arrow[l, "q'_i"] \arrow[u, "\psi_{i+1}"] & Y_{i+1} \arrow[l, "q'_{i+1}"] \arrow[u, "\psi_{i+2}"] & \cdots \arrow[l] \end{tikzcd}$$

the squares are u. i ([3], p. 33) *and the second sequence is a f'-sequence then the first one is a f'-sequence.*

By theorems 1 and 3 we can prove :

Theorem 4. *Let $X_1 \xrightarrow{q_1} X_2 \xrightarrow{q_2} X_3$ be a f'-sequence with $X_i (i = 1, 2, 3)$. Hausdorff locally compact spaces and Y any topological space. Then the sequence.*

$$(10)\qquad Y^{X_3} \xrightarrow{p_1} Y^{X_2} \xrightarrow{p_2} Y^{X_1} \quad \text{with } p_i(g) = g \circ q_i$$

is a fH-sequence.

Proof. Let $\rho : X_3 \times I \to \overline{X}_3$ be retracting function for the sequence given above (theorem 3). We define the map

$$\rho' : Y^{\overline{X}_3} \to Y^{X_3 \times I} \text{ with } \rho'(g) = g \circ \rho.$$

Since X_2 and X_3 are locally compact spaces $\overline{X}_3$ is also a locally compact space. We have $Y^{X_3 \times I} \cong (Y^{X_3})^{I}$ with $\alpha(\varphi)(t)(x_3) = \varphi(x_3, t)$. Let be the space

$$Y^{\overline{X}_2} = \{ (g, G) \in Y^{X_3} \times (Y^{X_2})^I / G(0) = g \circ q_2 \}.$$

We can define a map

$$\theta : Y^{\overline{X}_2} \to Y^{\overline{X}_3} \text{ by}$$

$\theta(g, G)\,[x_2, t] = G(t)\,(x_2)$ and $\theta(g, G)[x_3, 0] = g(x_3)$.

If $[x_2, 0] = [q_2(x_2), 0]$ then $\theta(g, G)\,[x_2, 0] = G(0)(x_2) = g(q_2(x_2)) = \theta(g, G)\,[q_2(x_2), 0]$ and therefore θ is well defined. Now we define the map

$$\lambda : \overline{Y^{X_2}} \to (Y^{X_3})^I$$

by the composition

$$\overline{Y^{X_2}} \xrightarrow{\theta} Y^{\overline{X}_3} \xrightarrow{\rho'} Y^{X_3 \times I} \xrightarrow{\alpha} (Y^{X_3})^I$$

therefore $\lambda(g, G) = \alpha\rho'\theta\,(g, G)$. Then we have

$\lambda(g, G)(0)(x_3) = (\alpha\rho'\theta(g, G)\,(0)(x_3) = \alpha\,(\theta(g, G) \circ \rho)\ (0)(x_3) = (\theta(g, G) \circ \rho)\ (x_3, 0) = \theta(g, G)\,[x_3, 0] = g(x_3)$ therefore $\lambda(g, G)(0) = g$. Moreover $p_2 p_1\ \lambda(g, G)(t)(x_1) = (\lambda(g, G)(t) \circ q_2 \circ q_1)(x_1) = \lambda(g, G)\ (t)\ (q_2 q_1\,(x_1)) = (\theta(g, G) \circ \rho)\ (q_2 q_1(x_1), t) = \theta(g, G)(\rho(q_2 q_1(x_1), t) = \theta(g, G)\,[q_1(x_1), t] = G(t)(q_1(x_1))) = (G(t) \circ q_1)\ (x_1) = p_2(G(t)\,(x_1))$ therefore $p_2 p_1\ \lambda\,(g, G)(t) = p_2 G(t)$ and hence λ is a lifting function for (10) and by theorem 1 this is a fH-sequence.

Corollary. *If the sequence*

$$X_1 \xleftarrow{q_1} X_2 \xleftarrow{q_2} X_3 \leftarrow \ldots\ldots \leftarrow X_i \xleftarrow{q_i} X_{i+1} \xleftarrow{q_{i+1}} X_{i+2} \leftarrow \ldots\ldots$$

is a f'-sequence, X_i are locally compact Hausdorff spaces and Y any topological space, then the sequence

$$Y^{X_1} \xrightarrow{p_1} Y^{X_2} \xrightarrow{p_2} Y^{X_3} \to \ldots\ldots Y^{X_i} \xrightarrow{p_i} Y^{X_{i+1}} \xrightarrow{p_{i+1}} Y^{X_{i+2}} \xrightarrow{p_{i+2}} \ldots\ldots$$

with $p_i(g) = g \circ q_i$, is a fH-sequence.

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ASUPRA UNOR PROPRIETĂŢI DE RIDICARE ŞI EXTENSIE A HOMOTOPIEI ALE ŞIRURILOR DE APLICAŢII CONTINUE

Rezumat

În această Notă se definesc fH-şirurile în analogie cu f-şirurile [1], se dau condiţii necesare şi suficiente pentru ca un şir $X_1 \xrightarrow{p_1} X_2 \xrightarrow{p_2} X_3$ să fie fH-şir. Se dovedesc unele proprietăţi ale fH-şirurilor.

Se definesc șirurile de acoperire și ca o aplicație se arată că spațiile conexe și local liniar conexe formează o categorie ale cărei morfisme sînt proiecții de acoperire.

Se consideră noțiunea duală aceleia de fH-șir și anume f'-șir, care generalizează noțiunea de cofibrare. Se dau condiții necesare și suficiente ca un șir să fie f'-șir și se demonstrează o teoremă de dualitate.

SUR LA THÉORIE GLOBALE DES CONNEXIONS PROJECTIVES. LE GROUPE D'HOLONOMIE

PAR

GH. MURĂRESCU

1. Introduction. Soit $P(M,G)$ un espace fibré principal qui a M comme base et G comme groupe structural et $E(M, G, P(M, G))$ l'espace fibré associé, soudé à la base M[2], [4]. Soit $P(M, H)$, $H \subset G$, l'espace fibré principal réduit, dont l'existence est assurée par l'existence de la soudure. Sur $P(M, H)$ on définit une connexion Cartan par une 1-forme ω donnée sur $P(M, H)$, avec des valeurs dans g — l'algèbre de Lie de G — et satisfaisant aux conditions :

a) $\omega(A^*) = A$ pour chaque $A \in h$ — l'algèbre de Lie de $H \subset G$:

b) $R_a^* \, \omega = \mathrm{ad}\,(a^{-1})\omega$ pour chaque $a \in H$;

c) $\omega(X) = 0$ implique $x = 0$ sur $P(M, H)$.

2. Les connexions projectives. Soit $G = PL(n, R)$ le groupe projectif qui actionne sur P^n (l'espace projectif) par $y^i = (a^i + a^i_j x^j)/(1 + a_j x^j)$ et H le sous-groupe (d'isotropie pour $O(0, \ldots, 0) \in P^n$) défini localement par $a^i = 0$ [1]. Nous avons défini [7] une connexion projective sur $P = P(M, H)$ comme une connexion Cartan. Cette connexion est donnée par la 1-forme $\omega = (\omega^i, \omega^i_j, \omega_j)$ ayant des valeurs dans $g = pl(n, R) = R^n + gl(n, R) + R^{*n}$.

Les équations de structure pour la connexion projective sont [4], [7] :

$$
\begin{aligned}
& d\omega^i + \omega^i_k \wedge \omega^k = \Omega^i, \\
(1) \qquad & d\omega^i_j + \omega^i_k \wedge \omega^k_j + \omega^i \wedge \omega_j + \delta^i_j \, \omega^k \wedge \omega_k = \Omega^i_j, \\
& d\omega_j + \omega_k \wedge \omega^k_j = \Omega_j,
\end{aligned}
$$

$\Omega' = (\Omega^i)$ étant la forme de torsion et $\Omega'' = (\Omega^i_j, \Omega_j)$ la forme de courbure.

[1]) Pour la théorie locale des espaces à connexion projective, voir [1].