Comme dans le cas des connexions usuelles [5], on définit le *lift horizontal* d'un champ vectoriel (et d'une courbe sur M) et *le transport parallèle* comme l'isomorphisme $\tau : H_{x_0} \to H_{x_1}$ (le long de la courbe $\tau = x(t) \in M$). Ces isomorphismes forment un groupe lorsque τ est fermée.

D'après le théorème 1 on constate que pour chaque $a \in H$, nous avons : $\tau \circ R_a = R_a \circ \tau$.

Définition 3. *Le groupe des isomorphismes de la fibre locale :* $\pi^{-1}(x) \to \pi^{-1}(x)$ *définis par le transport parallèle le long d'une courbe fermée en* $x \in M$ *s'appelle le groupe d'holonomie de la connexion projective* ω *sur* P *en* x.

Dans le cas spécial de la courbe fermée homotope à zéro, on définit *le groupe réduit d'holonomie*.

Parce que l'isomorphisme $\tau : \pi^{-1}(x) \to \pi^{-1}(x)$ se réalise à l'aide du lift horizontal $\tau^* = u(t) = (x(t), a_j^i(t), a_j(t))$, $t \in [0, 1]$, nous avons le

Théorème 3. *Le groupe d'holonomie de la connexion projective dans chaque point de la variété* M *peut être réalisé comme un sous-groupe du groupe structural* H.

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ASUPRA TEORIEI GLOBALE A CONEXIUNILOR PROIECTIVE. GRUP DE OLONOMIE

Rezumat

Se definește [7] o conexiune proiectivă pe fibratul principal $P(M, H)$, unde H este subgrupul proiectiv de izotropie al punctului $O(0, \ldots, 0)$ din spațiul proiectiv P^n, cu ajutorul unei conexiuni Cartan [2], [4].

Se dau, din [7], expresiile formei de conexiune $\omega = (\omega^i, \omega^i_j, \omega_j)$, (2), (2'), (2''), ecuațiile de structură (1) și expresiile (4) pentru ad $(a^{-1})\omega$.

Se definesc cîmpuri și forme verticale și orizontale pe spațiul P în legătură cu care se demonstrează teoremele 1 și 2.

Ca și în cazul conexiunilor uzuale se definește transportul paralel în lungul unei curbe de pe M cu ajutorul căruia se definește apoi grupul de olonomie pentru conexiunea proiectivă ω de pe spațiul P. Se demonstrează (teorema 3) că acest grup se poate realiza ca un subgrup al grupului structural H.

SOME RESULTS CONCERNING THE ASYMPTOTIC EQUIVALENCE OF INTEGRO-DIFFERENTIAL EQUATIONS

BY

PAVEL TALPALARU

§ 1. Introduction. In this paper, we shall give some general results on the asymptotic relationship between the solutions of a linear differential system and its perturbed integro-differential system. The relationship between the asymptotic behavior of a homogeneous differential equation and a nonhomogeneous perturbation of that differential equation has been widely investigated. The problem considered in this article is in the general spirit of the investigations at of Ja. V. Bykov ([3] quoted after [10]). Ju. A. Vedi [10], T. G. Hallam and J. W. Heidel [6] and F. Brauer and J. S. W. Wong [4]. Recently D. L. Lovelady [7], [8] studied this problem in a more general case, taking into account functional differential equations in Banach spaces.

The hypotheses which will be imposed upon our equations are extensions of those suggested by [4] and [6].

Consider the homogeneous linear system

$$\dot{x} = A(t)x, \tag{1}$$

where $A(t)$ is an $n \times n$ matrix, whose real-valued components are defined and continuous for $t \in R_{t_0} = [t_0, \infty)$, and the perturbed integro-differential system

$$\dot{y} = A(t)y + f(t, y) + \int_{t_0}^{t} k(t, s, y(s))ds, \tag{2}$$

where $f(t, y)$ is a continuous function for $t \in R_{t_0}$ and $\|y\| < \infty$, and $k(t, s, y)$ is a continuous function of t, s, y for $t_0 \leqslant s \leqslant t < \infty$ and $\|y\| < \infty$.

Everywhere we consider x, y, f, k real n-vectors. The norm of a matrix or a vector is defined as the sum of the absolute value of each of the components and will be denoted, as above, by $\|\cdot\|$.

We shall denote by $X(t)$ the fundamental matrix of (1) which satisfies the initial condition $X(t_0) = I$, where I is the $n \times n$ identity matrix.

Using some of the techniques of the above-mentioned authors, we shall extend some results concerning the asymptotic equivalence to the systems (1) and (2).

In the study of the asymptotic equivalence of two systems, many types of equivalence are known. In this paper we consider the notion of asymptotic equivalence given by

Definition 1. *Let $\Delta(t)$ be an $n \times n$ matrix, and $\alpha(t)$ a positive function; we say that the systems (1) and (2) are asymptotically equivalent if, corresponding to each solution $x = x(t)$ of system (1), there exists a solution $y = y(t)$ of (2) with the property that*

$$\| \Delta(t) [x(t) - y(t)] \| = o(\alpha(t)) \quad (t \to \infty), \tag{3}$$

and conversely, to each solution $y = y(t)$ of (2) there corresponds a solution $x = x(t)$ of (1) such that (3) holds.

Since our results are formulated in terms of an arbitrary matrix $\Delta(t)$ and an arbitrary function $\alpha(t)$, they offer greater versatility in obtaining various asymptotic properties for specific integro-differential equations.

We shall use the following lemma [3].

Lemma. *Let there exist functions $u(t)$, $v(t)$, $h(t, \tau)$ and $\omega(r)$ such that:*

a) $u(t)$, $v(t)$, $h(t, \tau)$ are non-negative and integrable functions on $t \geq \tau \geq t_0$:

b) $\omega(r)$ is positive, continuous and non decreasing function on $r > 0$;

c) for any $t \geq t_0$ we have the inequality

$$u(t) \leq c + \int_{t_0}^{t} \left| v(s)\, \omega(u(s)) + \int_{t_0}^{s} h(s, \tau)\, \omega(u(\tau))\, d\tau \right| ds. \tag{4}$$

Then, for any $t \geq t_0$ we have

$$\int_{0}^{u(t)} \frac{ds}{\omega(s)} \leq \int_{t_0}^{t} \left| v(s) + \int_{t_0}^{s} h(s, \tau)\, d\tau \right| ds. \tag{5}$$

Proof. Define

$$b(t) = \int_{t_0}^{t} \left| v(s)\, \omega(u(s)) + \int_{t_0}^{s} h(s, \tau)\, \omega(u(\tau)) d\tau \right| ds + c.$$

Then

$$b'(t) = v(t) \left\| \frac{\omega(u(t))}{\omega(b(t))} \right\| + \int_{t_0}^{t} h(t, \tau) \frac{\omega(u(\tau))}{\omega(b(t))} d\tau,$$

hence

$$\frac{b'(t)}{\omega(b(t))} = v(t) \frac{\omega(u(t))}{\omega(b(t))} + \int_{t_0}^{t} h(t, \tau) \frac{\omega(u(\tau))}{\omega(b(t))} d\tau \leq v(t) + \int_{t_0}^{t} h(t, \tau) d\tau.$$

Integrating from t_0 to t, we obtain:

$$\int_{0}^{u(t)} \frac{ds}{\omega(s)} \leq \int_{t_0}^{t} \left| v(s) + \int_{t_0}^{s} h(s, \tau)\, d\tau \right| ds,$$

which yields the desired conclusion.

If we denote

$$\Omega(u) = \int^{u} \frac{ds}{\omega(s)},$$

then the inequality (5) can be written as

$$\Omega(u(t)) \leq \Omega(c) + \int_{t_0}^{t} \left| v(s) + \int_{t_0}^{s} h(s, \tau)\, d\tau \right| ds. \tag{6}$$

Remarks. 1. When $h(t, \tau) = 0$, the inequality (5) (or (6)) reduces to B e l l m a n's inequality under the form of B i h a r i [2].

2. If $\omega(r) = r$, the inequality (5) becomes

$$u(t) \leq c \exp\left\{ \int_{t_0}^{t} \left| v(s) + \int_{t_0}^{s} h(s, \tau)\, d\tau \right| ds \right\}.$$

Main results. § 2. Here we shall show that, under some conditions for a given solution $y = y(t)$ of (2), there exists a solution $x = x(t)$ of (1) such that (3) holds and conversely (but with an additional condition).

Theorem 1. *Suppose that the following conditions are satisfied:*

a) there exists a nonsingular continuous for $t \in R_{t_0}$ matrix such that

$$\| \Delta(t) X(t) \| \leq \alpha(t), \tag{7}$$

where $\alpha(t)$ is a positive and continuous function for $t \in R_{t_0}$;

b) there exists a continuous non-negative function $a(t)$ such that

$$\| X^{-1}(t) f(t, y) \| \leq a(t) \omega\left(\frac{\| \Delta(t) y \|}{\alpha(t)} \right), \text{ for } t \in R_{t_0}, \ \| y \| < \infty, \tag{8}$$

where $\omega(r)$ is the above function (see lemma) and in addition

$$\Omega(u)=\int_{u_0}^{u}\frac{d\tau}{\omega(\tau)}\to\infty,\ \textit{for}\ \ u\to\infty\,; \tag{9}$$

c) there exists a continuous, non-negative function $h(t,\tau)$ defined for $t_0\leqslant\tau\leqslant t<\infty$ such that

$$\|X^{-1}(t)\,k(t,\tau,y)\|\leqslant h(t,\tau)\,\omega\left(\frac{\|\Delta(\tau)\,y\|}{\alpha(\tau)}\right),\ t_0\leqslant\tau\leqslant t<\infty\,\|y\|<\infty\,; \tag{10}$$

d) *the functions $a(t)$ and $k(t,\tau)$ satisfy*

$$\int_{t_0}^{\infty}\left|\ a(s)+\int_{t_0}^{s}h(s,\tau)\,d\tau\right|ds<\infty\,; \tag{11}$$

then, to each solution $y=y(t)$ of (2) there exists a solution $x=x(t)$ of (1) such that (3) holds.

Proof. To prove the existence of a solution $x=x(t)$ which satisfies (3) is the same as to prove the existence of a constant vector c such that $x(t)=X(t)c$ satisfy (3). Let $y=y(t)$ be a solution of the system (2) satisfying the condition $y(t_0)=y_0$.

Then $y=y(t)$ can be written as

$$y(t)=X(t)y_0+X(t)\int_{t_0}^{t}X^{-1}(s)\left|f(s,y(s))+\int_{t_0}^{s}k(s,\tau,y(\tau))\,d\tau\right|ds, \tag{12}$$

from which, multiplying by $\Delta(t)$ and according to (7) and (8) we have:

$$\|\Delta(t)y(t)\|\leqslant\alpha(t)\,\|y_0\|+\alpha(t)\int_{t_0}^{t}\left[a(s)\,\omega\left(\frac{\|\Delta(s)\,y(s)\|}{\alpha(s)}\right)+\right.$$

$$\left.+\int_{t_0}^{s}h(s,\tau)\,\omega\left(\frac{\Delta(\tau)y(\tau)\|}{\alpha(\tau)}\right)d\tau\right|ds,$$

or

$$\frac{\|\Delta(t)y(t)\|}{\alpha(t)}\leqslant\|y_0\|+\int_{t_0}^{t}\left|\ a(s)\,\omega\left(\frac{\|\Delta(s)y(s)\|}{\alpha(s)}\right)+\int_{t_0}^{s}h(s,\tau)\right.\cdot$$

$$\left.\omega\left(\frac{\|\Delta(\tau)y\|}{\alpha(\tau)}\right)d\tau\right|ds.$$

If we apply now Lemma we find:

$$\Omega\left(\frac{\|\Delta(t)\ y(t)\|}{\alpha(t)}\right)\leqslant\Omega(\|y_0\|)+\int_{t_0}^{t}\left[a(s)+\int_{t_0}^{s}h(s,\tau)\,d\tau\right|ds.$$

It follows from condition (11) that there exists a constant $M>0$, so that

$$\frac{\|\Delta(t)\,y(t)\|}{\alpha(t)}\leqslant M,\ \text{for}\ t\in R_{t_0}.$$

Observe that

$$\left\|\int_{t_0}^{t}X^{-1}(s)\left|f(s,y(s))+\int_{t_0}^{s}k(s,\tau,y(\tau))d\tau\right|ds\ \right\|\leqslant\int_{t_0}^{t}\Big[\,\|X^{-1}(s)\,f(s,y(s))\|+$$

$$+\int_{t_0}^{s}\|X^{-1}(s)k(s,\tau,y(\tau))\|\,d\tau\ \Big|\,ds\leqslant\int_{t_0}^{t}\left[a(s)\,\omega\left(\frac{\|\Delta(s)\,y(s)\|}{\alpha(s)}\right)+\right.$$

$$\left.+\int_{t_0}^{s}h(s,\tau)\,\omega\left(\frac{\|\Delta(\tau)\,y(\tau)\|}{\alpha(\tau)}\right)d\tau\ \right|ds\leqslant\ \omega(M)\int_{t_0}^{t}\Big|\,a(s)+$$

$$+\int_{t_0}^{s}h(s,\tau)\,d\tau\,\Big|\,ds,\ t\in R_{t_0}.$$

As a consequence of the Lebesgue dominated convergence theorem, it results that

$$c=\lim_{t\to\infty}\int_{t_0}^{t}X^{-1}(s)\left[f(s,y(s))+\int_{t_0}^{s}k(s,\tau,y(\tau))d\tau\right]ds+y_0 \tag{13}$$

exists.

Using (13), we may rewrite (12) as

$$\Delta(t)y(t)=\Delta(t)X(t)y_0+\Delta(t)X(t)\int_{t_0}^{\infty}X^{-1}(s)\Big[f(s,y(s))+\int_{t_0}^{s}k(s,\tau,y(\tau))d\tau\ \Big|\,ds-$$

$$-\,\Delta(t)X(t)\int_{t}^{\infty}X^{-1}(s)\ \Big|f(s,y(s))+\int_{t_0}^{s}k(s,\tau,y(\tau))\,d\tau\ \Big|\,ds=\Delta(t)\,X(t)c-$$

$$-\,\Delta(t)X(t)\int^{\infty}X^{-1}(s)\ \Big|f(s,y(s))+\int_{t_0}^{s}k(s,\tau,y(\tau))\,d\tau\ \Big|\,ds.$$

Thus, it follows from (7) and (11) that

$$\|\Delta(t)(y(t)-X(t)c)\| \leqslant \alpha(t)\int_t^{\infty} X^{-1}(s)\left[f(s,y(s))+\int_{t_0}^{s} k(s,\tau,y(\tau))\,d\tau\right]ds = o(\alpha(t)),$$

i. e. the relation (3).

Remarks. 3. When $k(t,\tau,y)=0$, the theorem 1 reduces to the result of F. Brauer and J. S. W. Wong [4] with $\omega(t,r)=a(t)\omega(r)$. We observe only that the condition for the scalar equation

$$\dot{r}=\omega(t,r)=a(t)\,\omega(r),$$

to have a positive bounded solution for $t\in R_{t_0}$ (used by the above mentioned authors), is a consequence of our condition (11), which reduces to $a(t)\in L^1[t_0,\infty)$.

4. When $\alpha(t)=1$, the conditions (7), (8) and (10) may be written as

$$(7')\qquad \|\Delta(t)X(t)\|\leqslant 1,\ \text{ for } t\in R_{t_0};$$

$$(8')\qquad \|X^{-1}(t)f(t,y)\|\leqslant a(t)\,\omega(\|\Delta(t)y\|),\ \text{ for } t\in R_{t_0};$$

$$(10')\qquad \|X^{-1}(t)k(t,\tau,y(\tau))\|\leqslant h(t,\tau)\,\omega(\|\Delta(\tau)y\|),\ \text{for } t_0\leqslant\tau\leqslant t<\infty.$$

5. If $\Delta(t)=I$, (I—identity matrix) then (7) holds with $\alpha(t)=\|X(t)\|$ and the conditions (8) and (10) can be written now with

$$\omega(r)=\omega\left(\frac{\|y(t)\|}{\|X(t)\|}\right),$$

and the equivalence condition (3) becomes

$$(3')\qquad \frac{\|y(t)-X(t)c\|}{\|X(t)\|}=o(1),\ t\to\infty.$$

The investigation of the asymptotic equivalence using (3′) as definition, has been done by R. Bellman [1], Z. Szmydt [9], Choy-Tak-Taam [5].

6. If $\omega(r)=r$, the conditions of the theorem 1 may be weaken if we assume that

$$\|X^{-1}(t)f(t,y)\|\leqslant a(t)\|y\|,$$

which holds when, for example, $\|f(t,y)\|\leqslant\lambda(t)\|y\|$, $t\in R_{t_0}$ and therefore we can take $a(t)=\|X^{-1}(t)\|\lambda(t)$. The conditions (8) and (10) can be written then

$$(8'')\qquad \|X^{-1}(t)f(t,y)\|\leqslant a(t)\|X(t)\|\frac{\|y\|}{\|X(t)\|},$$

and

$$(10'')\qquad \|X^{-1}(t)k(t,\tau,y)\|\leqslant h(t,\tau)\|X(\tau)\|\cdot\frac{\|y\|}{\|X(\tau)\|}$$

respectively.

The last condition is satisfied if we assume, for example, $\|k(t,\tau,y)\|\leqslant h(t,\tau)\|y\|$.

7. Suppose that

a) $\liminf_{t\to\infty}\int_{t_0}^{t} A(s)\,ds>-\infty$;

b) the solutions of (1) are bounded;

c) we have;

$$(14)\qquad \|f(t,y)\|\leqslant a(t)\,\omega(\|y\|),$$

$$(15)\qquad \|k(t,\tau,y)\|\leqslant h(t,\tau)\,\omega(\|y\|);$$

d) $\displaystyle\int_{t_0}^{\infty}\left[a(s)+\int_{t_0}^{s}h(s,\tau)\,d\tau\right]ds<\infty.$

Then the conclusion of the theorem 1 is also true.

In fact, because $\|X(t)\|\leqslant M$, (7) holds with $\alpha(t)=M$ and $\Delta(t)=I$.

From a) and b) we have also $\|X^{-1}(t)\|\leqslant M_1$ and therefore (8) and (10) may be written

$$\|X^{-1}(t)f(t,y)\|\leqslant M_1\|f(t,y)\|\leqslant M_1\,a(t)\,\omega(\|y\|),$$

and

$$\|X^{-1}(t)\,k(t,\tau,y)\|\leqslant M_1\|k(t,\tau,y)\|\leqslant M_1h(t,\tau)\,\omega(\|y\|)$$

respectively.

§ 3. We now consider the converse of this theorem which holds if we take solutions $x(t)=X(t)c$ of (1) with c sufficiently small.

Theorem 2. *Let the hypotheses of theorem 1 be satisfied. Then, given any solution $x(t)=X(t)c$ of (1) with $\|c\|<M$, where*

$$M=\int_{t_0}^{\infty}\left|a(s)+\int_{t_0}^{s}h(s,\tau)\,d\tau\right|ds,$$

there exists a solution $y=y(t)$ of (2) such that (3) holds.

Proof. To prove this theorem we shall apply the fixed point theorem of Schauder-Tychonoff. Let c satisfy $\|c\|<M$, define $\eta=\dfrac{M-\|c\|}{2}$ and choose T_0 sufficiently large so that

$$\int_{T_0}^{\infty}\left| a(s)+\int_{t_0}^{s} h(s,\tau)\,d\tau\right| ds<\frac{\eta}{\omega(M)}.$$

We shall establish the existence of a solution (at least) of the integral equation

$$x(t)=X(t)c-X(t)\int_{t}^{\infty} X^{-1}(s)\left[f(s,x(s))+\int_{t_0}^{s} k(s,\tau,x(\tau))\,d\tau\right] ds,\ t\geq T_0.$$

Consider the set

$A=\left\{u\,;\,u(t)=\dfrac{\Delta(t)\,x(t)}{\alpha(t)},\right.$ *where* $x(t)$ *is continuous on* $[T_0,\infty)$ *and* $\|u\| \leq M-\eta\Big\}$. We define the operator T by

(16)
$$Tu(t)=\frac{\Delta(t)\,X(t)c}{\alpha(t)}-\frac{\Delta(t)\,X(t)}{\alpha(t)}\int_{t}^{\infty} X^{-1}(s)\left[f(s,\Delta^{-1}(s)\,\alpha(s)\,u(s))+\int_{t_0}^{s} k(s,\tau,\Delta^{-1}(\tau)\,\alpha(\tau)\,u(\tau))\,d\tau\right] ds,\ t\geq T_0.$$

We shall show that $T(A)\subset A$. If we use (7), (8) and (10) it follows that

$$\|Tu(t)\|\leq\|c\|+\int_{t}^{\infty}\Big[\|X^{-1}(s)f(s,\Delta^{-1}(s)\alpha(s)u(s))\|+\int_{t_0}^{s}\|X^{-1}(s)\,k(s,\tau,\Delta^{-1}(\tau).\alpha(\tau)u(\tau))\|\,d\tau\Big]ds\leq$$

$$\leq\|c\|+\int_{t}^{\infty}\left[a(s)\,\omega(\|u(s)\|)+\int_{t_0}^{s} h(s,\tau)\,\omega(\|u(\tau)\|)\,d\tau\right]ds.$$

Since $\omega(r)$ is a non-decreasing function and $u\in A$ (hence $\|u\|\leq M-\eta<M$), we have $\omega(\|u\|)\leq\omega(M)$. Therefore

$$\|Tu(t)\|\leq\|c\|+\omega(M)\int_{t}^{\infty}\left| a(s)+\int_{t_0}^{s} h(s,\tau)\,d\tau\right| ds\leq\|c\|+\eta=$$

$$=M-\eta,\ \text{for } t\geq T_0.$$

We show now that T is a continuous operator. Assume that the sequence $\{u_n\}$ converges uniformly to u on every compact interval of $[T_0,\infty)$ where $u_n,u\in A$.

Let $\varepsilon>0$ and choose $T_1>T_0$ sufficiently large so that

$$\int_{T_1}^{\infty}\left| a(s)+\int_{t_0}^{s} h(s,\tau)\,d\tau\right| ds<\frac{\varepsilon}{4\omega(M)}.$$

Then, using (16) we obtain

$$\|Tu(t)-Tu_n(t)\|\leq\int_{t}^{T_1}\|X^{-1}(s)\|\Big[\|f(s,\Delta^{-1}(s)\,\alpha(s)\,u(s))-f(s,\Delta^{-1}(s)\,\alpha(s)\,u_n(s))\|+\int_{t_0}^{s}\|k(s,\tau,\Delta^{-1}(\tau)\,\alpha(\tau)\,u(\tau))-$$

$$-k(s,\tau,\Delta^{-1}(\tau)\,\alpha(\tau)\,u_n(\tau))\|\,d\tau\Big]ds+\int_{T_1}^{\infty}\Big\{\Big\|X^{-1}(s)\Big[f(s,\Delta^{-1}(s)\,\alpha(s)\,u(s))+$$

$$+\int_{t_0}^{s} k(s,\tau,\Delta^{-1}(\tau)\,\alpha(\tau)\,u(\tau))\,d\tau\Big]\Big\|+\Big\|X^{-1}(s)\Big[f(s,\Delta^{-1}(s)\,\alpha(s)\,u_n(s))+$$

$$+\int_{t_0}^{s} k(s,\tau,\Delta^{-1}(\tau)\,\alpha(\tau)\,u_n(\tau))\,d\tau\Big]\Big\|\Big\}ds.$$

The last term (the integral from T_1 to ∞) is dominated by

$$\int_{T_1}^{\infty}\left[a(s)\,\omega(\|u(s)\|)+\int_{t_0}^{s} h(s,\tau)\,\omega(\|u(\tau)\|)d\tau\right]ds+\int_{T_1}^{\infty}\Big[a(s)\,\omega(\|u_n(s)\|)+$$

$$+\int_{t_0}^{s} h(s,\tau)\,\omega(\|u_n(\tau)\|)\,d\tau\Big]ds\leq\omega(M)\,\frac{\varepsilon}{2\omega(M)}=\frac{\varepsilon}{2}.$$

By the uniform convergence of $\{u_n\}$ on every compact subinterval of $[T_0, \infty)$, there exists $N = N(\varepsilon, T_1)$ such that for $n \geqslant N$

$$\| f(s, \Delta^{-1}(s)\, \alpha(s)\, u(s)) - f(s, \Delta^{-1}(s)\, \alpha(s)\, u_n(s)) \| < \frac{\varepsilon}{2K(1 + T_1)(T_1 - T_0)}$$

and

$$\| k(t, \tau, \Delta^{-1}(\tau)\, \alpha(\tau)\, u(\tau)) - k(s, \tau, \Delta^{-1}(\tau)\, \alpha(\tau) u_n(\tau)) \| < \frac{\varepsilon}{2K(1 + T_1)(T_1 - T_0)}$$

where

$$K = \sup_{[T_0, T_1]} \| X^{-1}(t) \| .$$

Combining these estimates we obtain:

$$\| Tu(t) - Tu_n(t) \| < \varepsilon, \text{ for } T_0 \leqslant t \leqslant T_1 \text{ and } n \geqslant N, \text{ i. e.}$$

the sequence $\{Tu_n\}$ is uniformly convergent to Tu on compact subintervals of $[T_0, \infty)$. Hence, T is a continuous operator. Since $T(A) \subset A$ it will follow that $T(A)$ is uniformly bounded.

Now, we shall show the equicontinuity of $T(A)$ at each point of $[T_0, \infty)$.

First, observe that for each $u \in A$ the function $z(t) = \Delta^{-1}(t)\, \alpha(t)\, Tu(t)$ is a solution of the integro-differential equation

$$z' = A(t)\, v + f(t, \Delta^{-1}(t)\, \alpha(t) u(t)) + \int_{t_0}^{t} k(t, \tau, \Delta^{-1}(\tau)\, \alpha(\tau)\, u(\tau))\, d\tau$$

and since

$$\left\| f(t, \Delta^{-1}(t)\, \alpha(t)\, u(t)) + \int_{t_0}^{t} k(t, \tau, \Delta^{-1}(\tau)\, \alpha(\tau)\, u(\tau))\, d\tau \right\|$$

is uniformly bounded for $u \in A$ on any compact subinterval of $[T_0, \infty)$, we see that z' is uniformly bounded on any compact subinterval of $[T_0, \infty)$.

To establish the equicontinuity of $T(A)$ at every point of $[T_0, \infty)$, consider

$$\| Tu(t_1) - Tu(t_2) \| = \| \Delta(t_1)\, \alpha^{-1}(t_1) z(t_1) - \Delta(t_2)\, \alpha^{-1}(t_2) z(t_2) \| \leqslant$$
$$\leqslant \| \Delta(t_1)\, \alpha^{-1}(t_1) \| \cdot \| z(t_1) - z(t_2) \| + \| \Delta(t_1)\, \alpha^{-1}(t_1) - \Delta(t_2)\, \alpha^{-1}(t_2) \| \cdot \| z(t_2) \|,$$

where t_1, t_2 are in some compact subinterval of $[T_0, \infty)$. The right side of this inequality can be made small enough by virtue of the equicontinuity of the family $\{z(t)\}$ and the continuity of $\Delta(t)\, \alpha^{-1}(t)$. Thus, $T(A)$ is an equicontinuous family of functions at each point of $[T_0, \infty)$. By the Schauder-Tychonoff theorem it follows that there exists a function $u \in A$ such that $u(t) = Tu(t)$, i. e. there exists a solution $x(t)$ of equation

$$x(t) = X(t)c - X(t) \int_{t}^{\infty} X^{-1}(s) \left[f(s, x(s)) + \int_{t_0}^{s} k(s, \tau, x(\tau))\, d\tau \right] ds.$$

Thus $x(t)$ is a solution of the integro-differential equation (2) and posseses the asymptotic behavior of (3).

This proves the theorem 2.

Remark 8. Observe that the nonsingularity of the matrix $\Delta(t)$ is essential only in the proof of theorem 2.

§ 4. Next, we shall give an analogue of the theorem 1 under stronger conditions.

Theorem 3. *Suppose that the following conditions are satisfied:*

a) *there exists a nonsingular continuous matrix $\Delta(t)$ such that (7) holds;*

b) *there exists functions $a(t)$ and $\omega(r)$ satisfying the same conditions as in theorem 1 and in addition*

$$(17) \quad \| X^{-1}(t)\, (f(t, y_1) - f(t, y_2)) \| \leqslant a(t)\, \omega\left(\frac{\| \Delta(t)\, (y_1 - y_2) \|}{\alpha(t)} \right), t \in R_{t_0};$$

c) *there exists a continuous non-negative function $h(t, \tau)$ defined on R_{t_0} so that*

$$(18) \quad \begin{aligned} &\| X^{-1}(t)\, (k(t, \tau, y_1(\tau)) - k(t, \tau, y_2(\tau))) \| \leqslant \\ &\leqslant h(t, \tau)\, \omega\left(\frac{\| \Delta(\tau)\, (y_1(\tau) - y_2(\tau)) \|}{\alpha(\tau)} \right), \end{aligned}$$

for $t_0 \leqslant \tau \leqslant t < \infty$;

d) *the functions $a(t)$ and $h(t, \tau)$ satisfy*

$$(19) \quad \int_{t_0}^{\infty} \left[a(s) + \int_{t_0}^{s} h(s, \tau)\, d\tau \right] ds < \infty,$$

and

$$(20) \quad \int_{t_0}^{\infty} \| X^{-1}(s) \| \left[\| f(s, 0) \| + \int_{t_0}^{s} \| k(s, \tau, 0) \|\, d\tau \right] ds < \infty .$$

Then, to each solution $y = y(t)$ of (2) there exists a solution $x = x(t)$ of (1) such that (3) holds.

The proof closely resembles to that of theorem 1; only few differences will be indicated. As above we have

$$\Delta(t)\, y(t) = \Delta(t)\, X(t)\, y_0 + \Delta(t)\, X(t) \int_{t_0}^{t} X^{-1}(s) \left[f(s, y(s)) + \int_{t_0}^{s} k(s, \tau, y(\tau))\, d\tau \right] ds,$$

from which it follows that:

$$\| \Delta(t)y(t)\| \leqslant \| \Delta(t)\, X(t)\| \, \| y_0 \| + \| \Delta(t)\, X(t) \| \int_{t_0}^{t} \Big| \, \| X^{-1}(s) f(s, 0) \| +$$

$$+ \int_{t_0}^{s} \| X^{-1}(s)\, k(s, \tau, 0) \| \, d\tau \, \Big| ds + \int_{t_0}^{t} \Big| \, a(s)\, \omega\left(\frac{\| \Delta(s)\, y(s) \|}{\alpha(s)} \right) +$$

$$+ \int_{t_0}^{s} h(s, \tau)\, \omega\left(\frac{\| \Delta(\tau)\, y(\tau) \|}{\alpha(\tau)} \right) d\tau \, \Big| ds .$$

From a) and d) it follows that

$$\frac{\| \Delta(t)\, y(t) \|}{\alpha(t)} \leqslant \| y_0 \| + \int_{t_0}^{t} \| X^{-1}(s) \| \Big| \, \| f(s, 0) \| + \int_{t_0}^{s} \| k(s, \tau, 0) \| \, d\tau \, \Big| ds +$$

$$+ \int_{t_0}^{t} \Big| \, a(s)\, \omega\left(\frac{\| \Delta(s)\, y(s) \|}{\alpha(s)} \right) + \int_{t_0}^{s} h(s, \tau)\, \omega\left(\frac{\| \Delta(\tau)\, y(\tau) \|}{\alpha(\tau)} \right) d\tau \, \Big| ds \leqslant$$

$$\leqslant M_1 + \int_{t_0}^{t} \Big| \, a(s)\, \omega\left(\frac{\| \Delta(s) y(s) \|}{\alpha(s)} \right) + \int_{t_0}^{s} h(s, \tau)\, \omega\left(\frac{\| \Delta(\tau)\, y(\tau) \|}{\alpha(\tau)} \right) d\tau \, \Big| ds .$$

Using our lemma, we obtain

$$\frac{\| \Delta(t)\, y(t) \|}{\alpha\,(t)} \leqslant M, \; t \in R_{t_0} .$$

To prove (3) we can proceed in the same manner as in theorem 1.

Remarks. 9. If we suppose that

a) all the solutions of (1) are bounded;

b) $\lim\limits_{t\to\infty} \inf \int_{t_0}^{t} \operatorname{tr} A(s)\, ds > -\infty$;

c) $\| f(t, y_1) - f(t, y_2) \| \leqslant a_1(t) \| y_1 - y_2 \|, \; t \in R_{t_0}; \; \| y_1 \| < \infty, \; \| y_2 \| < \infty$;

d) $\| k(t, \tau, y_1) - k(t, \tau, y_2) \| \leqslant h_1(t, \tau) \| y_1 - y_2 \|, \; t_0 \leqslant \tau \leqslant t < \infty$
$\| y_1 \| < \infty, \; \| y_2 \| < \infty$;

e) $\int_{t_0}^{t} \Big| \, a_1(s) + \int_{t_0}^{s} h_1(s, \tau)\, d\tau \, \Big| ds < \infty$ and $\int_{t_0}^{\infty} \Big| \, \| f(s, 0) \| +$

$$+ \int_{t_0}^{s} \| k(s, \tau, 0) \| \, d\tau \, \Big| ds < \infty,$$

where $a_1(t)$, $h_1(t, \tau)$ are analogous to $a(t)$ and $h(t, \tau)$ from theorem 1. Then, the conditions of theorem 2 are satisfied and we obtain a result of J u. A. V e d i [10].

In this case, $\omega(r) = r$, $\Delta(t) = I$, $\alpha(t) = M$ and $a(t) = a_1(t)$, $t \in R_{t_0}$. From a) and b) the estimate $\| X^{-1}(t) \| \leqslant M_1$, $t \in R_{t_0}$ can be found, and with this c) and d) give

$$\| X^{-1}(t)\,(f(t, y_1) - f(t, y_2)) \| \leqslant \| X^{-1}(t) \| \, \| f(t, y_1) - f(t, y_2) \| \leqslant$$
$$\leqslant M_1\, a_1(t) \, \| y_1 - y_2 \| ,$$

and also

$$\| X^{-1}(t)\,(k(t, \tau, y_1) - k(t, \tau, y_2)) \| \leqslant \| X^{-1}(t) \| \, \| k(t, \tau, y_1) -$$
$$- k(t, \tau, y_2) \| \leqslant M_1\, h_1(t, \tau) \| y_1 - y_2 \| , \text{ for } t_0 \leqslant \tau \leqslant t < \infty .$$

10. Suppose that, c) and d) (the previous remark) hold and in addition that

$$\int_{t_0}^{\infty} \| X^{-1}(s) \| \Big| \, \| X(s) \| \, a_1(s) + \int_{t_0}^{s} \| X(\tau) \| \, h_1(s, \tau)\, d\tau \, \Big| ds < \infty$$

and

$$\int_{t_0}^{\infty} \| X^{-1}(s) \| \Big| \, \| f(s, 0) \| + \int_{t_0}^{s} \| k(s, \tau, 0) \| \, d\tau \, \Big| ds < \infty .$$

Then, the theorem 2 is also true with $a(t) = a_1(t)$, $h(t, \tau) = h_1(t, \tau)$, $\alpha(t) = \| X(t) \|$, $\omega(r) = r$ and $\Delta(t) = I$.

In this case (3) will be replaced by

$$\frac{\| y(t) - X(t)c \|}{\| X(t) \|} = o(1). \tag{21}$$

11. If $f(t, y) = B(t)y$ and $k(t, \tau, y) = k(t, \tau)y$, then b) and c) from theorem 3 may be written:

$$\| X^{-1}(t)\,(f(t, y_1) - f(t, y_2)) \| = \| X^{-1}(t)\, B(t)\,(y_1 - y_2) \| \leqslant$$
$$\leqslant \| X^{-1}(t)\, B(t) \| \cdot \| y_1 - y_2 \| .$$

and

$$\| X^{-1}(t)\,(k(t,\tau,y_1) - k(t,\tau,y_2)) \| = \| X^{-1}(t)\,k(t,\tau)\,(y_1 - y_2) \| \leqslant$$
$$\leqslant \| X^{-1}(t)\,k(t,\tau) \| \cdot \| y_1 - y_2 \| ,$$

respectively, that is

$$a(t) = \| X^{-1}(t)\,B(t) \| , \quad h(t,\tau) = \| X^{-1}(t)\,k(t,\tau) \| ; \quad \omega(r) = r.$$

12. If we suppose that the kernel $k(t,\tau,y) = k(t-\tau)\,y$ and $\| f(t,y) \| \leqslant a(t)\,\| y \|$ with $f(t,0) = 0$, then the theorem 3 holds and the condition d) can be rewritten

$$\int_{t_0}^{\infty} \left| a(s) + \int_{t_0}^{s} k(\tau)\,d\tau \right| ds < \infty . \tag{19'}$$

Other applications of these results to the n order differential equation

$$x^{(n)} = \sum_{k=0}^{n-1} a_k\,(t)\,x^{(k-1)} \tag{22}$$

and its perturbed integro-differential equation

$$y^{(n)} = \sum_{k=0}^{n-1} a_k\,(t) y^{(k-1)} + f(t, y, y', \dots, y^{(n-1)}) + \int_{t_0}^{t} k(t, \tau, y(\tau), y'(\tau), \dots, y^{(n-1)}(\tau))\,d\tau \tag{23}$$

can be also given to obtain similar results to those of F. Brauer and J. S. W. Wong [4] and Ju. A. Vedi [10].

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CÎTEVA REZULTATE PRIVIND ECHIVALENŢA ASIMPTOTICĂ A ECUAŢIILOR INTEGRO-DIFERENŢIALE

Rezumat

În această lucrare se dau cîteva rezultate generale asupra echivalenţei asimptotice a sistemelor (1) şi (2) folosind în acest scop teorema de punct fix a lui Schauder-Tihonov şi un rezultat al lui I. V. Bîcov asupra inegalităţilor integrale.

Dacă $\Delta(t)$ este o matrice pătrată de ordinul n iar $\varphi(t)$ este o funcţie pozitivă, atunci sistemele (1) şi (2) se numesc asimptotic echivalente dacă la fiecare soluţie $x = x(t)$ a sistemului (1) corespunde o soluţie $y = y(t)$ a sistemului (2) cu o proprietate (3) şi reciproc. Rezultatele de bază ale lucrării sînt cuprinse în trei teoreme. Prima teoremă afirmă că dacă sînt îndeplinite condiţiile (7), (8), (10) şi (11) atunci la fiecare soluţie $y = y(t)$ a lui (2) corespunde o soluţie $x = x(t)$ a lui (1) încît (3) are loc.

Teorema 2 este o reciprocă a teoremei 1 care are loc pleeînd de la soluţii $x(t) = X(t)c$ ale lui (1) cu c suficient de mic. În sfîrşit, teorema 3 este analogă teoremei 1 în condiţii mai restrictive.