APPROXIMATING DIFFERENCE SCHEMES OF CAUCHY PROBLEMS FOR EVOLUTION SYSTEMS AND APPROXIMATION OF TWO-PARAMETER SEMIGROUPS OF LINEAR OPERATORS

BY

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This paper is concerned with the approximation by difference schemes, of the problem stated in [2], and the convergence of the approximate solutions by studying a sequence of two-parameter semigroups of linear operators.

1. The difference schemes for „evolution systems" in a Banach space B.

Let us consider the „evolution system"

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial \xi_1} = A_1 u, \\[2mm] \dfrac{\partial u}{\partial \xi_2} = A_2 u, \end{cases}$$

and the condition

$$(1')\qquad \lim_{\substack{\xi_1 \to 0 \\ \xi_2 \to 0}} \| u(x) - u_0 \| = 0,$$

in which A_1, A_2 are linear independent operators of

$$x = (\xi_1, \xi_2) \in E_2^+ = \{x = (\xi_1, \xi_2) : \xi_1 \geqslant 0, \ \xi_2 \geqslant 0, \ (\xi_1, \xi_2) \neq (0, 0)\}$$

having the domain $D \equiv D(A_1) \equiv D(A_2)$ and the ranges $R(A_i)$ $(i = 1, 2)$ in a Banach space B.

We shall assume that A_1 and A_2 commute on $R(A_1) \cap R(A_2) \cap D$ and $u_0 \in B$ is given.

In the work [2], it is introduced the notion of solution for the system (1) — (1′) and it is studied the relation between the Cauchy problem and the two-parameter semigroups.

Let $(A_{i,n})$ $i = 1, 2$, $n \in N$ be two sequences of linear bounded operators having the domain B and the range in B, satisfying the condition

$$A_{1,n} A_{2,n} = A_{2,n} A_{1,n}$$

for each n, and approximating A_i $(i = 1, 2)$ in the sense that:

$$\lim_{n\to\infty} A_{i,n} u = A_i u \quad \text{for} \quad u \in D \quad \text{and} \quad i = 1, 2. \tag{2}$$

We consider the semi-discrete difference scheme:

$$\begin{cases} \dfrac{\partial u_n}{\partial \xi_1} = A_{1,n} u_n, \\ \\ \dfrac{\partial u_n}{\partial \xi_2} = A_{2,n} u_n, \end{cases} \qquad u_n(0, 0) = u_0; \tag{3}$$

which is an approximation of (1) — (1′) in the sense of (2).

It follows from [2] (theorem III) that the problem (3) has the unique solution (in the sense given in [2]):

$$u_n(x : u_0) = e^{\xi_1 A_{1,n} + \xi_2 A_{2,n}} u_0 = \sum_{k=0}^{\infty} \frac{(\xi_1 A_{1,n} + \xi_2 A_{2,n})^k}{k!} u_0. \tag{4}$$

Define the operator $T_n(\xi_1, \xi_2)$ on B by the equality:

$$T_n(\xi_1, \xi_2)\, u_0 = u_n(x ; u_0) \quad \text{for} \quad u_0 \in B.$$

We have

$$\| T_n(x) \| \leqslant e^{K_n(\xi_1 + \xi_2)}, \tag{5}$$

where $x = (\xi_1, \xi_2) \in E_2^+$ and $K_n = \max\limits_{i=1,2} \{\| A_{i,n} \|\}$.

Now the question of convergence of the sequence of semigroups $T_n(x)$ which satisfies (5) is arising.

The problem (1) — (1′) may be also approximated by the scheme:

$$\begin{cases} u_n^{k+1, l} = T_{1,n}\, u_n^{k,l}, \\ u_n^{k, l+1} = T_{2,n}\, u_n^{k,l}, \qquad (k, l = 0, 1, 2, \ldots) \\ u_n^{(0,0)} = u_n^{0,0} = u_0, \end{cases} \tag{6}$$

where: $T_{1,n} = h_n^1 A_{1,n} + I$; $T_{2,n} = h_n^2 A_{2,n} + I$; $u_n^{k,l} = u(kh_n^1, lh_n^2)$; the two operators $A_{1,n}$, $A_{2,n}$ are given above satisfying (2), I is the identity operator on B, and $h_n^i > 0$ $(i = 1, 2)$ are the steps in the direction ξ_i.

Then $T_{1,n}$ and $T_{2,n}$ are bounded commuting linear operators on B. The solution of (6) is the following one:

$$u_n^{kl} = T_{1,n}^k\, T_{2,n}^l\, u_n^{00}. \tag{7}$$

Taking $u_n(x) = u_n^{kl}$ for $kh_n^1 \leqslant \xi_1 < (k+1)h_n^1$, $lh_n^2 \leqslant \xi_2 < (l+1)h_n^2$, it appears again the convergence question of solutions $u_n(x, u_0) = T_n(x)u_0$ of (6) to the solution $u(x ; u_0) = T(x)u_0$ of problem (1) — (1′) for $x \in E_2^+$, where $T_n(x)$ and $T(x)$ are operators in B, respectively defined by

$$\begin{aligned} T_n(x)u_0 &= u_n(x ; u_0), \\ T(x)u_0 &= u(x ; u_0), \end{aligned} \qquad \text{for} \quad u_0 \in D,$$

and they can be extended on whole B by continuity. We shall say that the scheme (6) is stable if

$$\begin{cases} \| T_{1,n}^k \| \leqslant M\, e^{Kkh_n^1}, \\ \| T_{2,n}^k \| \leqslant M\, e^{Kkh_n^2}, \end{cases} \tag{8}$$

for some constants K and M independent of n and k. For example, if $\| T_{1,n} \| \leqslant 1$ and $\| T_{2,n} \| \leqslant 1$ then (8) is satisfied. If (6) is stable then the solution (7) will satisfy the estimate

$$\| u_n^{kl} \| \leqslant M^2\, e^{K(kh_n^1 + lh_n^2)}.$$

Suppose that the hypotheses of theorem V from [2] are satisfied: we shall generalize some results from [4] at the problem introduced in [2] The proofs are similar of those in [4]. Throughout this pape the covergence of operators is the strong convergence.

2. Two-parameter semigroups of linear operators, and resolvents.

Definition 1. *A family $\{T(x);\ x \in E_2^+\}$ of bounded operators on B is called a proper (two-parameter) semigroup if*

$$T(x_1 + x_2) = T(x_1)\, T(x_2) \ \textit{for any}\ x_1, x_2 \in E_2^+,$$

$$T(0,0) = I \ \textit{(identity operator on } B\textit{)}.$$

and, in addition,

$$\lim_{\substack{\xi_1 \to 0 \\ \xi_2 \to 0}} T(\xi_1, \xi_2) y = y \quad \textit{for any}\ y \in B, \tag{9}$$

and

$$\| T(x) \| \leqslant M \quad \textit{for all}\ x \in E_2^+. \tag{10}$$

where M is a constant.

Remark. From semigroup property and from (9) it results

$$\lim_{\eta\to\infty} \sup \frac{1}{\eta} \log \| T(\eta u_k) \| \leqslant \omega_k < \infty \qquad (k = 1, 2)$$

(where u_1, u_2 is a base in the Euclidean space E_2) and therefore in the general case

$$\| T(x) \| \leqslant \| T(\xi_1 u_1) \| \, \| T(\xi_2 u_2) \| \leqslant M e^{\omega_1 \xi_1 + \omega_2 \xi_2};$$

hence the semigroup $T'(x) = e^{-(\omega_1 \xi_1 + \omega_2 \xi_2)} T(x)$ will be proper.

If $T(x)$ is a proper semigroup then [3, ch. 10, § 2] the infinitesimal generator (i.g.)

$$A(x)y = \lim_{\tau \to 0+} \frac{1}{\tau} [T(\tau x) - I] y \tag{11}$$

is a linear closed operator and it is densely defined in B. Also, it is known that

$$\frac{d}{d\xi} T(\xi x) y = A(x) T(\xi x) y = T(\xi x) A(x) y, \text{ for } y \in D(A(x)), \tag{12}$$

and

$$\int_0^{\alpha} T(\xi x) A(x) y \, d\xi = [T(\alpha x) - I] y \text{ for } \alpha > 0, \tag{13}$$

and $y \in D(A(x))$.

From Hille-Yosida theory ([5], ch. 9, § 4) it results easily the following:

Proposition. *If $T(x)$ is a proper semigroup, then for any $\lambda > 0$ and $x \in E_2^+$ there is a bounded operator on B:*

$$\mathcal{J}(\lambda; A(x)) = (\lambda I - A(x))^{-1},$$

which satisfies the relations

$$\mathcal{J}(\lambda; A(x)) y = \int_0^{\infty} e^{-\lambda s} T(sx) y \, ds, \qquad y \in B \tag{14}$$

$$\lim_{\lambda \to \infty} \lambda \mathcal{J}(\lambda; A(x)) y = y, \qquad \text{for any } y \in B \tag{15}$$

uniformly w.r. to x,

$$\mathcal{J}(\lambda; A(x)) - \mathcal{J}(\mu; A(x)) = (\mu - \lambda) \mathcal{J}(\lambda; A(x)) \mathcal{J}(\mu; A(x)), \tag{16}$$

and $\| [\lambda \mathcal{J}(\lambda; A(x))]^m \| \leqslant M$ *where M is independent of $\lambda > 0$, $m = 1, 2, \ldots$ and $x \in E_2^+$.*

Definition 2. *The family of operators $\mathcal{J}(\lambda; x)$ defined on B is called a proper resolvent family if*

$$\mathcal{J}(\lambda; x) - \mathcal{J}(\mu; x) = (\mu - \lambda) \mathcal{J}(\lambda; x) \mathcal{J}(\mu; x) \tag{16}$$

$$\| [\lambda \mathcal{J}(\lambda; x)]^m \| \leqslant M \tag{17}$$

for any $\lambda > 0$, $m = 1, 2, \ldots$ and $x \in E_2^+$.

$$\lim_{\lambda \to \infty} \lambda \mathcal{J}(\lambda; x) y = y \tag{18}$$

uniformly with respect to x for any $y \in B$.

It can be shown that if $\mathcal{J}(\lambda; x)$ is a proper resolvent family, then $\mathcal{J}^{-1}(\lambda; x)$ exists and is a densely defined operator in B, for each λ and x (see [5]; ch. 8; § 3, 4).

It is easy to show that the operator

$$\Omega(x) = \lambda I - \mathcal{J}^{-1}(\lambda; x) \tag{19}$$

is independent of λ and that it is closed.

Theorem 1. *Suppose that:*

1) $\{T_k(x)\}$ is a uniformly proper sequence of semigroups (namely we have (10′) $\| T_k(x) \| \leqslant M$ where M is independent of k and x) and $\mathcal{J}(\lambda; \widetilde{\Omega}_k(x))$ is the sequence of associated resolvent families.

2) The operators defined by $\mathcal{J}(\lambda; x) y = \lim_{k \to \infty} \mathcal{J}(\lambda; \widetilde{\Omega}_k(x) y)$ form a proper resolvent family (where the limit is uniform with respect to x and λ on compact sets).

3) The operator (19) can be represented as $\Omega(x) = \xi_1 \Omega_1 + \xi_2 \Omega_2$ with Ω_1 and Ω_2 independent of $x = (\xi_1, \xi_2)$.

Then the sequence $\{T_k(x)\}$ converges strongly to $T(x)$, the proper semigroup having $\mathcal{J}(\lambda; x) = \mathcal{J}(\lambda; \Omega(x))$ as resolvent family.

Proof. (we use the Hille's method from [3, Theorem 12.3.1]).

Define

$$T(tx; n) = \left(I - \frac{t \, \Omega(x)}{n} \right)^{-n}.$$

From (17) we have

$$\| T(tx; n) \| \leqslant M, \tag{20}$$

because:

$$T(tx; n) = \left[\frac{n}{t} \mathcal{J}\left(\frac{n}{t}; \Omega(x) \right) \right]^n.$$

We have also, for $t = 1$,

$$\lim_{\substack{\xi_1 \to 0 \\ \xi_2 \to 0}} T(x; n) y = y \quad \text{for any} \quad y \in B.$$

For $y \in D(\Omega(x))$ we have

$$\frac{d T(tx; n)}{dt} y = \left(I - \frac{t \, \Omega(x)}{n} \right)^{-n-1} \Omega(x) y.$$

and for $y \in D(\Omega^2(x))$ we have, also:

$$T(tx:m)y - T(tx:n)y = \int_0^t \frac{d}{d\sigma}[T((t-\sigma)x:m)T(\sigma x:n)y]d\sigma =$$

$$= \int_0^t \left(\frac{\sigma}{n} - \frac{t-\sigma}{m}\right)\left(I - \frac{(t-\sigma)\Omega(x)}{m}\right)^{-m-1}\left(I - \frac{\sigma\Omega(x)}{n}\right)^{-n-1}\Omega^2(x)y\,d\sigma.$$

From (17) it follows

$$\|(I - \mu\Omega(x))^{-m}\| \leqslant M \text{ for } \mu > 0 \text{ and } m \in N.$$

Thus

$$\|T(tx:n)y - T(tx:m)y\| \leqslant \frac{M^2t^2}{2}\left(\frac{1}{n} + \frac{1}{m}\right)\|\Omega^2(x)y\|,\ y \in D(\Omega(x))$$

and for $t = 1$ we have

$$(\ast)\ \|T(x:n)y - T(x:m)y\| \leqslant \frac{M^2}{2}\left(\frac{1}{n} + \frac{1}{m}\right)\|(\xi_1\Omega_1 + \xi_2\Omega_2)^2_y\|,\quad y \in D(\Omega^2(x))$$

Then, for $y \in D(\Omega^2(x))$ the sequence $T(x:n)y$ converges as $n \to \infty$, uniformly with respect to x on compact sets from $E_2^+ \cup \{(0,0)\}$.

We denote $T(x)y = \lim\limits_{n\to\infty} T(x:n)y$ $(y \in D(\Omega^2(x)))$.

Also, $T(tx:n)y \to T(tx)y$ as $n \to \infty$, uniformly over every bounded t-interval.

$D(\Omega^2(x))$ is dense in B because $\mathcal{J}(\lambda : \Omega(x))$ is a proper resolvent family. Hence, from $\|T(x:n)\| \leqslant M$ and by the Banach-Steinhaus theorem we have

$$T(x)y = \lim_{n\to\infty} T(x:n)y \text{ for all } y \in B.$$

uniformly with respect to x on compact sets. Furthermore, $T(x)$ is strongly continuous for $x \in E_2^+$, $\|T(x)\| \leqslant M$ and $T(0) = I$.

Indeed, we have $\lim\limits_{x\to 0} T(x:n) = I$ and we define in E_2^+ the following partial order relation:

$$\text{for } x = (\xi_1, \xi_2),\ x' = (\xi_1', \xi_2') \in E_2^+, \text{ we put}$$

$$x \leqslant x' \text{ if } \xi_1 \leqslant \xi_1' \text{ and } \xi_2 \leqslant \xi_2'.$$

Then, by Moore's lemma ([1], 1.7.6) we have

$$\lim_{n\to\infty}\lim_{x\to 0} T(x:n) = \lim_{x\to 0}\lim_{n\to\infty} T(x:n) = \lim_{x\to 0} T(x) = I.$$

Given an arbitrary $\varepsilon > 0$ there is a positive integer $N = N(\varepsilon)$ and a number $\delta(\varepsilon) > 0$ such that:

$$\|T(x_0)y - T(x)y\| \leqslant \|T(x_0)y - T(x_0:N)y\| + \|T(x_0:N)y -$$

$$- T(x:N)y\| + \|T(x:N)y - T(x)y\| < \varepsilon \quad \text{for } \|x - x_0\| < \delta(\varepsilon).$$

Remark. If $T_1(x)$ is a proper semigroup and $\mathcal{J}(\lambda : \widetilde{\Omega}_1(x))$ is the associated resolvent, then we can put

$$T_0(x)y = \lim_{n\to\infty}\left(I - \frac{\widetilde{\Omega}_1(x)}{n}\right)^{-n} y.$$

Also, $T_1(tx)$ is a one-parameter semigroup with respect to t for fixed x, with the resolvent $\mathcal{J}(\lambda : \widetilde{\Omega}_1(x))$ and it is known that

$$\lim_{n\to\infty}\left(I - \frac{t\widetilde{\Omega}_1(x)}{n}\right)^{-n} y = T_0(tx)y = T_1(tx)y.$$

Hence $T_0(tx)y = T_1(tx)y$ for all $t > 0$ and each fixed x; therefore $T_0(x) = T_1(x)$ and then we have the exponential formula

$$T_1(x) = \lim_{n\to\infty}\left(I - \frac{\widetilde{\Omega}_1(x)}{n}\right)^{-n} y.$$

Now, we show that $\lim\limits_{k\to\infty} T_k(x)y = T(x)y$.

Letting $m \to \infty$ in $(\ast)$ (n fixed), we obtain the estimates

$$\|T(x:n)y - T(x)y\| \leqslant \frac{M^2}{2n}\|(\xi_1\Omega_1 + \xi_2\Omega_2)^2 y\| \quad \text{for } y \in D(\Omega^2(x)):$$

also, for each k and each $y \in D(\widetilde{\Omega}_k^2(x))$ we have:

$$\|T_k(x:n)y - T_k(x)y\| \leqslant \frac{M^2}{2n}\|(\xi_1\widetilde{\Omega}_{1,k} + \xi_2\widetilde{\Omega}_{2,k})^2 y\|$$

(here $\widetilde{\Omega}_k(x) = \xi_1\widetilde{\Omega}_{1,k} + \xi_2\widetilde{\Omega}_{2,k}$).

We consider $y = \mathcal{J}^2(\mu : \Omega(x))z$ and $y_k = \mathcal{J}^2(\mu : \widetilde{\Omega}_k(x))z$ for a fixed μ and all $z \in B$. Then we have

$$\|T(x)y - T_k(x)y\| \leqslant \|T(x)y - T(x:n)y\| + \left\|\left(I - \frac{\Omega(x)}{n}\right)^{-n} y -\right.$$

$$\left. - \left(I - \frac{\widetilde{\Omega}_k(x)}{n}\right)^{-n} y\right\| + \|T_k(x:n)(y - y_k)\| + \|T_k(x:n)y_k - T_k(x)y_k\| +$$

$$+\| T_k(x)(y_k - y)\| \leqslant \frac{M^2}{2n}\|\Omega^2(x)y\| + \|[n\mathcal{J}(n;\Omega(x))]^n y - [n\mathcal{J}(n;\widetilde{\Omega}_k(x)]y\| +$$

$$+\frac{M^2}{2n}\|\widetilde{\Omega}_k^2(x)y_k\| + 2M\|y - y_k\|.$$

Since

$$\|\Omega^2(x)y\| = \|(\Omega\mathcal{J})^2 z\| = \|\lambda\mathcal{J}(\lambda;\Omega(x)) - I)^2 z\| \leqslant (M+1)^2\|z\|,$$

and

$$\|\widetilde{\Omega}_k^2(x)y_k\| = \|(\lambda\mathcal{J}(\lambda;\widetilde{\Omega}_k(x)) - I)^2 z\| \leqslant (M+1)^2\|z\|,$$

we have

$$\|T(x)y - T_k(x)y\| \leqslant \frac{1}{n}M^2(M+1)^2\|z\| + \|[n\mathcal{J}(n;\Omega(x))]^n y -$$

$$-[n\mathcal{J}(n;\widetilde{\Omega}_k(x))]^n y\| + 2M\|y - y_k\|.$$

Hence, by the hypothesis 2) we have

$$\limsup_{k\to\infty}\|T(x)y - T_k(x)y\| \leqslant \frac{M'}{n}\|z\|$$

uniformly with respect to x. Letting here $n \to \infty$ it follows

$$T(x)y = \lim_{k\to\infty} T_k(x)y \text{ for all } y \in D(\Omega^2(x))$$

uniformly on compact sets from E_2^+. Since $D(\Omega^2(x))$ is dense in B, the foregoing relation is valid on whole B.

We show now that $T(x)$ has the semigroup property: we have

$$\|T(x_1+x_2)y - T(x_1)T(x_2)y\| \leqslant \|T(x_1+x_2)y - T_k(x_1+x_2)y\| +$$
$$+\|T_k(x_1)T_k(x_2)y - T_k(x_1)T(x_2)y\| + \|T_k(x_1)T(x_2)y -$$
$$-T(x_1)T(x_2)y\| \leqslant \|T(x_1+x_2)y - T_k(x_1+x_2)y\| + M\|T_k(x_2)y -$$
$$-T(x_2)y\| + \|[T_k(x_1) - T(x_1)]T(x_2)y\|.$$

and letting $k \to \infty$ we obtain:

$$T(x_1+x_2)y = T(x_1)T(x_2)y \text{ for all } x_1, x_2 \in E_2^+ \text{ and } y \in B.$$

Since letting $k \to \infty$ in the relation

$$\mathcal{J}(\lambda;\widetilde{\Omega}_k(x))y = \int_0^\infty e^{-\lambda t}T_k(tx)y\,dt$$

we obtain

$$\mathcal{J}(\lambda;\Omega(x))y = \int_0^\infty e^{-\lambda t}T(tx)y\,dt.$$

then the proper semigroup $T(x)$ has $\mathcal{J}(\lambda;\Omega(x))$ as its resolvent family and therefore $\Omega(x)$ is the infinitesimal generator of $T(x)$.

Definition 3. *A sequence of resolvent families $\{\mathcal{J}_n(\lambda;x)\}$ is called uniformly proper, if each $\mathcal{J}_n(\lambda;x)$ is proper and the constant M in condition (17) can be taken independent of n.*

Now, we recall the lemmas 2.4 and 5.1 of [4] with some small modifications, necessary for our purpose:

Lemma 1. *If, 1) for each n, the operator A_n is defined on $D(A_n)\subset B$ and its range is in B, 2), the inverse C_n of A_n exists and is defined on whole B, 3) $\|C_n\| \leqslant M$ for all n, 4) the domain and range of A defined by $Ay = \lim_{n\to\infty} A_n y$ are dense in B, then:*

a) the operator $C = \lim_{n\to\infty} C_n$ is defined on whole B and has a dense range,

b) C has an inverse if and only if A has a closed extension and then C^{-1} is the closure of A.

Lemma 2. *If $\{\mathcal{J}_n(\lambda;x)\}$ is a uniformly proper sequence of resolvent families, such that for some positive μ, the strong (uniform in x) limit*

$$\lim_{n\to\infty}\mathcal{J}_n(\mu;x) = \mathcal{J}(\mu;x)$$

is densely defined and has a dense range, then for every $\lambda > 0$, the strong (uniform with respect to x) limit

$$\lim_{n\to\infty}\mathcal{J}_n(\lambda;x) = \mathcal{J}(\lambda;x)$$

is defined on whole B, and $\mathcal{J}(\lambda;x)$ is a proper resolvent.

Theorem 2. *Let $\{T_n(x)\}$ be a sequence of semigroups. Suppose that*

$$\lim_{\substack{\xi_1\to 0\\ \xi_2\to 0}} T_n(x)y = y \quad \text{for all } y \in B. \tag{21}$$

and

$$\|T_n(x)\| \leqslant Me^{K(\xi_1+\xi_2)} \tag{22}$$

for some constants M and K independent of n and $x = (\xi_1, \xi_2)$.

Let $\widetilde{\Omega}_n(x) = \xi_1\widetilde{\Omega}_{1,n} + \xi_2\widetilde{\Omega}_{2,n}$ be the i. g. of semigroup $T_n(x)$ and define

$$\Omega(x)y = \lim_{n\to\infty}\widetilde{\Omega}_n(x)y = (\xi_1\lim_{n\to\infty}\widetilde{\Omega}_{1,n} + \xi_2\lim_{n\to\infty}\widetilde{\Omega}_{2,n})y = (\xi_1\Omega_1 + \xi_2\Omega_2)y.$$

If 1) $\Omega(x)$ is densely defined,

2) for each x, for some $\lambda_0 > K(\xi_1+\xi_2)$ the range $R(\lambda_0 - \Omega(x))$ is dense in B, then the closure of $\Omega(x)$ is the i. g. of a semigroup $T(x)$ which satisfies the relation

$$\lim_{\substack{\xi_1\to 0\\ \xi_2\to 0}} T(x)y = y$$

and

$$T(x)y = \lim_{n\to\infty} T_n(x)y \quad for\ all \quad y \in B.$$

Proof. Define

$$T'_n(x) = e^{-K(\xi_1+\xi_2)}\, T_n(x) \text{ and } \widetilde{\Omega}'_n(x) = \widetilde{\Omega}_n(x) - K(\xi_1 + \xi_2).$$

Then $\widetilde{\Omega}'_n(x)$ is the i. g. of $T'_n(x)$ and the semigroups $T'_n(x)$ form, by (22), a uniformly proper sequence. Evidently, $\Omega'(x) = \lim_{n\to\infty} \widetilde{\Omega}'_n(x) = \Omega(x) - K(\xi_1 + \xi_2)$ is densely defined and $R(\lambda_0 - K(\xi_1 + \xi_2) - \Omega'(x))$ is dense in B. The sequence $\mathcal{J}_n(\lambda_0 - K(\xi_1 + \xi_2) : x) = (\lambda_0 - K(\xi_1 + \xi_2) - \widetilde{\Omega}'_n(x))^{-1}$ is uniformly bounded with respect to n and x. From lemma 1, with $A_n = \lambda_0 - K(\xi_1 + \xi_2) - \widetilde{\Omega}'_n(x)$ and $C_n = \mathcal{J}_n(\lambda_0 - K(\xi_1 + \xi_2) : x)$, (defined on B by [5] chapter 8, § 1) we conclude that the operator defined by

$$\mathcal{J}'(\lambda_0 - K(\xi_1 + \xi_2) : x)y = \lim_{n\to\infty} \mathcal{J}_n(\lambda_0 - K(\xi_1 + \xi_2) : x)y$$

(uniformly with respect to $x = (\xi_1, \xi_2)$ for $\xi_1 + \xi_2 \leqslant \alpha < \lambda_0/K$, since $\| \mathcal{J}_n(\lambda_0 - K(\xi_1 + \xi_2) : x) \| \leqslant M(\lambda_0 - K(\xi_1 + \xi_2))^{-1}$) has the domain on whole B and the dense range. From lemma 2, for $\mu = \lambda_0 - K(\xi_1 + \xi_2)$ it follows that

$$\mathcal{J}'_n(\lambda : x) = \lim_{n\to\infty} \mathcal{J}_n(\lambda : x)$$

forms a proper resolvent.

Since $\mathcal{J}'(\lambda_0 - K(\xi_1 + \xi_2) : x)$ has an inverse, then (by lemma 1) $A = \lambda_0 - K(\xi_1 + \xi_2) - \Omega'(x)$ has a closed extension and $\mathcal{J}'^{-1}(\lambda_0 - K(\xi_1 + \xi_2) : x)$ is the closure of $\lambda_0 - K(\xi_1 + \xi_2) - \Omega'(x)$, therefore $\mathcal{J}'(\lambda ; x)$ is the resolvent of closure of $\Omega'(x)$. By theorem 1, the operators $\mathcal{J}'(\lambda ; x)$ form a resolvent family of a proper semigroup $T'(x)$ such that $T'(x) = \lim_{n\to\infty} T'_n(x)$ and therefore the closure of $\Omega'(x) = \Omega(x) - K(\xi_1 + \xi_2)$ is i. g. of $T'(x)$. The statements for $T(x)$ follow now immediately.

Lemma 3. *Let h, l be some positive numbers, and T_1, T_2 two commuting operators on B such that $\| T_1^k \|$, $\| T_2^k \| \leqslant M$ for all $k \in N$. Then $\Omega(x) = h^{-1}\xi_1(T_1 - I) + l^{-1}\xi_2(T_2 - I)$ is the infinitesimal generator of a proper semigroup $S(x)$ (with $\| S(x) \| \leqslant M^2$).*

Proof. Put $\Omega_1(x) = h^{-1}\xi_1(T_1 - I)$ and $\Omega_2(x) = l^{-1}\xi_2(T_2 - I)$.

Define the semigroup

$$S(x) = e^{\Omega_1(x) + \Omega_2(x)} = \left[\sum_{k=0}^{\infty} \frac{(h^{-1}\xi_1)^k (T_1 - I)^k}{k\,!}\right]\left[\sum_{k=0}^{\infty} \frac{(l^{-1}\xi_2)^k (T_2 - I)^k}{k\,!}\right] =$$

$$= \left[e^{-\xi_1 h^{-1}} \sum_{k=0}^{\infty} \frac{(h^{-1}\xi_1)^k\, T_1^k}{k\,!}\right]\left[e^{-\xi_2 l^{-1}} \sum_{k=0}^{\infty} \frac{(l^{-1}\xi_2)^k\, T_2^k}{k\,!}\right].$$

From

$$S(tx) = \left[\sum_{k=0}^{\infty} \frac{(th^{-1}\xi_1)^k (T_1 - I)^k}{k\,!}\right]\left[\sum_{k=0}^{\infty} \frac{(tl^{-1}\xi_2)^k (T_2 - I)^k}{k\,!}\right]$$

we have

$$\frac{dS(tx)}{dt} = S(tx)\,\Omega_1 + S(tx)\,\Omega_2 = S(tx)\,\Omega(x) = \Omega(x)\,S(tx),$$

and hence $\Omega(x)$ is the i. g. of $S(x)$. From the last expression of $S(x)$ it follows that $\| S(x) \| \leqslant M^2$ for all $x \in E_2^+$.

Lemma 4. *Let h, l, T_1, T_2, $\Omega(x)$, $\Omega_1(x)$, $\Omega_2(x)$ and $S(x)$ be as in lemma 3. We put $k_1 = [h^{-1}\xi_1]$, $k_2 = [l^{-1}\xi_2]$ (the integer part) and $\alpha_1 = h\xi_1^{-1}$, $\alpha_2 = l\xi_2^{-1}$. Then for any $y \in B$ we have:*

$$\| S(\xi_1 u_1)y - T_1^{k_1} y \| \leqslant \frac{\alpha_1}{2} M^5 \| \Omega_1^2 y \| + M^4 \| \Omega_1 y \| (1 - k_1\alpha_1).$$

and

$$\| S(\xi_2 u_2)y - T_2^{k_2} y \| \leqslant \frac{\alpha_2}{2} M^5 \| \Omega_2^2 y \| + M^4 \| \Omega_2 y \| (1 - k_2\alpha_2).$$

Proof. It is obvious that $\alpha_i \to 0$ and $1 - k_i\alpha_i \to 0$ $(i = 1, 2)$ when $h, l \to 0$. From (13), for any $y \in B$, we have:

$$S(\alpha_1\xi_1 u_1)y = y + \int_0^{\alpha_1} S(\xi\xi_1 u_1)\,\Omega_1(x)y\,d\xi = y + \alpha_1\Omega_1(x)y +$$

$$+ \int_0^{\alpha_1}\!\!\int_0^{\xi} S(t\xi_1 u_1)\,\Omega_1^2(x)y\,dt\,d\xi = T_1 y + \int_0^{\alpha_1}\!\!\int_0^{\xi} S(t\xi_1 u_1)\,\Omega_1^2(x)y\,dt\,d\xi.$$

Hence

$$\| S(\alpha_1\xi_1 u_1)y - T_1 y \| \leqslant M^2 \frac{\alpha_1^2}{2} \| \Omega_1^2 y \|.$$

Afterwards

$$S^{k_1}(\alpha_1\xi_1 u_1)\,y - T_1^{k_1} y = \sum_{i=0}^{k_1-1} S(\alpha_1\xi_1 u_1)^{k_1-i-1}\, T_1^i [S(\alpha_1\xi_1 u_1) - T_1]\, y$$

and

$$\| S^{k_1}(\alpha_1\xi_1 u_1) \| = \| S(k_1\alpha_1\xi_1 u_1) \| \leqslant M^2.$$

Therefore

$$\| S(k_1\alpha_1\xi_1 u_1)y - T_1^{k_1} y \| \leqslant \frac{\alpha_1^2 k_1}{2} M^5 \| \Omega_1^2(x)\,y \|.$$

We have

$$\| S(\xi_1 u_1)y - S(k_1 \alpha_1 \xi_1 u_1) y \| = \| S(k_1 \alpha_1 \xi_1 u_1)\{S(\xi_1 u_1 - k_1 \alpha_1 \xi_1 u_1) - I\}y \| \leqslant$$
$$\leqslant M^2 \left\| \int_0^{1-k_1\alpha_1} S(t \xi_1 u_1)\Omega_1(x) y dt \right\| \leqslant M^4(1 - k_1 \alpha_1) \| \Omega_1(x) y \| .$$

and hence, using these estimates, the conclusions follow.

Similarly we establish the second inequality.

Theorem 3. *Let (h_n) and (l_n) la two sequences of positive numbers converging to zero, and $T_{1,n}$, $T_{2,n}$ be two sequences of operators defined on B and satisfying the following conditions:*

$$T_{1,n} T_{2,n} = T_{2,n} T_{1,n} \ \textit{for any} \ n \in N$$
$$\| T_{1,n}^{k_1} \| \leqslant M e^{K k_1 h_n}, \qquad \| T_{2,n}^{k_2} \| \leqslant M e^{K k_2 l_n}$$

where M and K are constants independent of n, k_1, $k_2 \in N$.

Let be $\Omega_{1,n}(x) = h_n^{-1}\xi_1(T_{1,n} - I)$, $\Omega_{2,n}(x) = l_n^{-1}\xi_2(T_{2,n} - I)$, $\Omega_n(x) = \Omega_{1,n}(x) + \Omega_{2,n}(x)$ and define

$$\Omega(x) = \lim_{n\to\infty} \Omega_{1,n}(x) + \lim_{n\to\infty} \Omega_{2,n}(x).$$

If 1) *$\Omega(x)$ is densely defined and*

2) *for each x, for some $\lambda > K(\xi_1 + \xi_2)$, $R(1 - \Omega(x))$ is dense in B, then the closure of $\Omega(x)$ is the i. g. of a semigroup $T(x)$ and*

$$T(x)y = \lim_{n\to\infty} T_{1,n}^{[\xi_1 h_n^{-1}]} T_{2,n}^{[\xi_2 l_n^{-1}]}$$

Proof. It is obvious that $\Omega(x)$ has the form $\Omega(x) = \xi_1 A_1 + \xi_2 A_2$ where A_1 and A_2 are independent of x. Define $T'_{1,n} = e^{-Kh_n} T_{1,n}$ and $T'_{2,n} = e^{-Kl_n} T_{2,n}$ such that for all k_1 and k_2 we have $\| T'^{k_1}_{1,n} \| \leqslant M$ and $\| T'^{k_2}_{2,n} \| \leqslant M$. Then $\Omega'_n(x) = h_n^{-1} \xi_1 (T'_{1,n} - I) + l_n^{-1} \xi_2 (T'_{2,n} - I)$ converges to $\Omega'(x) = \Omega(x) - K(\xi_1 + \xi_2)$ when $n \to \infty$, on $D(\Omega(x))$. Hence $\Omega'(x)$ is densely defined and $R(\lambda - K(\xi_1 + \xi_2) - \Omega'(x))$ is dense in B. Let $S_n(x)$ be the semigroup generated by $\Omega'_n(x)$ (lemma 3). Hence $\| S_n(x) \| \leqslant M^2$. By theorem 2, there is a proper semigroup $T'(x) = \lim_{n\to\infty} S_n(x)$ its i. g. being the closure of $\Omega'(x)$. We put $k_{1,n} = [h_n^{-1} \xi_1]$; $k_{2,n} = [l_n^{-1}\xi_2]$; $\alpha_{1,n} = h_n \xi_1^{-1}$; $\alpha_{2,n} = l_n \xi_2^{-1}$.

Then

$$\| S_n(x) y - T'^{k_{1,n}}_{1,n} T'^{k_{2,n}}_{2,n} y \| \leqslant \| S_n(\xi_1 u_1) S_n(\xi_2 u_2) y - S_n(\xi_1 u_1) T'^{k_{2,n}}_{2,n} y \| +$$
$$+ \| S_n(\xi_1 u_1) T'^{k_{2,n}}_{2,n} y - T'^{k_{1,n}}_{1,n} T'^{k_{2,n}}_{2,n} y \| \leqslant M^2 \| S_n(\xi_2 u_2) y - T'^{k_{1,n}}_{2,n} y \| +$$
$$+ M \| S_n(\xi_1 u_1) y - T'^{k_{1,n}}_{1,n} y \| \leqslant \frac{M^7 \alpha_{2,n}}{2} \| \Omega_{2,n}^2 y \| + M^6 \| \Omega_{2,n} y \| (1 - k_{2,n} \alpha_{2,n}) +$$
$$+ \frac{M^6 \alpha_{1,n}}{2} \| \Omega_{1,n}^2 y \| + M^5 \| \Omega_{1,n} y \| (1 - k_{1,n} \alpha_{1,n}).$$

For $y \in D(\Omega(x))$ the sequences $(\| \Omega_{i,n}^2 y \|)$, $(\| \Omega_{i,n} y \|)$ $(i = 1, 2)$ converge and hence they are bounded.

Thus

$$\lim_{n\to\infty} S_n(x) y = \lim_{n\to\infty} T'^{k_{1,n}}_{1,n} T'^{k_{2,n}}_{2,n} y = T'(x) y$$

for any $y \in D(\Omega(x))$ and by the Banach-Steinhaus theorem, for any $y \in B$.

Now, from $T^{k_{1,n}}_{1,n} T^{k_{2,n}}_{2,n} = e^{K(k_{1,n} h_n + k_{2,n} l_n)} T'^{k_{1,n}}_{1,n} T'^{k_{2,n}}_{2,n}$ it follows $\lim_{n\to\infty} T^{k_{1,n}}_{1,n} T^{k_{2,n}}_{2,n} = e^{K(\xi_1+\xi_2)} T'(x) = T(x)$ and the closure of $\Omega(x) = \Omega'(x) + K(\xi_1 + \xi_2)$ is the i. g. of semigroup $T(x)$.

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SCHEME CU DIFERENȚE CARE APROXIMEAZĂ PROBLEME CAUCHY PENTRU SISTEME DE EVOLUȚIE ȘI APROXIMAREA SEMIGRUPURILOR BIPARAMETRICE DE OPERATORI LINIARI

Rezumat

Sînt considerate aproximări prin scheme cu diferențe ale problemei puse în [2] și se studiază convergența (ca în [4]) soluțiilor aproximative reducînd-o la cercetarea șirurilor de semigrupuri biparametrice de operatori liniari.