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ASUPRA UNOR REZULTATE PRIVIND PRODUSUL DE FUNCŢII *H*

Rezumat

În lucrare se stabilesc două rezultate privind produsul de funcţii *H* de două variabile, care constituie generalizări ale rezultatelor obţinute recent de R. U. V e r m a [6] relativ la funcţii *G* de două variabile şi care la rîndul lor sînt generalizări ale rezultatelor lui P. N. R a t h i e [4].

FOURIER SERIES FOR THE *H*-FUNCTION OF TWO VARIABLES

BY

S. D. GUPTA

§ 1. The object of the present note is to establish the following results of Fourier series for the *H*-function of two variables.

$$\sum_{\gamma=0}^{\infty}\sum_{s=0}^{\infty}\frac{(n+\gamma)!}{n!\,\gamma!}\,\frac{(m+s)!}{m!\,s!}\sin(n+2r+1)\theta.\sin(m+2s+1)\varphi.$$

$$P\left[\begin{array}{l|l} \left|\begin{matrix} m_1, & 0 \\ p_1-m_1, & q_1\end{matrix}\right| & \{(ap_1,\ \alpha p_1)\};\ \{(bq_1,\ \beta q_1)\} \\ \begin{pmatrix} m_2+1, & n_2+1 \\ p_2+2-m_2, & q_2+1-n_2\end{pmatrix} & (1-\gamma,1)\{(cp_2,\gamma p_2)\}(1,1)(n+r+2,1);\ \left(1+\frac{n}{2},1\right)\left(\frac{3}{2}+\frac{n}{2},1\right) \quad x \\ & \{(dq_2,\ \delta q_2)\}(1,1) \\ \begin{pmatrix} m_3+1, & n_3+1 \\ p_3+2-m_3, & q_3+1-n_3\end{pmatrix} & (1-s,1)\{(ep_3,\varepsilon p_3)\}(1,1)(m+s+2,1);\ \left(1+\frac{m}{2},1\right)\left(\frac{3}{2}+\frac{m}{2},1\right) \quad y \\ & \{(fq_3,\ \tau q_3)\}(1,1) \end{array}\right]$$

$$=\frac{\pi}{4}\sum_{k=0}^{n}\sum_{p=0}^{m}\frac{\sin\theta}{k!}\,\frac{(\cos\theta-1)^k}{(n-k)!}\cdot\frac{\sin\varphi}{p!}\cdot\frac{(\cos\varphi-1)^p}{(m-p)!}\times$$

$$
(1.1)\qquad P\left[\begin{array}{c}\left[\begin{matrix} m_1 & 0 \\ p_1-m_1 & q_1\end{matrix}\right] \\ \begin{pmatrix} m_2,\ n_2+2 \\ p_2+2-m_2,\ q_2-n_2\end{pmatrix} \\ \begin{pmatrix} m_3,\ n_3+2 \\ p_3+2-m_3,\ q_3-n_3\end{pmatrix}\end{array}\right|
$$

$$
\left.\begin{array}{l}
\{(ap_1, \alpha p_1)\}; \qquad\qquad \{(bq_1, \beta q_1)\} \\
\{(cp_2, \gamma p_2)\}\ (1,1)\left(\frac{3}{2}+k,1\right);\left(1+\frac{n+k}{2},1\right)\left(\frac{3}{2}+\frac{n+k}{2},1\right) \\
\qquad\qquad \{(dq_2, \delta q_2)\} \\
\{(ep_3, \varepsilon p_3)\}\ (1,1)\left(\frac{3}{2}+p,1\right);\left(1+\frac{m+p}{2},1\right)\left(\frac{3}{2}+\frac{m+p}{2},1\right) \\
\qquad\qquad \{(fq_3, \tau q_3)\}
\end{array}\right|\begin{array}{c}\frac{x}{\sin^2\theta} \\ \\ \frac{y}{\sin^2\varphi}\end{array}\Bigg]
$$

where,

$$0 \leq \theta \leqslant \pi\,;\ 0 \leqslant \varphi \leqslant \pi,\ \ \frac{3}{2}+\frac{n}{2} > 0,\,;\ \ \frac{3}{2}+\frac{m}{2} > 0$$

$$T = \sum_{\tau=1}^{m_1}\alpha_\tau + \sum_{\tau=1}^{m_2}\gamma_\tau + \sum_{\tau=1}^{n_2}\delta_\tau - \sum_{\tau=m_1+1}^{p_1}\alpha_\tau - \sum_{\tau=1}^{q_1}\beta_\tau - \sum_{\tau=m_2+1}^{p_2}\varphi_\tau - \sum_{\tau=n_2+1}^{q_2}\delta_\tau > 0;$$

$$(1.\mathrm{i})\qquad |\arg x| < \frac{T\pi}{2};$$

$$T' = \sum_{\tau=1}^{m_1}\alpha_\tau + \sum_{\tau=1}^{m_2}\varepsilon_\tau + \sum_{\tau=1}^{n_2}\tau_\tau - \sum_{\tau=m_1+1}^{p_1}\alpha_\tau - \sum_{\tau=1}^{q_1}\beta_\tau - \sum_{\tau=m_3+1}^{p_3}\varepsilon_\tau - \sum_{\tau=n_3+1}^{q_3}\tau_\tau > 0;$$

$$|\arg y| < \frac{T'\pi}{2},$$

$$(1.\mathrm{ii})\qquad \begin{aligned} Q &= \sum_{1}^{m_1}\alpha_\tau + \sum_{1}^{p_2}\gamma_\tau - \sum_{1}^{q_2}\delta_\tau - \sum_{m_1+1}^{p_1}\alpha_\tau - \sum_{1}^{q_1}\beta_\tau \leqslant 0, \\ Q' &= \sum_{1}^{m_1}\alpha_\tau + \sum_{1}^{p_3}\varepsilon_\tau - \sum_{1}^{q_3}\tau_\tau - \sum_{m_1+1}^{p_1}\alpha_\tau - \sum_{1}^{q_1}\beta_\tau \leqslant 0. \end{aligned}$$

§ 2. We shall use the following contracted notation for the H-function of two variables defined by P a t h a k [5],

$$
P\left[\begin{array}{c}\left[\begin{matrix} m_1, & 0 \\ p_1-m_1, & q_1\end{matrix}\right] \\ \begin{pmatrix} m_2, & n_2 \\ p_2-m_2, & q_2-n_2\end{pmatrix} \\ \begin{pmatrix} m_3, & n_3 \\ p_3-m_3, & q_3-n_3\end{pmatrix}\end{array}\right|\begin{array}{ll} \{(ap_1, \alpha p_1)\}: & \{(bq_1, \beta q_1)\} \\ \{(cp_2, \gamma p_2)\}: & \{(dq_2, \delta q_2)\} \\ \{(ep_3, \varepsilon p_3)\}; & \{(fq_3, \tau q_3)\}\end{array}\left|\begin{array}{c} x \\ \\ y\end{array}\right]
$$

$$= \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(\xi+\eta)\,\Psi(\xi,\eta)\ x^{\xi}\,y^{\eta}\,d\xi\,d\eta$$

where,

$$\Phi(\xi+\eta) = \frac{\prod_{\tau=1}^{m_1}\Gamma(a_\tau+\alpha_\tau(\xi+\eta))}{\prod_{\tau=m_1+1}^{p_1}\Gamma(1-a_\tau-\alpha_\tau(\xi+\eta))\prod_{\tau=1}^{q}\Gamma(b_\tau+\beta_\tau(\xi+\eta))}$$

$$\Psi(\xi,\eta) = \frac{\prod_{\tau=1}^{m_2}\Gamma(1-c_\tau+\gamma_\tau\xi)\prod_{\tau=1}^{n_2}(\Gamma(d_\tau-\delta_\tau\xi)\prod_{\tau=1}^{m_3}\Gamma(1-e_\tau+\varepsilon_\tau\eta)\prod_{\tau=1}^{n_3}\Gamma(f_\tau-\tau_\tau\eta)}{\prod_{\tau=m_2+1}^{p_2}\Gamma(c_\tau-\gamma_\tau\xi)\prod_{\tau=n_2+1}^{q_2}\Gamma(1-d_\tau+\delta_\tau\xi)\prod_{\tau=m_3+1}^{p_3}\Gamma(e_\tau-\varepsilon_\tau\eta)\prod_{\tau=n_3+1}^{q_3}\Gamma(1-f_\tau+\tau_\tau\eta)}$$

the contour L_1 in the ξ-plane runs from $-i\infty$ to $+i\infty$ curving if necessary so as to ensure that the poles of $\Gamma(d_\tau-\delta_\tau\xi)$, $\tau=1,\ldots\ldots,n_2$ lie to the right and the poles of $\Gamma(1-c_\tau+\gamma_\tau\xi)$, $\tau=1,\ldots\ldots,m_2$ and $\Gamma(a_\tau+\alpha_\tau(\xi+\eta))$, $\tau=1,\ldots\ldots,m_1$ lie to the left of the contour. Similarly the contour L_2 in the η plane consists of the portion of the imaginary axis from $-i\infty$ to $+i\infty$ along with the necessary loops so as to ensure that the poles of $\Gamma(f_\tau-\tau_\tau\eta)$, $\tau=1,\ldots\ldots,n_3$ lie to the right and the poles of $\Gamma(1-e_\tau+\varepsilon_\tau\eta)$, $\tau=1,\ldots\ldots,m_3$ and $\Gamma(a_\tau+\alpha_\tau(\xi+\eta))$, $\tau=1,\ldots\ldots,m_1$ to the left of the contour. Greek letters α, β, γ, δ, ε, τ are all positive quantities. The positive integers p_1, p_2, p_3, q_1, q_2, q_3, m_1, m_2, m_3, n_2 and n_3 satisfy the following inequalities,

$$q_2 \geqslant 1,\ q_3 \geqslant 1,\ p_1 \geqslant 0,\ q_1 \geqslant 0,\ 0 \leqslant m_1 \leqslant p_1,\ 0 \leqslant m_2 \leqslant p_2,\ 0 \leqslant n_2 \leqslant q_2$$

$$0 \leqslant m_3 \leqslant p_3,\ 0 \leqslant n_3 \leqslant q_3,\ p_1+p_2 \leqslant q_1+q_2 \text{ and } p_1+p_3 \leqslant q_1+q_3.$$

The above integral converges absolutely under the following conditions

$$T = \sum_{\tau=1}^{m_1} \alpha_\tau + \sum_{\tau=1}^{m_2} \gamma_\tau + \sum_{\tau=1}^{n_2} \delta_\tau - \sum_{\tau=m_1+1}^{p_1} \alpha_\tau - \sum_{\tau=1}^{q_1} \beta_\tau - \sum_{\tau=m_2+1}^{p_2} \gamma_\tau - \sum_{\tau=n_2+1}^{q_2} \delta_\tau > 0,$$

$$|\arg x| < \frac{T\pi}{2}$$

$$T' = \sum_{\tau=1}^{m_1} \alpha_\tau + \sum_{\tau=1}^{m_3} \varepsilon_\tau + \sum_{\tau=1}^{n_1} \tau_\tau - \sum_{\tau=m_1+1}^{p_1} \alpha_\tau - \sum_{\tau=1}^{q_1} \beta_\tau - \sum_{\tau=m_3+1}^{p_3} \varepsilon_\tau - \sum_{\tau=n_3+1}^{q_3} \tau_\tau > 0;$$

$$|\arg y| < \frac{T'\pi}{2}$$

also,

$P(x, y) = O(|x|^\alpha, |y|^\beta)$ for small values of x and y.

where

$$Q = \sum_{1}^{m_1} \alpha_\tau + \sum_{1}^{p_2} \gamma_\tau - \sum_{1}^{q_2} \delta_\tau - \sum_{m_1+1}^{p_1} \alpha_\tau - \sum_{1}^{q_1} \beta_\tau \leqslant 0; \quad \alpha = \min R\left(\frac{dh}{\delta h}\right)$$

and

$$h = 1, 2, \ldots\ldots, n_2$$

$$Q' = \sum_{1}^{m_1} \alpha_\tau + \sum_{1}^{p_3} \varepsilon_\tau - \sum_{1}^{q_3} \tau_\tau - \sum_{m_1+1}^{p_1} \alpha_\tau - \sum_{1}^{q_1} \beta_\tau \leqslant 0; \quad \beta = \min R\left(\frac{fh}{\tau h}\right)$$

$$h = 1, 2, \ldots\ldots, n_3.$$

§ 3. Using the definition of H-function of two variables as its Mellin-Barne's type integral, the expression on the left of (1.1) can be written as

$$\sum_{\gamma=0}^{\infty} \sum_{s=0}^{\infty} \frac{(n+r)!\,(m+s)!}{n!\,r!\quad m!\,s!} \sin(n+2r+1)\theta \sin+(m+2s+1)\varphi$$

$$\frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c'-i\infty}^{c'+i\infty}$$

$$\frac{\prod_{\tau=1}^{m_1} \Gamma(a_\tau + \alpha_\tau(\xi+\eta)) \prod_{\tau=1}^{m_2} \Gamma(1-c_\tau+\gamma_\tau\xi) \prod_{\tau=1}^{n_2} \Gamma(d_\tau-\delta_\tau\xi) \prod_{\tau=1}^{m_3} \Gamma(1-e_\tau+\varepsilon_\tau\eta)}{\prod_{\tau=m_1+1}^{p_1} \Gamma(1-a_\tau-\alpha_\tau(\xi+\eta)) \prod_{\tau=1}^{q_1} \Gamma(b_\tau+\beta_\tau(\xi+\eta)) \prod_{\tau=m_2+1}^{p_2} \Gamma(c_\tau-\gamma_\tau\xi) \prod_{\tau=n_2+1}^{q_2} \Gamma(1-d_\tau+\delta_\tau\xi)}$$

$$\frac{\prod_{\tau=1}^{n_3} \Gamma(f_\tau-\tau_\tau\eta)\Gamma\left(1+\frac{n}{2}-\xi\right)\Gamma\left(\frac{3}{2}+\frac{n}{2}-\xi\right)\Gamma(\gamma+\xi)\Gamma\left(1+\frac{m}{2}-\eta\right)\Gamma\left(\frac{3}{2}+\frac{m}{2}-\eta\right)\Gamma(s+\eta)}{\prod_{\tau=m_3+1}^{p_3} \Gamma(e_\tau-\varepsilon_\tau\eta) \prod_{\tau+n_3+1}^{v_3} \Gamma(1-f_\tau+\tau_\tau\eta)\Gamma(\xi)\Gamma(1-\xi)\Gamma(n+\gamma+2-\xi)\Gamma(\eta)\Gamma(1-\eta)\Gamma(m+s+2-\eta)} \times$$

$$\times\, x^\xi y^\eta\, d\xi\, d\eta,$$

hence on changing the order of integration and summation which is permissible due to absolute convergence, proceeding on the lines of Parihar [4] and again using the definition of H-function of two variables, we establish (1,1).

§ 4. From (1.1) we can easily deduce the following integral,

$$\int_0^\pi \int_0^\pi \sum_{k=0}^{n} \sum_{p=0}^{m} \frac{\sin\theta}{k!} \frac{\sin(n+2r+1)\theta}{(n-k)!} \frac{\sin\varphi}{p!} \frac{\sin(m+2s+1)\varphi}{(m-p)!} \cdot (\cos\theta-1)^k$$

$$(\cos\Phi-1)^q . P\left[\begin{array}{c|c|c} \begin{bmatrix} m_1, & 0 \\ p_1-m_1, & q_1 \end{bmatrix} & \{(ap_1, \alpha p_1)\};\{(bq_1, \beta q_1)\} & x \\ \begin{pmatrix} m_2, & n_2+2 \\ p_2+2-m_2, & q_2-n_2 \end{pmatrix} & \begin{array}{l} \{(cp_2, \gamma p_2)\}(1,1)\left(\frac{3}{2}+k, 1\right); \\ \left(1+\frac{n+k}{2}, 1\right), \left(\frac{3}{2}+\frac{n+k}{2}, 1\right)\{(dq_2, \delta q_2)\} \end{array} & \frac{x}{\sin^2\theta} \\ \begin{pmatrix} m_3, & n_3+2 \\ p_3+2-m_3, & q_3-n_3 \end{pmatrix} & \begin{array}{l} \{(ep_3, \varepsilon p_3)\}(1,1)\left(\frac{3}{2}+p, 1\right); \\ \left(1+\frac{m+p}{2}, 1\right), \left(\frac{3}{2}+\frac{m+p}{2}, 1\right), \{(fq_3, \tau q_3)\} \end{array} & \frac{y}{\sin^2\varphi} \end{array}\right] =$$

$$= \Pi \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(n+r)!}{(n!\,r!)} \frac{(m+s)!}{m!\,s!}$$

$$P\left[\begin{array}{c|l} \begin{vmatrix} m_1, & 0 \\ p_1-m_1, & q_1 \end{vmatrix} & \{(ap_1, \alpha p_1)\}; \\ \begin{pmatrix} m_2+1, & n_2+2 \\ p_2+2-m_2, & q_2+1-n_2 \end{pmatrix} & (1-r, 1)\{(cp_2, \gamma p_2)\}, \\ \begin{pmatrix} m_3+1; & n_3+2 \\ p_3+2-m_3, & q_3+1-n_3 \end{pmatrix} & (1-s, 1)\{(ep_3, \varepsilon p_3)\}, \end{array}\right.$$

$$
\left.\begin{array}{l}
\qquad\qquad\qquad :\{(bq_1,\ \beta q_1)\} \\
(1:1)(n+r+2,\ 1):\ \left(\frac{3}{2}+\frac{n}{2},1\right)\left(1+\frac{n}{2},1\right),\ \{(dq_2,\ \delta q_2)\}\,(1,1) \\
(1,1)(m+s+2,1):\left(\frac{3}{2}+\frac{m}{2},1\right)\left(1+\frac{m}{2},1\right)\{(fq_3,\ \tau q_3)\}\,(1,1)
\end{array}\right|\left.\begin{array}{c} x \\ \\ y \end{array}\right]
$$

provedid that,

$$
0\leqslant\theta\leqslant\pi,\quad 0\leqslant\varphi\leqslant\pi,\ \frac{3}{2}+\frac{n}{2}>0,\ \frac{3}{2}+\frac{m}{2}>0,
$$

$$
T>0:|\arg x|<\frac{T\pi}{2},\ T'>0;\ |\arg y|<\frac{T'\pi}{2},
$$

$$
Q\leqslant 0,\ Q'\leqslant 0
$$

where T and T' are given by (1.2) and Q, Q' are given by (1.ii).

Many more results can be obtained by specialising the parameters.

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SERII FOURIER PENTRU FUNCȚIA *H* DE DOUĂ VARIABILE

Rezumat

Utilizînd definiția funcției *H* de două variabile dată de R. S. Pathak [5] și forma ei integrală dată de Mellin-Barne, autorul stabilește formula (1.1) de reprezentare printr-o serie Fourier.

SOME INFINITE SERIES FOR THE *H*-FUNCTION OF TWO VARIABLES

BY

S. D. GUPTA

§ 1. Macrobert [2] in 1959 has summed few series for the Macrobert *E*-function. In this paper an attempt has been made to sum the series of similar types with the *H*-function of two variables defined by Pathak [1].

For the definition of *H*-function of two variables see Pathak 1] and for other notations and results see Macrobert [2].

§ 2. **Infinite Series.** In this section we shall obtain following series

Series 1:

$$
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\alpha^m\ \beta^n}\frac{(xt)^m}{m!}\frac{(yz)^n}{n!}
$$

$$
P\left[\begin{array}{c}
\left|\begin{matrix} m_1,\ 0 \\ p_1-m_1,\ q_1\end{matrix}\right] \\
\begin{pmatrix} m_2+1,\ n_2 \\ p_2-m_2,\ q_2-n_2\end{pmatrix} \\
\begin{pmatrix} m_3+1,\ n_3 \\ p_3-m_3,\ q_3-n_3\end{pmatrix}
\end{array}\right|\begin{array}{l}
\{(ap_1-(m+n)\,\alpha p_1,\ \alpha p_1)\}:\ \{(bq_1-(m+n)\beta q_1,\beta q_1)\} \\
(1,1)\{(cp_2+m\,\gamma p_2,\ \gamma p_2)\};\ \{(dq_2+m\delta q_2,\ \delta q_2)\} \\
(1,1)\{(ep_3+n\ \varepsilon p_3,\ \varepsilon p_3)\};\ \{(fq_3+n\tau q_3,\ \tau q_3)\}
\end{array}\left|\begin{array}{c} \frac{\alpha}{x^2} \\ \\ \frac{\beta}{y^2}\end{array}\right]
$$

$$
=P\left[\begin{array}{c}
\left|\begin{matrix} m_1,\ 0 \\ p_1-m_1,\ q_1\end{matrix}\right| \\
\begin{pmatrix} m_2+1,\ n_2 \\ p_2-m_2,\ q_2-n_2\end{pmatrix} \\
\begin{pmatrix} m_3+1,\ n_3 \\ p_3-m_3,\ q_3-n_3\end{pmatrix}
\end{array}\right|\begin{array}{ll}
\{(ap_1,\ \alpha p_1)\}; & \{(bq_1,\ \beta q_1)\} \\
(1,1)\{(cp_2,\ \gamma p_2)\}; & \{(dq_2,\ \delta q_2)\} \\
(1,1)\{(ep_3,\ \varepsilon p_3)\}; & \{(fq_3,\ \tau q_3)\}
\end{array}\left|\begin{array}{c} \frac{\alpha}{(x^2-tx)} \\ \\ \frac{\beta}{(y^2-yz)}\end{array}\right]
$$