ON A MICROPOLAR THEORY FOR BENDING OF PLATES

BY

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1. Introduction. In recent years, methods of asymptotic expansion have yielded a consistent way of obtaining plate and shell theories from the three-dimensional equations of classical elasticity (see, for example [1], [3], [4]). Similar methods were used for micropolar theory [5], [7].

In what follows, we consider a derivation of two-dimensional plate theory from the three-dimensional equations of the linear theory of micropolar elasticity, by means of the same asymptotic approach which has been used in [3]. In the system of stress-strain relations, the coefficients involving the effect of couple stresses are assumed to be of order h^2 with regard to the coefficients of classical elasticity, where $2h$ is the plate thickness. The investigation is based on the well-known concept of partition of the state of stress into an interior state of stress and a boundary-layer. Integrals of the differential equations of the theory of elasticity may be developped by using three iterative processes. The one, corresponding to the interior state of stress will be defined as the basic process, and the first approximation in this process is equivalent to Love-Kirchkoff theory of classical elasticity. Neither estimation of the formal asymptotic errors, nor proof of asymptotic convergence of the iterative processes are given (cf. the related works).

2. Basic Equations. The Three-Dimensional Problem. According to references [1], [8] the equations of equilibrium of micropolar elasticity are given by:

$$
\begin{aligned}
&\sigma_{\lambda\alpha,\lambda} + \sigma_{3\alpha,3} = 0, \qquad \mu_{\lambda\alpha,\lambda} + \mu_{3\alpha,3} + \varepsilon_{\alpha\lambda}(\sigma_{\lambda 3} - \sigma_{3\lambda}) = 0,\\
&\sigma_{\lambda 3,\lambda} + \sigma_{33,3} = 0, \qquad \mu_{\lambda 3,\lambda} + \mu_{33,3} + \varepsilon_{\lambda\mu}\sigma_{\lambda\mu} = 0,
\end{aligned}
\tag{2.1}
$$

where σ_{ij} and μ_{ij} $(i,j = 1, 2, 3)$ are respectively stress tensor components and couple stress components; for the sake of simplicity, volume forces and moments are assumed to be absent. Throughout this paper, we employ the rectangular coordinates x_1, x_2, x_3. We use the usual summation convention for repeated indices. Greek indices run over the range (1, 2), and latin ones over the range (1, 2, 3). Indices following a comma indicate partial differentiation with respect to $x_i (i = 1, 2, 3)$. Finally, $\varepsilon_{\alpha 3}$ is the alternating symbol in two dimensions.

The constitutive equations are :

$$\sigma_{ij} = \lambda \beta_{hh} \delta_{ij} + 2\mu \beta_{(ij)} + 2\alpha \beta_{[ij]}, \qquad (2.2)$$
$$\mu_{ij} = \beta \varphi_{hh} \delta_{ij} + 2\gamma \varphi_{(ij)} + 2\varepsilon \varphi_{[ij]},$$

where the coefficients λ, μ, α, β, γ, ε are elastic constants appropriate to the theory. The symbols ( ) and [ ] denote the symmetric, respectively the skew-symmetric part of a tensor. The strain and micropolar strain tensor are respectively given by :

$$\beta_{ij} = u_{j,i} - \varepsilon_{ijk} \varphi_k, \quad \varphi_{ij} = \varphi_{j,i}, \qquad (2.3)$$

where u_i, φ_i are respectively the displacement and microrotation components : ε_{ijk} $(i, j, k = 1, 2, 3)$ is the permutation symbol.

A form of the constitutive equations which is convenient in connection with the application of the proposed procedure, is the following

$$u_{3,3} = \frac{\sigma_{33} - \lambda u_{\lambda,\lambda}}{\lambda + 2\mu}, \qquad \varphi_{3,3} = \frac{\mu_{33} - \beta \varphi_{\lambda,\lambda}}{\beta + 2\gamma},$$
$$\sigma_{(\alpha\beta)} = \frac{\nu}{1-\nu} \sigma_{33} \delta_{\alpha\beta} + \frac{E}{1-\nu^2} [\nu u_{\lambda,\lambda} + (1-\nu) u_{(\alpha\beta)}],$$
$$\sigma_{[\alpha\beta]} = 2\alpha (u_{[\beta\alpha]} - \varepsilon_{\alpha\beta} \varphi_3), \qquad (2.4)$$
$$\sigma_{(\alpha 3)} = \mu (u_{\alpha,3} + u_{3,\alpha}), \qquad \sigma_{[\alpha 3]} = \alpha (u_{3,\alpha} - u_{\alpha,3} + 2\varepsilon_{\alpha\lambda} \varphi_\lambda),$$
$$\mu_{\alpha\beta} = \beta (\varphi_{\lambda,\lambda} + \varphi_{3,3}) \delta_{\alpha\beta} + (\gamma - \varepsilon) \varphi_{\alpha,\beta} + (\gamma + \varepsilon) \varphi_{\beta,\alpha},$$
$$\mu_{\alpha 3} = (\gamma - \varepsilon) \varphi_{\alpha,3} + (\gamma + \varepsilon) \varphi_{3,\alpha},$$
$$\mu_{3\alpha} = (\gamma - \varepsilon) \varphi_{3,\alpha} + (\gamma + \varepsilon) \varphi_{\alpha,3}.$$

Consider a thin plate of constant thickness $2h$ bounded by the planes $x_3 = \pm h$. Equations (2.1), (2.4) are further complemented by six boundary conditions for each of the two faces of the plate :

$$\sigma_{3\alpha} = 0, \quad \sigma_{33} = \pm \frac{p}{2}, \quad \mu_{3\alpha} = \pm \frac{q_\alpha}{2}, \quad \mu_{33} = 0 \quad (x_3 = \pm h), \qquad (2.5)$$

where p and q are functions of x_1 and x_2 only. We suppose that surface stresses are prescribed on the edge surface S.

In the following sections we employ the foregoing equations to develop a theory of bending (or flexural) of the plate, assuming that u_3, φ_α, $\sigma_{\alpha 3}$, $\sigma_{3\alpha}$, $\mu_{\alpha\beta}$, μ_{33} are even functions of x_3, and u_α, φ_3, $\sigma_{\alpha\beta}$, σ_{33}, $\mu_{\alpha 3}$, $\mu_{3\alpha}$ are odd functions of x_3.

3. The Interior Domain Problem. We introduce a new thickness variable ζ defined by

$$\zeta = \frac{x_3}{h}, \qquad (3.1)$$

and we assume that all displacements, microrotations, stresses and couple stresses in (2.1) and (2.4) can be expanded in powers of h in the form

$$Q = h^q \sum_{s=1} h^{s-1} Q^{(s)}, \qquad (3.2)$$

where Q is any of the quantities mentioned above. We have introduced the scaling factor h^q, where

$$q = -3 \text{ for } u_3 \text{ and } \varphi_\alpha;$$
$$q = -2 \text{ for } u_\alpha, \varphi_3, \sigma_{\alpha\beta} \text{ and } \mu_{\alpha 3}; \qquad (3.3)$$
$$q = -1 \text{ for } \sigma_{3\alpha}, \sigma_{\alpha 3}, \mu_{\alpha\beta}, \text{ and } \mu_{33};$$
$$q = 0 \text{ for } \sigma_{33} \text{ and } \mu_{3\alpha}.$$

In what follows, we examine only the case when the micropolar coefficients are of the following order of magnitude :

$$(\alpha, \beta, \gamma, \varepsilon) = O(h^2(\lambda, \mu)), \qquad (3.4)$$

i. e.

$$\alpha = h^2 \bar{\alpha}, \quad \beta = h^2 \bar{\beta}, \quad \gamma = h^2 \bar{\gamma}, \quad \varepsilon = h^2 \bar{\varepsilon}, \qquad (3.5)$$

where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$ are of the same order of magnitude as λ, μ.

Inserting the expansions (3.2) into equations (2.1) and (2.4), taking into account (3.1) and (3.3), and equating the coefficients of the same powers of h, we obtain :

$$\sigma^{(s)}_{\lambda\alpha,\lambda} + \sigma^{(s)}_{3\alpha,3} = 0, \quad \mu^{(s)}_{\lambda\alpha,\lambda} + \mu^{(s)}_{3\alpha,3} + \varepsilon_{\alpha\lambda} (\sigma^{(s)}_{\lambda 3} - \sigma^{(s)}_{3\lambda}) = 0,$$
$$\sigma^{(s)}_{\lambda 3,\lambda} + \sigma^{(s)}_{33,3} = 0, \quad \mu^{(s)}_{\lambda 3,\lambda} + \mu^{(s)}_{33,3} + \varepsilon_{\lambda\mu} \sigma^{(s)}_{\lambda\mu} = 0,$$
$$u^{(s)}_{3,3} = \frac{\sigma^{(s-4)}_{33} - \lambda u^{(s-2)}_{\lambda,\lambda}}{\lambda + 2\mu}, \quad \varphi^{(s)}_{3,3} = \frac{\mu^{(s)}_{33} - \beta \varphi^{(s)}_{\lambda,\lambda}}{\beta + 2\gamma}.$$

$$\sigma^{(s)}_{(\alpha\beta)} = \frac{\nu}{1-\nu}\sigma^{(s-2)}_{33}\delta_{\alpha\beta} + \frac{E}{1-\nu^2}[\nu u^{(s)}_{\lambda,\lambda}\delta_{\alpha\beta} + (1-\nu)u^{(s)}_{(\alpha,\beta)}], \tag{3.6}$$

$$\sigma^{(s)}_{[\alpha,\beta]} = 2\varkappa(u^{(s-2)}_{[\beta,\alpha]} + \varepsilon_{\alpha\beta}\varphi^{(s-2)}_3), \quad \sigma^{(s-2)}_{(\alpha3)} = \mu(u^{(s)}_{\alpha,3} + u^{(s)}_{3,\alpha}),$$

$$\sigma^{(s)}_{[\alpha3]} = \varkappa(u^{(s)}_{3,\alpha} - u^{(s)}_{\alpha,3} + 2\varepsilon_{\alpha\lambda}\varphi^{(s)}_\lambda),$$

$$\mu^{(s)}_{\alpha\beta} = \beta(\varphi^{(s)}_{\lambda,\lambda} + \varphi^{(s)}_{3,3})\delta_{\alpha\beta} + (\gamma-\varepsilon)\varphi^{(s)}_{\alpha,\beta} + (\gamma+\varepsilon)\varphi^{(s)}_{\beta,\alpha};$$

$$\mu^{(s)}_{\alpha3} = (\gamma-\varepsilon)\varphi^{(s)}_{\alpha,3} + (\gamma+\varepsilon)\varphi^{(s)}_{3,\alpha}, \quad \mu^{(s-2)}_{3\alpha} = (\gamma-\varepsilon)\varphi^{(s-2)}_{3,\alpha} + (\gamma+\varepsilon)\varphi^{(s)}_{\alpha,3},$$

where, for convenience, we have omitted the superposed bars on the quantities $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$. Also, the subscript 3 following a comma now denotes partial differentiation with respect to ζ. Here and further on we consider that the quantities $Q^{(s)} \equiv 0$ for $s < 1$.

The systems (3.6) ($s = 1, 2, \ldots$) define the basic iterative process. In finding the quantities $Q^{(s+1)}$ we suppose $Q^{(1)}, \ldots, Q^{(s)}$ to be known. To solve the system (3.6) for a given s we set:

$$Q^{(s)} = Q^{(s)}_i + Q^{*(s)}, \tag{3.7}$$

where $Q^{(s)}_i$ satisfy the „homogeneous" system obtained from (3.6) by discarding the terms with superscripts less than s, and $Q^{*(s)}$ is a particular solution of the system (3.6), when the quantities with superscripts less than s are considered to be known.

The homogeneous system can be integrated easily:

$$\begin{aligned}
&u^{(s)}_{3i} = w^{(s)}, && \sigma^{(s)}_{[\alpha\beta]i} = 0, && \mu^{(s)}_{\alpha\beta i} = \mu^{(s)}_{\alpha\beta0},\\
&\varphi^{(s)}_{\alpha i} = \psi^{(s)}_\alpha, && \sigma^{(s)}_{\alpha\beta i} = \sigma^{(s)}_{(\alpha\beta)i} = \sigma^{(s)}_{\alpha\beta1}\zeta, && \mu^{(s)}_{\alpha3i} = 0,\\
&u^{(s)}_{\alpha i} = -w^{(s)}_{,\alpha}\zeta, && \sigma^{(s)}_{3\alpha i} = \sigma^{(s)}_{3\alpha2}\zeta^2 + \sigma^{(s)}_{3\alpha0}, && \mu^{(s)}_{3\alpha i} = \mu^{(s)}_{3\alpha1}\zeta,\\
&\varphi^{(s)}_{3i} = \frac{1}{\beta+2\gamma}(-\beta\psi^{(s)}_{\lambda,\lambda} + \mu^{(s)}_{330})\zeta, && \sigma^{(s)}_{\alpha3i} = \sigma^{(s)}_{3\alpha2}\zeta^2 + \sigma^{(s)}_{\alpha30}, && \mu^{(s)}_{33i} = \mu^{(s)}_{330};\\
& && \sigma^{(s)}_{331} = \sigma^{(s)}_{333}\zeta^3 + \sigma^{(s)}_{331}\zeta, &&
\end{aligned} \tag{3.8}$$

where $w^{(s)}$, $\psi^{(s)}_\alpha$, $\sigma^{(s)}_{2\alpha0}$, $\mu^{(s)}_{330}$ are functions of x_α, independent of ζ, and

$$\sigma^{(s)}_{\alpha\beta1} = -\frac{E}{1-\nu^2}[\nu w^{(s)}_{,\lambda\lambda}\delta_{\alpha\beta} + (1-\nu)w^{(s)}_{,\alpha\beta}],$$

$$\sigma^{(s)}_{3\alpha2} = \frac{E}{2(1-\nu^2)}w^{(s)}_{,\lambda\lambda\alpha},$$

$$\sigma^{(s)}_{\alpha30} = \sigma^{(s)}_{3\alpha0} + 4\varkappa(w^{(s)}_{,\alpha} + \varepsilon_{\alpha\lambda}\psi^{(s)}_\lambda),$$

$$\sigma^{(s)}_{333} = -\frac{E}{6(1-\nu^2)}w^{(s)}_{,\lambda\lambda\mu\mu},$$

$$\sigma^{(s)}_{331} = -[\sigma^{(s)}_{3\lambda0,\lambda} + 4\varkappa(w^{(s)}_{,\lambda\lambda} + \varepsilon_{\alpha\lambda}\psi^{(s)}_{\lambda,\alpha})], \tag{3.9}$$

$$\mu^{(s)}_{\alpha\beta0} = \frac{\beta}{\beta+2\gamma}(2\gamma\psi^{(s)}_{\lambda,\lambda} + \mu^{(s)}_{330})\delta_{\alpha\beta} + (\gamma-\varepsilon)\psi^{(s)}_{\alpha,\beta} + (\gamma+\varepsilon)\psi^{(s)}_{\beta,\alpha},$$

$$\mu^{(s)}_{3\alpha1} = \left[\varepsilon - \frac{\gamma(2\gamma+3\beta)}{\beta+2\gamma}\right]\psi^{(s)}_{\lambda,\lambda\alpha} - (\gamma+\varepsilon)\psi^{(s)}_{\alpha,\lambda\lambda} - 4\varkappa(\varepsilon_{\alpha\lambda}w^{(s)}_{,\lambda} - \psi^{(s)}_\alpha) -$$

$$-\frac{\beta}{\beta+2\gamma}\mu^{(s)}_{330,\alpha}.$$

We can obtain a particular solution of the system (3.6) in the form

$$u^{*(s)}_3 = \frac{1}{\lambda+2\mu}\int_0^\zeta(\sigma^{(s-4)}_{33} - \lambda u^{(s-2)}_{\lambda,\lambda})\,d\zeta, \quad \mu^{*(s)}_{\alpha3} = \frac{\gamma-\varepsilon}{\gamma+\varepsilon}\mu^{(s-2)}_{3\alpha} + \frac{4\gamma\varepsilon}{\gamma+\varepsilon}\varphi^{(s-2)}_{3,\alpha},$$

$$\varphi^{*(s)}_\alpha = \frac{1}{2\gamma}\int_0^\zeta(\mu^{*(s)}_{\alpha3} + \mu^{(s-2)}_{3\alpha} - 2\gamma\varphi^{(s-2)}_{3,\alpha})\,d\zeta,$$

$$u^{*(s)}_\alpha = \frac{1}{2\mu}\int_0^\zeta(\sigma^{(s-2)}_{\alpha3} + \sigma^{(s-2)}_{3\alpha} - 2\mu u^{*(s)}_{3,\alpha})\,d\zeta.$$

$$\sigma^{*(s)}_{(\alpha\beta)} = \frac{\nu}{1-\nu}\sigma^{(s-2)}_{33}\delta_{\alpha\beta} + \frac{E}{1-\nu^2}[\nu u^{*(s)}_{\lambda,\lambda}\delta_{\alpha\beta} + (1-\nu)u^{*(s)}_{(\alpha,\beta)}],$$

$$\sigma^{*(s)}_{(\alpha\beta)} = 2\varkappa(u^{(s-2)}_{[\beta,\alpha]} + \varepsilon_{\alpha\beta}\varphi^{(s-2)}_3), \quad \sigma^{*(s)}_{3\alpha} = -\int_0^\zeta\sigma^{*(s)}_{\lambda\alpha,\lambda}\,d\zeta, \tag{3.10}$$

$$\sigma^{*(s)}_{\alpha3} = \sigma^{*(s)}_{3\alpha} + 2\varkappa(u^{*(s)}_{3,\alpha} - u^{*(s)}_{\alpha,3} + 2\varepsilon_{\alpha\gamma}\varphi^{*(s)}_\lambda),$$

$$\sigma^{*(s)}_{33} = -\int_0^\zeta\sigma^{*(s)}_{\lambda3,\lambda}\,d\zeta, \quad \mu^{*(s)}_{33} = -\int_0^\zeta(\mu^{*(s)}_{\lambda3,\lambda} + \varepsilon_{\lambda\mu}\sigma^{*(s)}_{\lambda\mu})\,d\zeta,$$

$$\varphi^{*(s)}_3 = \frac{1}{\beta+2\gamma}\int_0^\zeta(\mu^{*(s)}_{33} - \beta\varphi^{*(s)}_{\lambda,\lambda})\,d\zeta,$$

$$\mu^{*(s)}_{\alpha\beta} = \frac{\beta}{\beta+2\gamma}\,\mu^{*(s)}_{33}\,\delta_{\alpha\beta} + \frac{2\beta\gamma}{\beta+2\gamma}\,\varphi^{*(s)}_{\lambda,\lambda}\,\delta_{\alpha\beta} + (\gamma-\varepsilon)\varphi^{*(s)}_{\alpha,\beta} + (\gamma+\varepsilon)\varphi^{*(s)}_{\beta,\alpha},$$

$$\mu^{*(s)}_{3\alpha} = -\int_0^{\zeta}\left[\mu^{*(s)}_{\alpha\lambda,\lambda} + \varepsilon_{\alpha\lambda}(\sigma^{*(s)}_{\lambda 3} - \sigma^{*(s)}_{3\lambda})\right] d\zeta.$$

Here the stared quantities are functions of x_α and ζ; the ones without stars and with superscripts less than s are supposed to be known. Since $Q^{(s)} \equiv 0$ for $s < 1$, it follows easily that $Q^{*(1)} = Q^{*(2)} = 0$. For $s > 2$, $Q^{*(s)}$ are some polynomials in ζ, given by the recurrence formulae (3.10).

Using (3.2), (3.3), (3.7) and (3.8), the boundary conditions (2.5) give

$$h^{s-2}\left(\sigma^{(s)}_{3\alpha 2} + \sigma^{(s)}_{3\alpha 0} + \sigma^{*(s)}_{3\alpha}\right) = 0,$$

$$h^{s-1}\left(\sigma^{(s)}_{333} + \sigma^{(s)}_{331} + \sigma^{*(s)}_{33}\right) = \frac{1}{2}\,p^{(s)},$$

(3.11) $$\text{for } \zeta = 1$$

$$h^{s-1}\left(\mu^{(s)}_{3\alpha 1} + \mu^{*(s)}_{3\alpha}\right) = \frac{1}{2}\,q^{(s)}_{\alpha},$$

$$h^{s-2}\left(\mu^{(s)}_{330} + \mu^{*(s)}_{33}\right) = 0,$$

where $p^{(1)} = p$, $q^{(1)}_\alpha = q_\alpha$, $p^{(s)} = q^{(s)}_\alpha = 0$ for $s > 1$.

From (3.11.) we immediately have

$$\sigma^{(1)}_{3\alpha 0} = -\frac{E}{2(1-\nu^2)}\,w^{(1)}_{,\lambda\lambda\alpha}, \qquad \sigma^{(s)}_{3\alpha 0} = -\sigma^{(s)}_{3\alpha 2} - \sigma^{*(s)}_{3\alpha},$$

(3.12) $$\sigma^{(1)}_{331} = -\sigma^{(1)}_{333} + \frac{p}{2}, \qquad \sigma^{(s)}_{331} = -\sigma^{(s)}_{333} - \sigma^{*(s)}_{33}, \qquad (\zeta = 1,\ s > 1)$$

$$\mu^{(1)}_{3\alpha 1} = \frac{1}{2}\,q_\alpha, \qquad \mu^{(s)}_{3\alpha 1} = -\mu^{*(s)}_{3\alpha},$$

$$\mu^{(1)}_{330} = 0, \qquad \mu^{(s)}_{330} = -\mu^{*(s)}_{33}.$$

Using (3.12), the relations (3.7) lead to a system of three partial differential equations for three unknowns w and ψ_α, which for $s = 1$ can be written as

$$\frac{E}{1-\nu^2}\,\Delta\Delta w^{(1)} - 4\alpha(\Delta w^{(1)} + \varepsilon_{\mu\lambda}\psi^{(1)}_{\lambda,\mu}) = \frac{p}{2},$$

(3.13)

$$\left[\frac{E^*}{2(1+\nu^*)} + \varepsilon\right]\Delta\psi^{(1)}_\alpha + \left[\frac{E^*}{2(1-\nu^*)} - \varepsilon\right]\psi^{(1)}_{\lambda,\lambda\alpha} + 4\alpha(\varepsilon_{\alpha\lambda}\,w^{(1)}_{,\lambda} - \psi^{(1)}_\alpha) = -\frac{q_\alpha}{2},$$

where we have put (cf. [6])

(3.14) $$E^* = \frac{\gamma(3\beta + 2\gamma)}{\beta+\gamma}, \qquad \nu^* = \frac{\beta}{2(\beta+\gamma)}.$$

These equations have been derived without explicitly using the conditions which define the applied stresses and couple stresses acting over the edge surface of the plate.

4. The Boundary-Layer Problem. We wish to examine the boundary-layer equations in the neighbourhood of a normal section S of the plate (the edge surface) through a curve C in the middle surface of the plate, where C is sufficiently smooth so that sharp corners are excluded.

More precisely, we assume that the curve C is given by $x_1 = 0$. If not, a procedure mentioned in [2], [4] lead immediately to such a form. Let the domain of the plate be given by $x_1 < 0$.

Now, we introduce a scaling of the co-ordinates which is suitable for our discussion of boundary-layer equations.

Let be:

(4.1) $$\xi = \frac{x_1}{h}, \qquad \zeta = \frac{x_3}{h}.$$

Next, for all displacement, microrotation, stress and couple stress components we assume two expansions of type (3.2) where, in the case of:

Problem I

(4.2)
$q = -2$ for $\varphi_1, \varphi_3, \sigma_{12}, \sigma_{21}, \sigma_{23}, \sigma_{32}$;
$q = -1$ for $u_2, \varphi_2, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{13}, \sigma_{31}, \mu_{11}, \mu_{22}, \mu_{33}, \mu_{13}, \mu_{31}$;
$q = 0$ for $u_1, u_3, \mu_{12}, \mu_{21}, \mu_{23}, \mu_{32}$.

Problem II

(4.3)
$q = -1$ for $\varphi_2, \sigma_{11}, \sigma_{22}, \sigma_{13}, \sigma_{31}$;
$q = 0$ for $u_1, u_3, \varphi_1, \varphi_3, \sigma_{12}, \sigma_{21}, \sigma_{23}, \sigma_{32}, \mu_{12}, \mu_{21}, \mu_{23}, \mu_{32}$;
$q = 1$ for $u_2, \mu_{11}, \mu_{22}, \mu_{33}, \mu_{13}, \mu_{31}$.

Now we consider each problem in detail.

Problem I

Inserting the expansions (3.2) into (2.1) and (2.4), taking into account (4.1) and (4.2), and equating the coefficients of the same powers of h, we get

$$
\begin{aligned}
&\sigma^{(s)}_{12,1} + \sigma^{(s-2)}_{22,2} + \sigma^{(s)}_{32,3} = 0, && \mu^{(s)}_{11,1} + \mu^{(s-2)}_{21,2} + \mu^{(s)}_{31,3} - 2\sigma^{(s)}_{[32]} = 0,\\
&\sigma^{(s)}_{(12)} = \mu\left(u^{(s)}_{2,1} + u^{(s-2)}_{1,2}\right), && \mu^{(s)}_{13,1} + \mu^{(s-2)}_{23,2} + \mu^{(s)}_{33,3} + 2\sigma^{(s)}_{[12]} = 0,\\
&\sigma^{(s)}_{[12]} = \alpha\left(u^{(s-2)}_{2,1} - u^{(s-4)}_{1,2} - 2\varphi^{(s-2)}_{3}\right), && \mu^{(s)}_{11} = (\beta + 2\gamma)\,\varphi^{(s)}_{1,1} + \beta\left(\varphi^{(s)}_{3,3} + \varphi^{(s-2)}_{2,2}\right),\\
(4.4)\quad &\sigma^{(s)}_{(32)} = \mu\left(\mu^{(s)}_{2,3} + u^{(s-2)}_{3,2}\right), && \mu^{(s)}_{33} = (\beta + 2\gamma)\varphi^{(s)}_{3,3} + \beta\left(\varphi^{(s)}_{1,1} + \varphi^{(s-2)}_{2,2}\right),\\
&\sigma^{(s)}_{[32]} = \alpha\left(u^{(s-2)}_{2,3} - u^{(s-4)}_{3,2} + 2\varphi^{(s-2)}_{1}\right), && \mu^{(s)}_{13} = (\gamma - \varepsilon)\,\varphi^{(s)}_{1,3} + (\gamma + \varepsilon)\varphi^{(s)}_{3,1},\\
& && \mu^{(s)}_{31} = (\gamma - \varepsilon)\varphi^{(s)}_{3,1} + (\gamma + \varepsilon)\varphi^{(s)}_{1,3};\\
& && \mu^{(s)}_{22} = (\beta + 2\gamma)\varphi^{(s-2)}_{2,2} + \beta\left(\varphi^{(s)}_{1,1} + \varphi^{(s)}_{3,3}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\sigma^{(s)}_{11,1} + \sigma^{(s)}_{21,2} + \sigma^{(s)}_{31,3} = 0, && \mu^{(s)}_{12,1} + \mu^{(s)}_{22,2} + \mu^{(s)}_{32,3} - 2\sigma^{(s)}_{[13]} = 0,\\
&\sigma^{(s)}_{13,1} + \sigma^{(s)}_{23,2} + \sigma^{(s)}_{33,3} = 0, && \mu^{(s)}_{12} = (\gamma - \varepsilon)\varphi^{(s)}_{1,2} + (\gamma + \varepsilon)\varphi^{(s)}_{2,1};\\
&\sigma^{(s)}_{11} = \frac{\nu}{1-\nu}\,\sigma^{(s)}_{33} + \frac{E}{1-\nu^2}\left(u^{(s)}_{1,1} + \nu u^{(s)}_{2,2}\right), && \mu^{(s)}_{21} = (\gamma - \varepsilon)\varphi^{(s)}_{2,1} + (\gamma + \varepsilon)\varphi^{(s)}_{1,2},\\
(4.5)\quad &\sigma^{(s)}_{22} = \frac{\nu}{1-\nu}\,\sigma^{(s)}_{33} + \frac{E}{1-\nu^2}\left(\nu u^{(s)}_{1,1} + u^{(s)}_{2,2}\right), && \mu^{(s)}_{32} = (\gamma - \varepsilon)\varphi^{(s)}_{3,2} + (\gamma + \varepsilon)\varphi^{(s)}_{2,3},\\
&\sigma^{(s)}_{33} = (\lambda + 2\mu)u^{(s)}_{3,3} + \lambda\left(u^{(s)}_{1,1} + u^{(s)}_{2,2}\right), && \mu^{(s)}_{23} = (\gamma - \varepsilon)\varphi^{(s)}_{2,3} + (\gamma + \varepsilon)\varphi^{(s)}_{3,2},\\
&\sigma^{(s)}_{(13)} = \mu\left(u^{(s)}_{3,1} + u^{(s)}_{1,3}\right),\\
&\sigma^{(s)}_{[13]} = \alpha\left(u^{(s-2)}_{3,1} - u^{(s-2)}_{1,3} + 2\varphi^{(s-2)}_{2}\right),
\end{aligned}
$$

where now the subscripts 1 and 3 following a comma denote partial differentiation with respect to ξ and ζ, respectively

To solve the systems (4.4), (4.5) we set:

$$
R^{(s)} = R^{(s)}_{I} + R^{*(s)}_{I}. \tag{4.6}
$$

Here $R^{(s)}$ stands for any unknown in (4.4) and (4.5); $R^{(s)}_{I}$ denotes the general solution of the „homogeneous" systems (4.4) and (4.5), i. e. the systems without the terms with superscripts less than s; $R^{*(s)}_{I}$ denotes a particular solution of those systems. In fact, it is easy to show that, without less in generality, we can consider:

1) $\sigma^{(s)}_{12\mathrm{I}}$; $\sigma^{(s)}_{21\mathrm{I}}$, $\sigma^{(s)}_{32\mathrm{I}}$, $\sigma^{(s)}_{23\mathrm{I}}$, $u^{(s)}_{2\mathrm{I}}$, $\mu^{(s)}_{11\mathrm{I}}$, $\mu^{(s)}_{13\mathrm{I}}$; $\mu^{(s)}_{31\mathrm{I}}$; $\mu^{(s)}_{33\mathrm{I}}$, $\mu^{(s)}_{22\mathrm{I}}$, $\varphi^{(s)}_{1\mathrm{I}}$, $\varphi^{(s)}_{3\mathrm{I}}$

— the general solution of the homogeneous system (4.4);

2) $\sigma^{(s)}_{11\mathrm{I}}$, $\sigma^{(s)}_{13\mathrm{I}}$, $\sigma^{(s)}_{31\mathrm{I}}$, $\sigma^{(s)}_{33\mathrm{I}}$, $\sigma^{(s)}_{22\mathrm{I}}$, $u^{(s)}_{1\mathrm{I}}$, $u^{(s)}_{3\mathrm{I}}$, $\mu^{(s)}_{12\mathrm{I}}$, $\mu^{(s)}_{21\mathrm{I}}$, $\mu^{(s)}_{32\mathrm{I}}$, $\mu^{(s)}_{23\mathrm{I}}$, $\varphi^{(s)}_{2\mathrm{I}}$.

— a particular solution of (4.5), where we have replaced all the other unknowns by the quantities with the subscripts I:

3) $\sigma^{*(s)}_{12\mathrm{I}}, \ldots, \varphi^{*(s)}_{3\mathrm{I}}$

— a particular solution of (4.4), where the terms with superscripts less then s are considered to be known:

$$\sigma^{*(s)}_{11\mathrm{I}}, \ldots, \varphi^{*(s)}_{2\mathrm{I}}$$

— a particular solution of (4.5), where we have replaced all the other unknowns by the stared quantities with the subscript I.

Since $R^{(s-2)} = 0$ for $s = 1, 2$, one can set $R^{*(1)}_{I} = R^{*(2)}_{I} = 0$.

The main system is (4.4), which resembles the one for the torsion problem of a prismatic bar, having x_2-axis as its axis. We therefore call this problem „quasi-torsion".

The equations (4.4) are fulfilled by

$$
\begin{aligned}
&\sigma^{(s)}_{12\mathrm{I}} = \Psi^{(s)}_{\mathrm{I},11} && \mu^{(s)}_{11\mathrm{I}} = F^{(s)}_{1,13}, && \mu^{(s)}_{31\mathrm{I}} = -F^{(s)}_{1,11},\\
&\sigma^{(s)}_{21\mathrm{I}} = \Psi^{(s)}_{\mathrm{I},11}, && \mu^{(s)}_{13\mathrm{I}} = F^{(s)}_{3,13}, && \mu^{(s)}_{33\mathrm{I}} = -F^{(s)}_{3,11};\\
&\sigma^{(s)}_{32\mathrm{I}} = \Psi^{(s)}_{\mathrm{I},13}, && \mu^{(s)}_{22\mathrm{I}} = \frac{\beta}{2(\beta+\gamma)}\left(F^{(s)}_{1,13} - F^{(s)}_{3,11}\right),\\
(4.7)\quad &\sigma^{(s)}_{23\mathrm{I}} = \Psi^{(s)}_{\mathrm{I},13},\\
&u^{(s)}_{2\mathrm{I}} = \frac{1}{\mu}\,\Psi^{(s)}_{\mathrm{I},1}, && \varphi^{(s)}_{1\mathrm{I}} = \frac{(\beta + 2\gamma)F^{(s)}_{1,3} + \beta F^{(s)}_{3,1}}{4\gamma(\beta+\gamma)},\\
& && \varphi^{(s)}_{3\mathrm{I}} = \frac{(\gamma - \varepsilon)F^{(s)}_{1,1} + (\gamma + \varepsilon)F^{(s)}_{3,3}}{4\gamma\varepsilon}.
\end{aligned}
$$

Here the functions $\Psi^{(s)}_{\mathrm{I}}$, $F^{(s)}_{1}$ and $F^{(s)}_{3}$ must satisfy the equations

$$\Delta\Psi^{(s)} = 0,$$

$$
4\varepsilon(1-\nu^{*})\,\Delta F^{(s)}_{\alpha} + \left[\frac{E^{*}}{2(1+\nu^{*})} - \varepsilon(1 - 2\nu^{*})\right] F^{(s)}_{\lambda,\lambda\alpha} = 0, \tag{4.8}
$$

where now the Greek indices take the values 1, 3, $\Delta f = f_{,11} + f_{,33}$, and we used (3.14).

We can take as a particular solution of (4.5)

$$
(4.9)\quad
\begin{aligned}
&\sigma^{(s)}_{11\mathrm{I}} = -2\Psi^{(s)}_{,12}, && \mu^{(s)}_{12\mathrm{I}} = F^{(s)}_{1,23} + \frac{\beta}{2(\beta+\gamma)}F^{(s)}_{3,12},\\
&\sigma^{(s)}_{13\mathrm{I}} = -\Psi^{(s)}_{,23}, && \mu^{(s)}_{21\mathrm{I}} = \frac{\gamma(\beta+\gamma+\varepsilon)}{(\beta+\gamma)(\gamma+\varepsilon)}F^{(s)}_{1,23} + \frac{\beta}{2(\beta+\gamma)}F^{(s)}_{3,12},\\
&\sigma^{(s)}_{31\mathrm{I}} = -\Psi^{(s)}_{,23}, && \mu^{(s)}_{32\mathrm{I}} = -\frac{3\beta+2\gamma}{2(\beta+\gamma)}F^{(s)}_{1,12},\\
&\sigma^{(s)}_{33\mathrm{I}} = 0, && \mu^{(s)}_{23\mathrm{I}} = F^{(s)}_{3,23} - \frac{\beta(\gamma-\varepsilon)}{2(\beta+\gamma)(\gamma+\varepsilon)}F^{(s)}_{1,12},\\
&\sigma^{(s)}_{22\mathrm{I}} = 2\Psi^{(s)}_{,12}, && \varphi^{(s)}_{2\mathrm{I}} = -\frac{\gamma-\varepsilon}{4\gamma\varepsilon}F^{(s)}_{3,2} - \left[\frac{1}{4\gamma\varepsilon} + \right.\\
&u^{(s)}_{1\mathrm{I}} = -\frac{1}{\mu}\Psi^{(s)}_{,2}; && \left. + \frac{\beta\gamma-\beta\varepsilon-\gamma\varepsilon}{2\gamma(\beta+\gamma)(\gamma+\varepsilon)}\right]\int F^{(s)}_{1,12}\,d\zeta.\\
&u^{(s)}_{3\mathrm{I}} = 0, &&
\end{aligned}
$$

Problem II

An analogous procedure gives

$$
(4.10)\quad
\begin{aligned}
&\sigma^{(s)}_{11,1} + \sigma^{(s-2)}_{21,2} + \sigma^{(s)}_{31,3} = 0, && \mu^{(s)}_{12,1} + \mu^{(s-2)}_{22,2} + \mu^{(s)}_{32,3} - 2\sigma^{(s)}_{[13]} = 0,\\
&\sigma^{(s)}_{13,1} + \sigma^{(s-2)}_{23,2} + \sigma^{(s)}_{33,3} = 0, && \mu^{(s)}_{12} = (\gamma-\varepsilon)\varphi^{(s-2)}_{1,2} + (\gamma+\varepsilon)\varphi^{(s)}_{2,1},\\
&\sigma^{(s)}_{11} = \frac{\nu}{1-\nu}\sigma^{(s)}_{33} + \frac{E}{1-\nu^2}\left(u^{(s)}_{1,1} + \nu u^{(s-2)}_{2,2}\right), && \mu^{(s)}_{21} = (\gamma-\varepsilon)\varphi^{(s)}_{2,1} + (\gamma+\varepsilon)\varphi^{(s-2)}_{1,2},\\
&\sigma^{(s)}_{33} = (\lambda+2\mu)u^{(s)}_{3,3} + \lambda\left(u^{(s)}_{1,1} + u^{(s)}_{2,2}\right), && \mu^{(s)}_{32} = (\gamma-\varepsilon)\varphi^{(s-2)}_{3,2} + (\gamma+\varepsilon)\varphi^{(s)}_{2,3},\\
&\sigma^{(s)}_{(13)} = \mu\left(u^{(s)}_{3,1} + u^{(s)}_{1,3}\right), && \mu_{23} = (\gamma-\varepsilon)\varphi^{(s)}_{2,3} + (\gamma+\varepsilon)\varphi^{(s-2)}_{3,2}.\\
&\sigma^{(s)}_{[13]} = \alpha\left(u^{(s-2)}_{3,1} - u^{(s-2)}_{1,3} + 2\varphi^{(s-2)}_{2}\right), &&\\
&\sigma^{(s)}_{22} = \frac{\nu}{1-\nu}\sigma^{(s)}_{33} + \frac{E}{1-\nu^2}\left(\nu u^{(s)}_{1,1} + u^{(s-2)}_{2,2}\right). &&
\end{aligned}
$$

$$
(4.11)\quad
\begin{aligned}
&\sigma^{(s)}_{12,1} + \sigma^{(s)}_{22,2} + \sigma^{(s)}_{32,3} = 0, && \mu^{(s)}_{11,1} + \mu^{(s)}_{21,2} + \mu^{(s)}_{31,3} - 2\sigma^{(s)}_{[32]} = 0,\\
&\sigma^{(s)}_{(12)} = \mu\left(u^{(s)}_{2,1} + u^{(s)}_{1,2}\right), && \mu^{(s)}_{13,1} + \mu^{(s)}_{23,2} + \mu^{(s)}_{33,3} + 2\sigma^{(s)}_{[12]} = 0,\\
&\sigma^{(s)}_{[12]} = \alpha\left(u^{(s-2)}_{2,1} - u^{(s-2)}_{1,2} - 2\varphi^{(s-2)}_{3}\right), && \mu^{(s)}_{11} = (\beta+2\gamma)\varphi^{(s)}_{1,1} + \beta\left(\varphi^{(s)}_{3,3} + \varphi^{(s)}_{2,2}\right),\\
&\sigma^{(s)}_{(32)} = \mu\left(u^{(s)}_{2,3} + u^{(s)}_{3,2}\right), && \mu^{(s)}_{33} = (\beta+2\gamma)\varphi^{(s)}_{3,3} + \beta\left(\varphi^{(s)}_{1,1} + \varphi^{(s)}_{2,2}\right),\\
&\sigma^{(s)}_{[32]} = \alpha\left(u^{(s-2)}_{2,3} - u^{(s-2)}_{3,2} + 2\varphi^{(s-2)}_{1}\right), && \mu^{(s)}_{13} = (\gamma-\varepsilon)\varphi^{(s)}_{1,3} + (\gamma+\varepsilon)\varphi^{(s)}_{3,1},\\
& && \mu^{(s)}_{31} = (\gamma-\varepsilon)\varphi^{(s)}_{3,1} + (\gamma+\varepsilon)\varphi^{(s)}_{1,3},\\
& && \mu^{(s)}_{22} = (\beta+2\gamma)\varphi^{(s)}_{2,2} + \beta\left(\varphi^{(s)}_{1,1} + \varphi^{(s)}_{3,3}\right).
\end{aligned}
$$

$$
(4.12)\qquad R^{(s)} = R^{(s)}_{\mathrm{II}} + R^{*(s)}_{\mathrm{II}} \quad \left(R^{*(1)}_{\mathrm{II}} = R^{*(2)}_{\mathrm{II}} = 0\right).
$$

The system (4.10) defines a problem that we call „quasiplane-strain" (in the ξ, ζ-plane).

Using a similar notation to the one we have introduced in the previous section, the general solution of (4.10) and a particular solution of (4.11) can be written, respectively, as

$$
(4.13)\quad
\begin{aligned}
&\sigma^{(s)}_{11\mathrm{II}} = \Phi^{(s)}_{,133}, && \mu^{(s)}_{12\mathrm{II}} = G^{(s)}_{,11},\\
&\sigma^{(s)}_{13\mathrm{II}} = \sigma^{(s)}_{31\mathrm{II}} = -\Phi^{(s)}_{,113}, && \mu^{(s)}_{32\mathrm{II}} = G^{(s)}_{,13},\\
&\sigma^{(s)}_{33\mathrm{II}} = \Phi^{(s)}_{,111}, && \mu^{(s)}_{21\mathrm{II}} = \frac{\gamma-\varepsilon}{\gamma+\varepsilon}G^{(s)}_{,11},\\
&\sigma^{(s)}_{22\mathrm{II}} = \nu\left(\Phi^{(s)}_{,111} + \Phi^{(s)}_{,133}\right), && \mu^{(s)}_{23\mathrm{II}} = \frac{\gamma-\varepsilon}{\gamma+\varepsilon}G^{(s)}_{,13};\\
&u^{(s)}_{1\mathrm{II}} = \frac{1-\nu^2}{E}\Phi^{(s)}_{,33} - \frac{\nu(1+\nu)}{E}\Phi^{(s)}_{,11}, && \varphi^{(s)}_{2\mathrm{II}} = \frac{1}{\gamma+\varepsilon}G^{(s)}_{,1}.\\
&u^{(s)}_{3\mathrm{II}} = \frac{1-\nu^2}{E}\int\Phi^{(s)}_{,111}\,d\zeta - \frac{\nu(1+\nu)}{E}\Phi^{(s)}_{,13}, &&
\end{aligned}
$$

$$
(4.14)\quad
\begin{aligned}
&\sigma^{(s)}_{12\mathrm{II}} = (1-\nu)\Phi^{(s)}_{,332} - \nu\Phi^{(s)}_{,112}, && \mu^{(s)}_{11\mathrm{II}} = -\frac{2\gamma}{\gamma+\varepsilon}G^{(s)}_{,12},\\
&\sigma^{(s)}_{21\mathrm{II}} = (1-\nu)\Phi^{(s)}_{,332} - \nu\Phi^{(s)}_{,112}; && \mu^{(s)}_{22\mathrm{II}} = \frac{2\gamma}{\gamma+\varepsilon}G^{(s)}_{,12},\\
&\sigma^{(s)}_{32\mathrm{II}} = -\Phi^{(s)}_{,123}, && \mu^{(s)}_{33\mathrm{II}} = 0,\\
&\sigma^{(s)}_{23\mathrm{II}} = -\Phi^{(s)}_{,123}, && \mu^{(s)}_{31\mathrm{II}} = -G^{(s)}_{,23},\\
&u^{(s)}_{3\mathrm{II}} = \frac{1-\nu^2}{E}\int\Phi^{(s)}_{,111}\,d\zeta - \frac{\nu(1+\nu)}{E}\Phi^{(s)}_{,13}, && \mu^{(s)}_{13\mathrm{II}} = -\frac{\gamma-\varepsilon}{\gamma+\varepsilon}G^{(s)}_{,23},\\
& && \varphi^{(s)}_{1\mathrm{II}} = -\frac{1}{\gamma+\varepsilon}G^{(s)}_{,2}.\\
& && \varphi^{(s)}_{3\mathrm{II}} = 0.
\end{aligned}
$$

The functions Φ and G fulfil the equations

$$
(4.15)\qquad \Delta\Delta\Phi^{(s)} = 0, \quad \Delta G^{(s)} = 0,
$$

where

$$
\Delta\Delta\Phi^{(s)} = \Phi^{(s)}_{,1111} + 2\Phi^{(s)}_{,1133} + \Phi^{(s)}_{,3333}; \quad \Delta G^{(s)} = G^{(s)}_{,11} + G^{(s)}_{,33}.
$$

5. Edge Conditions. In this way we have constructed two auxiliary iterative processes. In section 3, equations for the „interior" problem were obtained for plate theory in which the conditions on the faces $x_3 = \pm h$ were satisfied, but the equations were not, in general, uniformly valid on to the surface S. Now we seek solutions of the three-dimensional equations of elasticity which, together with the equations of the section 3, enable us to satisfy arbitrary edge conditions over S. Since the basic equations are linear in the dependent variables, solutions of different types may be added and it is sufficient to seek boundary layer equations, which give zero applied stresses on the surfaces $x_3 = \pm h$, that is homogeneous conditions (2.5) are satisfied. Moreover, we require boundary-layer solutions which are such that the sum of the interior and boundary-layer stresses and couple stresses rapidly reduce to the interior ones as we go away from S. Hence, for a given $x_1 < 0$, we require boundary-layer stress and couple stresses to be small as h tends to zero. Since $x_1 = h\xi$, we therefore look for solutions of the three-dimensional equations which are such that boundary-layer stresses and couple stresses vanish as $\xi \to -\infty$ (so called „decreasing conditions" at infinity).

If $R^{*(s)}_{\mathrm{I}}$ and $R^{*(s)}_{\mathrm{II}}$ are chosen in such a way as to satisfy for every s the homogeneous conditions (2.5) and the decreasing conditions as $\xi \to -\infty$, the solutions $R^{(s)}_{\mathrm{I}}$ and $R^{(s)}_{\mathrm{II}}$ must fulfil the same conditions and, in addition, maintain certain arbitrariness in order to satisfy the edge conditions on the boundary surface S. In fact, we have to solve two systems, (4.8) and (4.15), in a semi-infinite strip given by $\xi \leqslant 0$, $-1 \leqslant \zeta \leqslant 1$. Therefore, besides the homogeneous conditions on the faces and the decreasing conditions, we impose for both $R^{(s)}_{\mathrm{I}}$ and $R^{(s)}_{\mathrm{II}}$ three conditions of the type:

$$f^{(s)}_k = \bar{f}^{(s)}_k \quad \text{for} \quad \xi = 0 \quad (k = 1, 2, 3), \tag{5.1}$$

where $f^{(s)}_k$ are linear homogeneous functions of the quantities $R^{(s)}_{\mathrm{I}}$ and $R^{(s)}_{\mathrm{II}}$, respectively, and $\bar{f}^{(s)}_k$ are arbitrary functions of η, ζ.

From (4.7) and (4.9) it is obvious that the homogeneous conditions (2.5) will be satisfied by the adequate quantities $R^{(s)}_{\mathrm{I}}$ if we impose

$$\Psi^{(s)}_{,3} = F^{(s)}_{,\alpha} = 0, \text{ for } \zeta = \pm 1 \quad (\alpha = 1, 3). \tag{5.2}$$

Likewise, from (4.10) and (4.11) one can see that the same conditions will be satisfied by the corresponding quantities $R^{(s)}_{\mathrm{II}}$, if

$$\varphi^{(s)} = \Phi^{(s)}_{,3} = G^{(s)}_{,3} = 0 \quad \text{for} \quad \zeta = \pm 1. \tag{5.3}$$

It can be shown that the systems (4.8) and (4.15) have unique solutions which satisfy (5.1), (5.2) or (5.3) respectively, and are such that all components of stress vanish as $\xi \to -\infty$, in a manner to be specified later, if some extra compatibility conditions are fulfilled. We notice that the variable η occurs in these systems merely as a parameter.

With the help of the homogenous equilibrium equations that enter into (4.4) and (4.10), as well as the homogeneous conditions (2.5) we can prove that

$$\int_{-1}^{1} \mu^{(s)}_{11\mathrm{I}}\, d\zeta = 0 \quad \text{for} \quad \xi = 0, \tag{5.4}$$

$$\int_{-1}^{1} d\zeta \int_{-1}^{\zeta} \sigma^{(s)}_{11\mathrm{I}}\, d\zeta = 0 \quad \text{or} \quad \int_{-1}^{1} \zeta \sigma^{(s)}_{11\mathrm{I}}\, d\zeta = 0 \quad \text{for} \quad \xi = 0, \tag{5.5}$$

$$\int_{-1}^{1} d\zeta \int_{-1}^{\zeta} \sigma^{(s)}_{13\mathrm{I}}\, d\zeta = 0 \quad \text{or} \quad \int_{-1}^{1} \sigma^{(s)}_{13\mathrm{I}}\, d\zeta = 0 \quad \text{for} \quad \xi = 0, \tag{5.6}$$

$$\int_{-1}^{1} \mu^{(s)}_{12\mathrm{I}}\, d\zeta = 0 \quad \text{for} \quad \xi = 0, \tag{5.7}$$

provided that $\mu^{(s)}_{11\mathrm{I}}$, $\sigma^{(s)}_{11\mathrm{I}}$, $\xi\sigma^{(s)}_{13\mathrm{I}}$, $\mu^{(s)}_{12\mathrm{I}}$ vanish uniformly with respect to ζ as $\xi \to -\infty$. At this point it is useful to remark that other analogous relations can as well be obtained, but in our procedure they will lead to relations that either will be identically satisfied, or will impose some extra conditions in finding the quantities $R^{*(s)}_{\mathrm{I}}$ and $R^{*(s)}_{\mathrm{II}}$.

We suppose now that surface stresses and couple stresses are prescribed on the boundary surface $S(\xi = 0)$. For convenience we consider the case of the free edge, hence the zero applied loading is assumed on S:

$$\sigma_{11} = \sigma_{12} = \sigma_{13} = \mu_{11} = \mu_{12} = \mu_{13} = 0 \quad \text{for} \quad \xi = 0. \tag{5.8}$$

As σ_{13} vanishes on $\zeta = \pm 1$, it can be shown that the conditions

$$\sigma_{12} = \sigma_{13} = 0 \quad \text{for} \quad \xi = 0 \tag{5.9}$$

are equivalent to

$$\sigma_{12} = 0, \quad \tau \equiv \sigma_{12,2} - h^{-1}\sigma_{13,3} = 0 \quad \text{for} \quad \xi = 0. \tag{5.10}$$

6. The Matching Procedure. We shall consider that every stress and couple stress component consists of a sum of three terms corresponding to the basic iterative process and, respectively, the two auxiliary iterative processes. Hence for these stress components we have expansions of type

$$S = h^{q}\sum_{s=1} h^{s-1}\left(Q_{i}^{(s)} + Q^{*(s)}\right) + h^{q'}\sum_{s=1} h^{s-1}\left(R_{\mathrm{I}}^{(s)} + R_{\mathrm{I}}^{*(s)}\right) + \tag{6.1}$$
$$+ h^{q''}\sum_{s=1} h^{s-1}\left(R_{\mathrm{II}}^{(s)} + R_{\mathrm{II}}^{*(s)}\right).$$

Here S stands for the system of all stresses and couple stresses, and q, q', q'' take the adequate values. A similar expansion is obtained for τ from (5.10). With the help of (4.7) to (4.9) we can show that

$$\tau_{1}^{(s)} \equiv \sigma_{121,2}^{(s)} - \sigma_{131,3}^{(s)} = 0. \tag{6.2}$$

We shall also use the notations

$$\tau_{1}^{(s)} \equiv \sigma_{121,2}^{(s)} - 2\sigma_{132}^{(s)}, \quad \tau^{*(s)} \equiv \sigma_{12,2}^{*(s)} - \sigma_{13,3}^{*(s)}. \tag{6.3}$$

Inserting the expansions (6.1) into boundary conditions (5.8), taking into account (5.9) and (5.10), and equating the coefficients of the same powers of h, we get

$$\zeta\sigma_{111}^{(1)} = 0, \quad \zeta\sigma_{111}^{(2)} + \sigma_{11\mathrm{I}}^{(1)} + \sigma_{11\mathrm{II}}^{(1)} = 0, \tag{6.4 a}$$
$$\zeta\sigma_{111}^{(3)} + \sigma_{11}^{*(3)} + \sigma_{11\mathrm{I}}^{(2)} + \sigma_{11\mathrm{II}}^{(2)} = 0, \ldots$$

$$\zeta\sigma_{121}^{(1)} + \sigma_{12\mathrm{I}}^{(1)} = 0, \quad \zeta\sigma_{121}^{(2)} + \sigma_{12\mathrm{I}}^{(2)} = 0, \tag{6.4 b}$$
$$\zeta\tau_{121}^{(3)} + \sigma_{12}^{*(3)} + \sigma_{12\mathrm{I}}^{(3)} + \sigma_{12\mathrm{I}}^{*(3)} + \sigma_{12\mathrm{II}}^{(1)} = 0, \ldots$$

$$\zeta\tau_{1}^{(1)} - \sigma_{13\mathrm{II},3}^{(1)} = 0, \quad \zeta\tau_{1}^{(2)} - \sigma_{13\mathrm{II},3}^{(2)} = 0, \tag{6.4 c}$$
$$\zeta\tau_{1}^{(3)} + \tau^{*(3)} + \tau_{\mathrm{I}}^{*(3)} - \left(\sigma_{13\mathrm{II}}^{(3)} + \sigma_{13\mathrm{II}}^{*(3)}\right)_{,3} + \sigma_{12\mathrm{II},2}^{(1)} = 0, \ldots$$

$$\mu_{110}^{(1)} + \mu_{11\mathrm{I}}^{(1)} = 0, \quad \mu_{110}^{(2)} + \mu_{11\mathrm{I}}^{(2)} = 0, \tag{6.4 d}$$
$$\mu_{110}^{(3)} + \mu_{11}^{*(3)} + \mu_{11\mathrm{I}}^{(3)} + \mu_{11\mathrm{I}}^{*(3)} + \mu_{11\mathrm{II}}^{(1)} = 0, \ldots$$

$$\mu_{120}^{(1)} = 0, \quad \mu_{120}^{(2)} + \mu_{121}^{(1)} + \mu_{12\mathrm{II}}^{(1)} = 0, \quad \mu_{120}^{(3)} + \mu_{12}^{*(3)} + \mu_{121}^{(2)} + \mu_{12\mathrm{II}}^{(2)} = 0, \ldots \tag{6.4 e}$$

$$\mu_{13\mathrm{I}}^{(1)} = 0, \quad \mu_{13}^{*(3)} + \mu_{13\mathrm{I}}^{(2)} = 0, \quad \mu_{13}^{*(4)} + \mu_{13\mathrm{I}}^{(3)} + \mu_{13\mathrm{I}}^{*(3)} + \mu_{13\mathrm{II}}^{(1)} = 0, \ldots, \tag{6.4 f}$$

Using the compatibility conditions (5.4)—(5.7) in (6.4 a), (6.4 c), (6.4 d) and (6.4 e), it follows that

$$-\mu_{110}^{(1)} = 0, \quad -\mu_{110}^{(2)} = 0, \quad -2\mu_{110}^{(3)} - \int_{-1}^{1}\left(\mu_{11}^{*(3)} + \mu_{111}^{*(3)} + \mu_{11\mathrm{II}}^{(1)}\right) d\zeta = 0, \ldots \tag{6.5 a}$$

$$\sigma_{111}^{(1)} = 0, \quad -\frac{2}{3}\sigma_{111}^{(2)} - \int_{-1}^{1}\zeta\sigma_{111}^{(1)}\, d\zeta = 0, \quad -\frac{2}{3}\sigma_{111}^{(3)} - \int_{-1}^{1}\zeta\left(\sigma_{11}^{*(3)} + \right. \tag{6.5 b}$$
$$\left. + \sigma_{111}^{(2)}\right) d\zeta = 0, \ldots$$

$$-\frac{2}{3}\tau_{1}^{(1)} = 0, \quad -\frac{2}{3}\tau_{1}^{(2)} = 0, \quad -\frac{2}{3}\tau_{1}^{(3)} + \int_{-1}^{1} d\zeta \int_{-1}^{\zeta}\left(\tau^{*(3)} + \tau_{1}^{*(3)} - \right. \tag{6.5 c}$$
$$\left. - \sigma_{1311,3}^{(3)} + \sigma_{1211,2}^{(1)}\right) d\zeta = 0, \ldots$$

$$\mu_{120}^{(1)} = 0, \quad -2\mu_{120}^{(2)} - \int_{-1}^{1}\mu_{121}^{(1)}\, d\zeta = 0, \quad -2\mu_{120}^{(3)} - \int_{-1}^{1}\left(\mu_{12}^{*(3)} + \mu_{121}^{(2)}\right) d\zeta = 0, \ldots \tag{6.5 d}$$

In this way the functions $w^{(s)}$ and $\psi_{\alpha}^{(s)}$ can be determined by solving a system similar to (3.13) with the boundary conditions (6.5 a) — (6.5 d). Likewise the functions $\Psi^{(s)}$, $F_{1}^{(s)}$ and $F_{2}^{(s)}$ are determined by solving the system (4.8) in our semi-infinite strip with the boundary conditions (6.4 b), (6.4 d) and (6.4 e), and $\Phi^{(s)}$ and $G^{(s)}$ are determined from (4.15) with the boundary conditions (6.4 a), (6. 4 c) and (6.4f), if, naturaly, the appropriate conditions on $\zeta = \pm 1$ and at $\xi = -\infty$ are fulfilled. With regard to the system (3.13), this involves some extra conditions for p and q_{α}, namely :

$$\int_{D} p\, d\sigma = \int_{D} q_{\alpha}\, d\sigma = \int_{D} x_{\alpha} p\, d\sigma = 0. \tag{6.6}$$

where D is the initial interior domain of the plate.

7. Relation with the Classical Theory. Let us introduce the notations

$$M_{\alpha\beta}^{(s)} = h^{s-3}\int_{-h}^{h} z\sigma_{\alpha\beta t}^{(s)}\, dz, \quad \mathfrak{M}_{\alpha\beta}^{(s)} = h^{s-2}\int_{-h}^{h}\mu_{\alpha\beta t}^{(s)}\, dz,$$
$$V_{\alpha}^{(s)} = h^{s-2}\int_{-h}^{h}\sigma_{\alpha 3t}^{(s)}\, dz, \qquad \mathcal{F}_{\alpha}^{(s)} = h^{s-4}\psi_{\alpha}^{(s)}. \tag{7.1}$$
$$W^{(s)} = h^{s-1} w^{(s)}.$$

Using (3.1), (3.8), (3.9) and (3.12), one gets

$$
\begin{aligned}
M_{\alpha\beta}^{(s)} &= -\frac{2\,E\,h^3}{3\,(1-\nu^2)}\left[\nu\, W_{,\lambda\lambda}^{(s)}\,\delta_{\alpha\beta} + (1-\nu)\,W_{,\alpha\beta}^{(s)}\right], \\
V_{\alpha}^{(s)} &= -\frac{2\,E\,h^3}{3\,(1-\nu^2)}\,W_{,\lambda\lambda\alpha}^{(s)} + 8\alpha h\,(W_{,\alpha}^{(s)} + \varepsilon_{\alpha\lambda}\mathcal{F}_{\lambda}^{(s)}) - h^{s-1}\int_{-h}^{h}\sigma_{3\alpha}^{*(s)}\,d\zeta, \\
\mathfrak{M}_{\alpha\beta}^{(s)} &= 2h\left[\frac{2\beta\gamma}{\beta+2\gamma}\,\mathcal{F}_{\lambda,\lambda}^{(s)}\,\delta_{\alpha\beta} + (\gamma-\varepsilon)\,\mathcal{F}_{\alpha,\beta}^{(s)} + (\gamma+\varepsilon)\,\mathcal{F}_{\beta,\alpha}^{(s)}\right] - \\
&\quad - h^{s-1}\frac{\beta}{\beta+2\gamma}\int_{-h}^{h}\mu_{33}^{*(s)}\,d\zeta,
\end{aligned}
\tag{7.2}
$$

as well the equations

$$
\begin{aligned}
&V_{\lambda,\lambda}^{(s)} = -\,p h^{s-1}, \\
&\mathfrak{M}_{\lambda\alpha,\lambda}^{(s)} + \varepsilon_{\alpha\lambda}\,V_{\lambda}^{(s)} - \varepsilon_{\alpha\lambda}\,M_{\lambda\mu,\mu}^{(s)} = -\,q_{\alpha}h^{s-1}.
\end{aligned}
\tag{7.3}
$$

For $s = 1$, if we denote

$$
V_{\alpha}^{(1)} = \overline{V}_{\alpha}^{(1)} + \overline{\overline{V}}_{\alpha}^{(1)},
\tag{7.4}
$$

where

$$
\overline{V}_{\alpha}^{(1)} = -\frac{2\,E\,h^3}{3\,(1-\nu^2)}\,W_{,\lambda\lambda\alpha}^{(1)}\,,\quad \overline{\overline{V}}_{\alpha}^{(1)} = 8\,\alpha h\,(W_{,\alpha}^{(1)} + \varepsilon_{\alpha\lambda}\mathcal{F}_{\lambda}^{(1)})\,,
\tag{7.5}
$$

one can write (7.3) in a form that generalizes the wellknown equations of the classical theory:

$$
M_{\lambda\mu,\lambda\mu}^{(1)} + \overline{V}_{\lambda,\lambda}^{(1)} = -\,p\,,\quad \mathfrak{M}_{\lambda\alpha,\lambda}^{(1)} + \varepsilon_{\alpha\lambda}\,\overline{\overline{V}}_{\lambda,\lambda}^{(1)} = -\,q_{\alpha}\,.
\tag{7.6}
$$

Finally, from (6.5 a) to (6.5 d), (6.3) and (7.2) one can obtain the boundary conditions in the case of the free edge, which also generalize the classical results:

$$
\left.
\begin{aligned}
&M_{11}^{(1)} = 0,\; M_{12,2}^{(1)} + V_{1}^{(1)} = 0, \\
&\mathfrak{M}_{11}^{(1)} = 0,\quad \mathfrak{M}_{12}^{(1)} = 0
\end{aligned}
\right\}\ \text{on } x_1 = 0.
\tag{7.7}
$$

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ASUPRA UNEI TEORII MICROPOLARE A ÎNCOVOIERII PLĂCILOR

Rezumat

Folosind un procedeu asemănător celui utilizat în [3], se construieşte o teorie a plăcilor plane, cu ajutorul integrării asimptotice a ecuaţiilor tridimensionale ale teoriei liniare a elasticităţii micropolare.