ON THE BENDING OF ELASTIC PLATES IN ASYMMETRICAL ELASTICITY

BY

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1. Introduction. During the past years, many problems have been investigated within the theory of asymmetrical elasticity with independent local rotations. Among these, the construction of a theory of plates has received considerable attention. The difficulties encountered in attempting to solve this problem are well known from the symmetrical elasticity. Thus it is not surprising that all the known theories have been obtained by methods of approximation which appear to be justified provided the thickness of the plate is sufficiently small. From this point of view the investigators may be divided into two categories. Starting with certain *a priori* hypotheses some authors approach the problem of plates in classical manner. In this category we recall the contributions of E r i n g e n [1] and R e i s s n e r [8]. Others derive the plate equations by using asymptotic methods of integration which are similar to those in symmetrical elasticity. A partial result in this direction is due to G r e e n and N a g h d i [3].

In the present paper we derive a theory of linear bending of plates which is simpler than the theories in [1, 8] and more complete than the first approximation of the plate theory given in [3].

Our method of analysis is based on five hypotheses concerning the displacements and microrotations and the stresses and couple-stresses which enable to express all quantities which characterize the bending by means of three functions of two variables. The first of these, $w(x_1, x_2)$, denotes the deflection of the middle plane of the plate, the others, $\Phi_1(x_1, x_2)$ and $\Phi_2(x_1, x_2)$, specify the local rotations in this plane. By using the L a g r a n g e's variational principle, we establish the system of equilibrium equations (three equations) and natural boundary conditions (four conditions) which completely define the present theory.

More precisely, the content of the paper is the following: in Section 2 we give the notations, the basic equations which we use and we formulate the three-dimensional boundary value problem for elastic plate subjected to bending. The fundamental hypotheses of the theory and their consequences are exposed in Section 3. Section 4 contains the variational derivation of the equilibrium equations and of the boundary conditions. In Section 5 some theorems are established: the necessary and sufficient conditions for the internal energy of the plate to be positive definite (1), the Clapeyron's theorem (2), the Betti's formulae and the reciprocity relations (3), and the uniqueness theorem (4).

2. Notations and basic equations. 2.1. Consider an elastic plate of uniform thickness $2h$ the middle plane of which, in undeformed state, coincides with x_1Ox_2-plane. We denote by $\mathcal{D}$ the domain of the midlle-section and by $\mathcal{C}$ a curve without multiple points which design the boundary of $\mathcal{D}$. We suppose that $\mathcal{D}$ is a regular domain, i. e. a domain for which the divergence theorem is applicable.

The plate is assumed to consist of isotropic centro-symmetric material. The effects of body forces and body couples are disregarded.

Throughout the paper the following notations are used: σ_{ij}, μ_{ij} $(i, j = 1, 2, 3)$ — the components of stress and couple-stress tensors, β_{ij}, φ_{ij} — the infinitesimal strain components, u_i, φ_i — the components of the displacement and microrotation vectors, $\hat{P}_i$, $\hat{M}_i$ — the components of surface tractions and couples, t, n — the tangential and normal directions to an element of the cylindrical boundary of the plate, M_n, M_{nt}, Q_n — the bending moment, the twisting moment and the shearing force parallel to x_3-axis measured per unit length of a section of plate perpendicular to the n direction, $R_{\rho n}$ $(\rho = 1, 2)$ — the component about x_ρ-axes of the couple-stress resultant acting upon the section of plate perpendicular to the n direction, α, β, γ, λ, μ, ε — the elastic constants of the material, D — the classical rigidity, D' — the micropolar rigidity, ε_{ijk}, δ_{ij} — the permutation operator and the Kronecker's delta, $(\dots)$, $[\dots]$ — the symmetric and skew-symmetric parts of a tensor and x_i — the Cartesian co-ordinates. Coma denotes partial differentiation with respect to variable x_i and the repeated indices imply summation.

2.2. The full three-dimensional field equations of the linear asymmetrical elasticity with independent local rotations and with no body forces and body couples are as follows [2, 4, 5]:

— equilibrium equations

$$\sigma_{ij,i} = 0, \quad \mu_{ij,i} + \varepsilon_{jkr}\,\sigma_{kr} = 0, \tag{2.1}$$

— kinematical relations

$$\beta_{ij} = u_{j,i} + \varepsilon_{kji}\,\varphi_k, \quad \varphi_{ij} = \varphi_{j,i}, \tag{2.2}$$

— constitutive equations

$$(2.3)\quad \begin{aligned} \beta_{ij} &= -\frac{\lambda\theta\delta_{ij}}{2\mu(3\lambda + 2\mu)} + \frac{1}{2\mu}\,\sigma_{(ij)} + \frac{1}{2\alpha}\,\sigma_{[ij]}, \quad \theta = \sigma_{11} + \sigma_{22} + \sigma_{33}; \\ \varphi_{ij} &= -\frac{\beta\psi\delta_{ij}}{2\gamma\,(3\beta + 2\gamma)} + \frac{1}{2\gamma}\,\mu_{(ij)} + \frac{1}{2\varepsilon}\,\mu_{[ij]}, \quad \psi = \mu_{11} + \mu_{22} + \mu_{33}; \end{aligned}$$

— boundary conditions

$$\sigma_{ij}n_i = \hat{P}_j, \qquad \mu_{ij}n_i = \hat{M}_j, \qquad n \equiv (n_1, n_2, n_3). \tag{2.4}$$

In what follows our attention will be confined to the case of a plate which is subjected to bending by an arbitrary system of stresses and couple-stresses acting upon the faces $x_3 = \pm h$ and by an arbitrary system of tractions and couples or displacements and microrotations distributed over the cylindrical surface of the plate. Further we consider the surface-stresses and couple-stresses which act upon the faces to be of the form:

$$\hat{P}_i\,(x_1, x_2, \pm h) = \pm \frac{1}{2}\,p(x_1, x_2)\,\delta_{i3}, \tag{2.5}$$

$$\hat{M}_\rho\,(x_1, x_2, \pm h) = \pm \frac{1}{2}\,q_\rho(x_1, x_2), \quad \hat{M}_3(x_1, x_2, \pm h) = 0, \quad (\rho = 1, 2), \tag{2.6}$$

where p and q_ρ are measured per unit area of the middle-surface.

3. Hypotheses and simplified equations. 3.1. We can prove that the full three-dimensional problem of a plate subjected to arbitrary conditions over the cylindrical surface and to conditions of type (2.5) and (2.6) on the faces, can be decomposed into two parts. One in which the displacements and the microrotations have the following properties of symmetry:

$$(\mathrm{S}_1)\quad \begin{aligned} \{u_1, u_2, \varphi_3\}\,(x_1, x_2, -x_3) &= \{u_1, u_2, \varphi_3\}\,(x_1, x_2, x_3), \\ \{\varphi_1, \varphi_2, u_3\}\,(x_1, x_2, -x_3) &= -\{\varphi_1, \varphi_2, u_3\}\,(x_1, x_2, x_3) \end{aligned}$$

and one of the complementary symmetry, namely:

$$(\mathrm{S}_2)\quad \begin{aligned} \{u_1, u_2, \varphi_3\}\,(x_1, x_2, -x_3) &= -\{u_1, u_2, \varphi_3\}\,(x_1, x_2, x_3), \\ \{\varphi_1, \varphi_2, u_3\}\,(x_1, x_2, -x_3) &= \{\varphi_1, \varphi_2, u_3\}\,(x_1, x_2, x_3). \end{aligned}$$

The symmetrics (S_1) characterize the extensional motion of the plate (or the plane state of stresses and couple-stresses) whereas (S_2) specify the flexure (or bending) of the plate.

There exists two ways of treatment of both problems: the exact treatment as three-dimensional problems and the two-dimensional treatment which consist in approximating them by two-dimensional ones.

Our method of investigation is based on the second variant.

In order to establish an approximate theory of bending, when the thickness of the plate is small as compared with its other dimensions and only small deflections are taken into account we introduce some hypotheses. Since the displacements and microrotation fields throughout the plate vary independently, it appears naturally to formulate these hypotheses as follows:

(a) in the middle plane of the plate the macrodeformations are absent. During the bending, the elements of the median section are subjected to microdeformations only,

(b) the points of the plate lying initially on a normal to the middle plane remain on the normal to the middle surface of the plate after bending and the normal suffer no extensions,

(c) there are no local rotations about x_3-axis,

(d) the local rotations do not vary in the direction transverse to the plate,

(e) the normal stress σ_{33} and the couple-stresses μ_{13}, μ_{23} and μ_{33} are small compared with other stresses and couple-stresses respectively and can be disregarded.

We note that the first, the second and the fifth (first part) of these assumptions are similar to those in Kirchhoff's theory of bending in symmetrical elasticity. The additional assumptions which involve the microrotations and the couple-stresses (the second part of (e)) are inserted in order to obtain a simple and easy applicable to concret problems theory.

3.2. Just as in the classical theory of plates, from (a) and (b), we derive that u_3 is independent of x_3, i. e.,

$$u_3(x_1, x_2, x_3) = w(x_1, x_2) \tag{3.1}$$

and

$$u_\rho(x_1, x_2, x_3) = -x_3 w_{,\rho}(x_1, x_2) \quad (\rho = 1, 2). \tag{3.2}$$

The third assumption implies

$$\varphi_3(x_1, x_2, x_3) = 0, \tag{3.3}$$

throughout the plate and, from (d), we deduce that φ_1 and φ_2 must be of the form:

$$\varphi_\rho(x_1, x_2, x_3) = \Phi_\rho(x_1, x_2), \quad (\rho = 1, 2). \tag{3.4}$$

Thus, by assuming (a) — (d), it follows that the displacements u_i and the microrotations φ_i have the properties of symmetry required by (S_2), whereas the equalities (3.1) — (3.4) indicate the explicit form with respect to x_3 of these functions as an immediate consequence of the specified assumptions. By using (a) — (d) and the remaining hypotheses, we can approximate (with certain error, which is not investigated here) the three-dimensional problem of bending involving six unknown functions u_i and φ_i by a two-dimensional one containing only w and Φ_ρ as unknowns.

Next we will investigate in detail the last theory. To this end we start with (3.1) — (3.4) and define the two-dimensional measures of strain $k_{\rho\delta}$, k_ρ and $\varkappa_{\rho\delta}$ in the form:

$$k_{\rho\delta} = k_{\delta\rho} = -w_{,\rho\delta}, \quad k_\rho = w_{,\rho} + \varepsilon_{\rho\delta}\Phi_\rho, \quad \varkappa_{\rho\delta} = \Phi_{\delta,\rho}, \quad (\varepsilon_{\rho\delta} = \varepsilon_{\rho 3 \delta}) \tag{3.5}$$

When (3.1) — (3.4) are inserted into (2.2) and the previous notations are used, we obtain the three-dimesional characteristics of strain as:

$$\beta_{\rho\delta} = x_3 k_{\rho\delta}, \quad \beta_{[\rho 3]} = k_\rho, \quad \varphi_{\rho\delta} = \varkappa_{\rho\delta}. \tag{3.5'}$$

Substituting this result into (2.3), we have

$$\begin{aligned} E x_3 k_{11} &= \sigma_{11} - \nu(\sigma_{22} + \sigma_{33}), & \bar{E}\varkappa_{11} &= \mu_{11} - \bar{\nu}(\mu_{22} + \mu_{33}), \\ E x_3 k_{22} &= \sigma_{22} - \nu(\sigma_{11} + \sigma_{33}), & \bar{E}\varkappa_{22} &= \mu_{22} - \bar{\nu}(\mu_{11} + \mu_{33}), \\ 4\mu\alpha x_3 k_{12} &= 2\alpha\sigma_{(12)} + 2\mu\,\sigma_{[12]}, & 2\gamma\varkappa_{(12)} &= \mu_{(12)}, \\ 2\alpha k_\rho &= \sigma_{[\rho 3]}, & 2\varepsilon\varkappa_{[12]} &= \mu_{[12]}, \end{aligned} \tag{3.5''}$$

where the elastic moduli E, ν $\bar{E}$, $\bar{\nu}$ are [6]:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}, \quad \bar{E} = \frac{\gamma(3\beta + 2\gamma)}{\beta + \gamma}, \quad \bar{\nu} = \frac{\beta}{2(\beta + \gamma)}.$$

In view of (e) we neglect in (3.5″) the terms in σ_{33} and μ_{33} with regard to others. Observing that $\sigma_{[12]} = 0$ (from $(3.5'')_3$) and solving (3.5″) with respect to stresses and couple-stresses we get:

$$\sigma_{\rho\lambda} = \frac{E x_3}{1 - \nu^2}\left[(1 - \nu)\, k_{\rho\lambda} + \nu\delta_{\rho\lambda} k\right], \quad k = k_{11} + k_{22}, \tag{3.6}$$

$$\sigma_{[\rho 3]} = 2\alpha k_\rho, \tag{3.7}$$

$$\mu_{\rho\rho} = \frac{\bar{E}}{1 - \bar{\nu}^2}\left[(1 - \bar{\nu})\,\varkappa_{\rho\rho} + \bar{\nu}\varkappa\right], \quad \varkappa = \varkappa_{11} + \varkappa_{22}, \qquad \mu_{(12)} = 2\gamma\varkappa_{(12)}, \quad \mu_{[12]} = 2\varepsilon\varkappa_{[12]}. \tag{3.8}$$

The remaining stresses and couple-stresses can be obtained by using the equilibrium equations.

Thus, integrating the first two equations (2.1) with respect to the variable x_3 and using (3.6) and (2.5) as well as (3.5) we obtain

$$\sigma_{3\rho} = -\frac{E(h^2 - x_3^2)}{2(1 - \nu^2)}\Delta w_{,\rho} \quad (\Delta w = w_{,11} + w_{,22}). \tag{3.9}$$

To find the shearing stresses $\sigma_{\rho 3}$, we insert (3.9) into (3.7); the result is:

$$\sigma_{\rho 3} = -\frac{E(h^2 - x_3^2)}{2(1 - \nu^2)}\, \Delta w_{,\rho} + 4\alpha(w_{,\rho} + \varepsilon_{\rho\lambda}\, \Phi_\lambda)\ . \tag{3.10}$$

In a like manner, by substituting (3.7) into the first two equations of equilibrium involving the couple-stresses, integrating them with respect to x_3 and observing that $\mu_{3\rho}$ are uneven functions of x_3, we find

$$\mu_{3\rho} = -x_3[\mu_{\delta\rho,\delta} + 4\alpha\ \varepsilon_{\rho\lambda}(w_{;\lambda} + \varepsilon_{\lambda\mu}\ \Phi_\mu)]\ . \tag{3.11}$$

Since by (3.3) and (3.4) we have $\varphi_{[\rho 3]} = 0$, we conclude from (2.3) that $\mu_{[\rho 3]} = 0$, or equivalently :

$$\mu_{\rho 3} = \mu_{3\rho}\ . \tag{3.12}$$

Thus, we have obtained the explicit form of all the stresses and couple-stresses which characterize the present theory.

3.3 Consider now sections of unit length perpendicular to the middle plane of the plate and define with respect to these sections, the following quantities :

$$M_{\rho\lambda} = \int_{-h}^{+h} x_3\, \sigma_{\rho\lambda}\, dx_3\ ,\quad Q_\rho = \int_{-h}^{+h} \sigma_{\rho 3} dx_3\ ,\quad R_{\rho\lambda} = \int_{-h}^{+h} \mu_{\rho\lambda} dx_3\ . \tag{3.13}$$

We note that the first six of these expressions are identical with the bending moments, the twisting moments and the shearing stress resultants known from the classical theory of plates. The last four quantities define the resultants of the couple-stresses across the thickness of the plate.

Now, let us point out the connection between (3.13) and the functions w and Φ_ρ.

To this end we multiply (3.6) by x_3 and integrate between $x_3 = -h$ and $x_3 = +h$. Taking into account (3.13) and (3.5), we get.

$$M_{\rho\lambda} = -D[(1 - \nu)\, w_{,\rho\lambda} + \nu\delta_{\rho\lambda}\Delta w]. \tag{3.14}$$

Further, the integration of (3.10) and (3.8) across the thickness of the plate between the same limits as before, together with (3.13) leads to the following supplementary equalities :

$$Q_\rho = -D\Delta w_{,\rho} + 8\alpha h(w_{,\rho} + \varepsilon_{\rho\lambda}\Phi_\lambda\ ), \tag{3.15}$$

$$\begin{aligned} R_{\rho\rho} &= D'[(1 - \bar{\nu})\, \Phi_{\rho,\rho} + \bar{\nu}\Phi],\quad \Phi = \Phi_{1;1} + \Phi_{2;2}; \\ R_{\rho\lambda} &= 2h(\gamma + \varepsilon)\, \Phi_{\lambda,\rho} + 2h(\gamma - \varepsilon)\, \Phi_{\rho,\lambda}\ ,\ (\rho \neq \lambda), \end{aligned} \tag{3.16}$$

where the constants D and D' have the values :

$$D = \frac{2h^3E}{3(1 - \nu^2)}\ ,\ D' = \frac{2h\bar{E}}{1 - \bar{\nu}^2}\ . \tag{3.17}$$

We define D' as *the micropolar rigidity* of the plate.

Let us consider now an elementary section having the normal $n(n_\rho)$. The stress moments and the couple-stress resultants acting on this section are given by :

$$\begin{aligned} M_n &= M_{\rho\lambda} n_\rho n_\lambda\ , \quad M_{nt} = \varepsilon_{\rho\lambda}\, M_{\rho\mu}\ n_\lambda n_\mu\ , \\ Q_n &= Q_\rho n_\rho\ , \quad R_{\rho n} = R_{\delta\rho} n_\delta \end{aligned} \tag{3.18}$$

These relations may be expressed in terms of actual functions by using (3.14) — (3.16). Then, we have :

$$\begin{aligned} M_n &= -D[\nu\Delta w + (1 - \nu)\, (w_{,11} n_1^2 + 2w_{,12} n_1 n_2 + w_{,22}\, n_2^2)], \\ M_{nt} &= -D(1 - \nu)\, [w_{,12}(n_1^2 - n_2^2) + (w_{,22} - w_{,11})\, n_1 n_2], \\ Q_n &= -D\Delta w_{,n} + 8\alpha h w_{,n} + 8\alpha h(\Phi_2 n_1 - \Phi_1 n_2), \\ R_{1n} &= D'\, (\Phi_{1,1} + \bar{\nu}\Phi_{2,2}) n_1 + 2h[(\gamma + \varepsilon)\Phi_{2,1} + (\gamma - \varepsilon)\Phi_{1,2}] n_2, \\ R_{2n} &= 2h[(\gamma + \varepsilon)\, \Phi_{1,2} + (\gamma - \varepsilon)\Phi_{1;2}] n_1 + D'[\Phi_{2,2} + \bar{\nu}\Phi_{1,1}] n_2 \end{aligned} \tag{3.19}$$

where $n_1 = \cos 0$, $n_2 = \sin 0$, 0 — being the angle of the external normal n with the x_1 — axis measured in clockwise direction and $(\ldots)_{,n} = (\ldots)_{,\rho}\, n_\rho$.

4. Equilibrium equations and boundary conditions. In order to obtain the differential equations which the unknown functions w and Φ_ρ must satisfy and make clear the form of the corresponding boundary conditions, we shall use the principle of virtual displacements. To this end we first establish the expression fot the internal energy of the bent plate when the hypotheses (a) — (c) are taken into consideration.

In asymmetrical elasticity with independent local rotations the internal energy in an infinitesimal element dv of an elastic solid is given in terms of stresses and couple-stresses as follows :

$$\begin{aligned} 2dV = \Big(&-\frac{\nu}{E}\, 0^2 + \frac{1 + \nu}{E}\, \sigma_{(ij)}\, \sigma_{(ij)} + \frac{1}{2\alpha}\, \sigma_{[ij]}\, \sigma_{[ij]} - \\ &-\frac{\bar{\nu}}{\bar{E}} K^2 + \frac{1 + \bar{\nu}}{\bar{E}}\, \mu_{(ij)}\mu_{(ij)} + \frac{1}{2\varepsilon}\mu_{[ij]}\ \mu_{[ij]}\Big)\, dv. \end{aligned} \tag{4.1}$$

In virtue of the assumptions (b) — (c) we shall disregard in (4.1) the terms which contain the symmetrical parts of the shearing stresses σ_{13}, σ_{23}, the stress σ_{33} and the components (see (3.12)), $\mu_{\rho 3}$, $\mu_{3\rho}$ and μ_{33} of the couple-stress tensor. Thus, using that $\sigma_{12} = \sigma_{21}$ the expression of the elementary energy becomes :

$$2dW = \left\{ \frac{1}{E} \left[\sigma_{11}^2 + \sigma_{22}^2 - 2\nu\sigma_{11}\sigma_{22} + 2(1+\nu)\,\sigma^2_{(12)} \right] + \frac{1}{\varkappa}\left(\sigma^2_{[13]} + \sigma^2_{[23]} \right) + \right. \tag{4.2}$$
$$\left. + \frac{1}{\bar E}\left[\mu_{11}^2 + \mu_{22}^2 - 2\bar\nu\mu_{11}\mu_{22} + 2(1+\bar\nu)\mu^2_{(12)} \right] + \frac{1}{\varepsilon}\mu^2_{[12]} \right\} dS\,dx_3$$

where dS denotes the element of the median surface. The total internal energy of the plate is then obtained by integrating (4.2) over the domain $\mathcal{D}\times[-h, +h]$. Substituting the values (3.6) — (3.8) of the stresses and couple-stresses and remembering the definitions (3.5) and (3.17), we get :

$$\mathcal{W} = \frac{1}{2}\int_{\mathcal{D}} \{ D[(\Delta w)^2 - 2(1-\nu)(w_{,11}w_{,22} - w^2_{,12})] + 8\varkappa h[(w_{,1} + \tag{4.3}$$
$$+\Phi_2)^2 + (w_{,2} - \Phi_1)^2] + D'[(\Phi_{1,1} + \Phi_{2,2})^2 - 2(1-\bar\nu)\Phi_{1,1}\Phi_{2,2} + \frac{1}{2}$$
$$(1-\bar\nu)(\Phi_{1,2} + \Phi_{2,1})^2] + 2h\varepsilon(\Phi_{2,1} - \Phi_{1,2})^2 \}\, dS.$$

According to Lagrange's variational principle the change of the internal energy of the plate corresponding to the infinitely small variations δw, $\delta w_{,n}$, $\delta\Phi_\rho$ must be equal to the work done by surface stresses and couple-stresses during the assumed virtual displacements. In our case we must have

$$\delta\mathcal{W} = \delta\mathcal{A}_1 + \delta\mathcal{A}_2 \tag{4.4}$$

where, in view if (2.4) — (2.6), $\delta\mathcal{A}_1$ and $\delta\mathcal{A}_2$ are :

$$\delta\mathcal{A}_1 = \int_{\mathcal{D}} (p\,\delta w + q_\rho\,\delta\Phi_\rho)\, dS, \tag{4.5}$$
$$\delta\mathcal{A}_2 = \int_{\mathcal{C}_p} [-\hat M_n \delta w_{,n} + (\hat Q_n - \hat M_{nt,s})\delta w + \hat R_{\rho n}\,\delta\Phi_\rho] ds,$$

ds — being the elementary length of $\mathcal{C}_p$ and :

$$\hat M_n = \int_{-h}^{+h} x_3 \hat P_\rho n_\rho\, dx_3, \quad \hat Q_n = \int_{-h}^{+h} \hat P_3 dx_3, \quad \hat M_{nt} = \int_{-h}^{+h} x_3\, \varepsilon_{\rho\lambda} \hat P_\rho n_\lambda\, dx_3,$$
$$\hat R_{\rho n} = \int_{-h}^{+h} \hat M_\rho\, dx_3, \qquad (\ldots)_{,s} = \partial(\ldots)/\partial s\,.$$

When the expressions (4.5) for $\delta\mathcal{A}_1$ and $\delta\mathcal{A}_2$ are inserted into (4.4) and the variation of (4.3) is taken, we obtain the following variational equation :

$$\int_{\mathcal{D}} \{ D[\Delta w\,\delta(\Delta w) - (1-\nu)(w_{,11}\delta w_{,22} + w_{,22}\delta w_{,11} - 2w_{,12}\delta w_{,12}) +$$
$$8\varkappa h[(w_{,1} + \Phi_2)\,\delta(w_{,1} + \Phi_2) + (w_{,2} - \Phi_1)\,\delta(w_{,2} - \Phi_1)] +$$
$$D'[(\Phi_{1,1} + \Phi_{2,2})\,\delta(\Phi_{1,1} + \Phi_{2,2}) - (1-\bar\nu)(\Phi_{1,1}\delta\Phi_{2,2} + \Phi_{2,2}\delta\Phi_{1,1} + \tag{4.6}$$
$$\frac{1}{2}(1-\bar\nu)(\Phi_{1,2} + \Phi_{2,1})\,\delta(\Phi_{1,2} + \Phi_{2,1}) + 2\varepsilon h(\Phi_{1,2} - \Phi_{2,1})\delta(\Phi_{1,2} - \Phi_{2,1}) -$$
$$- p\delta w - q_\rho\delta\Phi_\rho\}\, ds - \int_{\mathcal{C}} [-\hat M_n\,\delta w_{,n} + (\hat Q_n - \hat M_{nt,s})\,\delta w + \hat R_{\rho n}\,\delta\Phi_\rho\,] ds = 0.$$

We transform this equation and use the fact $\mathcal{D}$ is a regular domain. Without performing the elementary operations which invole the divergence Theorem and the integrations by parts (for details see, e. g. [7], p.88), we present only the final result in the form :

$$\int_{\mathcal{D}} [D\Delta\Delta w - 8\varkappa h\Delta w - 8\varkappa h(\Phi_{2,1} - \Phi_{1,2}) - p]\,\delta w\, dS + \tag{4.7}$$
$$\{-2h[(\frac{\bar E}{2(1+\bar\nu)} + \varepsilon)\,\Delta\Phi_\rho + (\frac{\bar E}{2(1-\bar\nu)} - \varepsilon)\Phi_{,\rho} + 4\varkappa\,\varepsilon_{\rho\lambda} w_{,\lambda} - 4\varkappa\Phi_\rho] -$$
$$- q_\rho\}\,\delta\Phi_\rho\; dS + \int_{\mathcal{C}} \{-D\Delta w_{,n} + 8\varkappa h(w_{,n} + \Phi_2 n_1 - \Phi_1 n_2) + D(1-\nu)\,[w_{,12}$$
$$(n_2^2 - n_1^2) + (w_{,11} - w_{,22})\, n_1 n_2]_{,s} - \hat V_n\}\,\delta w\, ds + \int_{\mathcal{C}} \{ D[\nu\Delta w + (1+\nu)$$
$$(w_{,11} n_1^2 + w_{,22} n_2^2 + 2w_{,12} n_1 n_2] + \hat M_n \}\,\delta w_{,n} ds + \int_{\mathcal{C}} \{D'(\Phi_{1,1} + \bar\nu\Phi_{2,2})\, n_1 +$$
$$+ 2h[(\gamma + \varepsilon)\,\Phi_{1,2} + (\gamma - \varepsilon)\Phi_{2,1}]\, n_2 - \hat R_{1n}\}\,\delta\Phi_1 ds + \int_{\mathcal{C}} \{2h[(\gamma + \varepsilon)\Phi_{2,1} +$$
$$+ (\gamma - \varepsilon)\,\Phi_{1,2}]\, n_1 + D'(\Phi_{2,2} + \bar\nu\Phi_{11})\, n_2 - \hat R_{2n}\}\;\delta\Phi_2\, ds = 0$$

where the notation $\hat V_n$ stands for generalized shearing force : $\hat V_n = \hat Q_n$

$-\widehat{M}_{n\rho s}$. Remembering that δw, $\delta w_{,n}$, $\delta\Phi_1$ and $\delta\Phi_2$ are arbitrary small quantities and using the standard techniques of the calculus of variations we conclude that Eq. (4.7) will be satisfied if the following two sets of equations are fulfilled:

$$D\Delta\Delta w - 8\varkappa h\Delta w - 8\varkappa h(\Phi_{2,1} - \Phi_{1,2}) = p(x_1, x_2), \tag{4.8}$$

$$-2h\left|\left(\frac{\bar{E}}{2(1+\bar{\nu})} + \varepsilon\right)\Delta\Phi_\rho + \left(\frac{\bar{E}}{2(1-\bar{\nu})} - \varepsilon\right)\Phi_{\lambda,\lambda\rho} + 4\varkappa\varepsilon_{\rho\lambda} w_{,\lambda} - 4\varkappa\Phi_\rho\right| = q_\rho(x_1, x_2),$$

$$-D\Delta w_{,n} + 8\varkappa h(w_{,n} + \Phi_2 n_1 - \Phi_1 n_2) + D(1-\nu)\,[w_{,12}(n_2^2 - n_1^2) +$$
$$+ (w_{,11} - w_{,22})\, n_1 n_2]_{,s} = \widehat{V}_n,$$

$$-D[\nu\Delta w + (1+\nu)(w_{,11}n_1^2 + w_{,22}n_2^2 + 2w_{,12}\, n_1 n_2)] = \widehat{M}_n\,.$$

$$D'(\Phi_{1,1} + \bar{\nu}\Phi_{2,2})\, n_1 + 2h[(\gamma+\varepsilon)\,\Phi_{1,2} + (\gamma-\varepsilon)\Phi_{2,1}]n_2 = \widehat{R}_{1n}, \tag{4.9}$$
$$2h[(\gamma+\varepsilon)\,\Phi_{2,1} + (\gamma-\varepsilon)\,\Phi_{1,2}]\, n_1 + D'(\Phi_{2,2} + \bar{\nu}\Phi_{1,1})\, n_2 = \widehat{R}_{2n}.$$

The equalities (4.8) can be regarded as equilibrium equations of the plate. Otherwise, they can be obtained by integrating Eqs. (2.1) between $x_3 = -h$ and $x_3 = +h$ and taking into account the definitions (3.13). The second set of equations characterizes the boundary conditions which the unknown functions must satisfy. Thus, (4.8) and (4.9) completely define the functions w and Φ_ρ. When these functions are known, we obtain by means of (3.5) — (3.12) the stress and couple-stress fields throughout the plate.

Let us observe that, in view of (3.19), the statical boundary conditions (4.9) can be written as:

$$V_n = \widehat{V}_n\,,\qquad M_n = \widehat{M}_n\,,\qquad R_{\rho n} = \widehat{R}_{\rho n}\,,\qquad (\rho = 1, 2) \text{ on } \mathscr{C}.$$

In this form, (4.9) appears as an extension of Kirchhoff's boundary conditions from symmetrical elasticity. In fact, if the effect of local rotations is not taken into consideration and $\varkappa = \beta = \gamma = \varepsilon = 0$ in (4.8) and (4.9), we find again the well-known equations of Kirchhoff's theory of bending. A discussion of possible boundary conditions of the present theory, when the classical theory is taken as comparison, is given in the Table from the following page.

Remark. Let us suppose, for the sake of simplicity, that $p = q_\rho = 0$. Then, the system of Eqs. (4.8) can be written as

$$\frac{1}{h}\, D\Delta\Delta w - 8\varkappa\Delta w - 8\varkappa(\Phi_{2,1} - \Phi_{1,2}) = 0,$$

$$\left(\frac{\bar{E}}{2(1+\bar{\nu})} + \varepsilon\right)\Delta\Phi_\rho + \left(\frac{\bar{E}}{2(1-\bar{\nu})} - \varepsilon\right)\Phi_{\lambda,\lambda\rho} + 4\varkappa\varepsilon_{\rho\sigma} w_{,\sigma} - 4\varkappa\,\Phi_\rho = 0$$

where the value of D is that given by (3.17).

The term $\frac{1}{h}\, D\,\Delta\Delta\, w$ in the first equation is an „edge effect" term. If $\varkappa \neq O(h^2)$, by omitting this term we obtain the system or differential equations

$$-8\varkappa\,(\Delta w^0 + \Phi^0_{2,1} - \Phi^0_{1,2}) = 0, \tag{4.10}$$
$$\left(\frac{\bar{E}}{2(1+\bar{\nu})} + \varepsilon\right)\Delta\Phi^0_\rho + \left(\frac{\bar{E}}{2(1-\bar{\nu})} - \varepsilon\right)\Phi^0_{\lambda,\lambda\rho} + 4\varkappa\varepsilon_{\rho\sigma}\, w_{,\sigma} - 4\varkappa\Phi^0_\rho = 0$$

which characterizes the „interior plate problem" of zero-th order. At the same time (4.10) can by regarded as equilibrium equations of an elastic membrane the shape of which is identical with the median section of the considered plate. From this point of view, Eqs. (4.10) are equivalent to those given in [3] for the first approximations of the micropolar plate theory.

Classical theory		Present theory
Equations of equilibrium (in $\mathscr{D}$)		
$D\Delta\Delta w = p$		The system (4.8)
Boundary conditions (on $\mathscr{C}$)		
$w = \widehat{w}$, $w_{,n} = \widehat{u}$	or	$w = \widehat{w}$, $w_{,n} = \widehat{u}$, $\Phi_\rho = \widehat{\Phi}_\rho$;
		$w = \widehat{w}$, $w_{,n} = \widehat{u}$, $R_{\rho n} = \widehat{R}_{\rho n}$;
$w = \widehat{w}$, $M_n = \widehat{M}_n$	or	$w = \widehat{w}$, $M_n = \widehat{M}_n$, $\Phi_\rho = \widehat{\Phi}_\rho$
		$w = \widehat{w}$, $M_n = \widehat{M}_n$, $R_{\rho n} = \widehat{R}_{\rho n}$;
$V_n = \widehat{V}_n$, $M_n = \widehat{M}_n$	or	$V_n = \widehat{V}_n$, $M_n = \widehat{M}_n$, $\Phi_\rho = \widehat{\Phi}_\rho$,
		$V_n = \widehat{V}_n$, $M_n = \widehat{M}_n$, $R_{\rho n} = \widehat{R}_{\rho n}$

5. Some Theorems. In this Section we establish some theorems which are important for applications. The procedures which we use are similar to those in [1].

5.1. Theorem 1. *The necessary and sufficient conditions for the internal energy of the plate subjected to bending to be positive definite are:*

$$\frac{E}{1+\nu} > 0,\quad \frac{E}{1-\nu} > 0,\quad \frac{\bar{E}}{1+\bar{\nu}} > 0,\quad \frac{\bar{E}}{1-\bar{\nu}} > 0,\quad \varkappa > 0,\quad \varepsilon > 0. \tag{5.1}$$

Proof. As we have seen in Sec. 4, the elementary energy of the plate is given in terms of nonvanishing stresses and couple-stresses by (4.2). By using the constitutive equations (3.5) and integrating (4.2) between $-h$ and $+h$ we obtain the internal energy of the plate measured per unit area of the median surface, in the form

$$2\mathcal{V} = \frac{2h^3}{3}\,\frac{E}{1-\nu^2}\,[k_{11}^2 + k_{22}^2 + 2\nu\, k_{11} k_{22} + 2(1-\nu)\, k_{12}^2] +$$

$$2h\,\frac{\bar{E}}{1-\bar{\nu}^2}\,[\varkappa_{11}^2 + \varkappa_{22}^2 + 2\bar{\nu}\,\varkappa_{11}\varkappa_{22} + 2(1-\bar{\nu})\,\varkappa_{(12)}^2] + \tag{5.2}$$

$$8\alpha\, h\,(k_1^2 + k_2^2) + 8\varepsilon\, h\, \varkappa_{[12]}^2 \,.$$

Consider (5.2) as a quadratic form, $2\mathcal{V} = a_{ij}\,\xi_i\,\xi_j$ in the nine-dimensional space of variables

$$\xi_1 = k_{11},\quad \xi_2 = k_{22},\quad \xi_3 = k_{12},\quad \xi_4 = \varkappa_{11},\quad \xi_5 = \varkappa_{22},$$
$$\xi_6 = \varkappa_{(12)},\quad \xi_7 = k_1,\quad \xi_8 = k_2,\quad \xi_9 = \varkappa_{[12]},$$

with

$$a_{11} = a_{22} = \frac{aE}{1-\nu^2},\quad a_{12} = a_{21} = \frac{a\nu E}{1-\nu^2},\quad a_{33} = \frac{2aE}{1+\nu},$$

$$a_{44} = a_{55} = 2h\,\frac{\bar{E}}{1-\bar{\nu}^2},$$

$$a_{45} = a_{54} = \frac{2h\bar{\nu}\bar{E}}{1-\bar{\nu}^2},\quad a_{66} = \frac{4h\bar{E}}{1+\bar{\nu}},\quad a_{77} = a_{88} = 8\alpha\, h,\quad a_{99} = 8\varepsilon\, h,$$

$$a = \frac{2h^3}{3},$$

and all other $a_{ij} = 0$.

In order that $a_{ij}\,\xi_i\,\xi_j$ be a positive definite quadratic form for all $\xi_j (j = 1, 2, \ldots, 9)$ it is necessary and sufficient that all the principal determinants of the matrix (a_{ij}) be essentially positive [9], p. 40. These conditions imply precisely the inequalities (5.1) and the proof is complete.

5.2 Theorem 2. (*Clapeyron's theorem*). *If the plate occupies an equilibrium configuration under a given system of surface forces and couples, then the internal energy is equal to one-half of the work of the external loads.*

Proof. To prove the theorem we start with the expression of the whole mechanical work of the external loads:

$$\int_{\mathcal{S}} (p\,\delta w + q_\rho\,\delta\Phi_\rho)\, dS + \int_{\mathcal{C}} (-\hat{M}_n\, w_{,n} + \hat{V}_n\, w + \hat{R}_{\rho n}\,\Phi_\rho)\, ds. \tag{5.3}$$

By integrating the equations of equilibrium (2.1) between $-h$ and $+h$, using the definitions (3.13) and taking into account (2.5), (2.6), (3.7) and (3.11), we deduce:

$$0 = M_{11,1} + M_{21,2} - Q_1',\qquad 0 = M_{12,1} + M_{22,2} - Q_2';\quad Q_\rho' = \int_{h}^{+h} \sigma_{3\rho}\, dx_3$$

$$p = -(Q_{1,1}' + Q_{2,2}') - 8\alpha h\,(w_{,1} + \Phi_2)_{,1} - 8\alpha h\,(w_{,2} - \Phi_1)_{,2}\,, \tag{5.3'}$$

$$q_\rho = -[R_{\delta\rho\delta} + 8\alpha\, h\varepsilon_{\rho\lambda}\,(w_{,\lambda} + \varepsilon_{\lambda\mu}\,\Phi_\mu)].$$

We multiply (5.3′) by $w_{,1}$, $w_{,2}$, w and Φ_ρ respectively and consider the integral

$$I = \int_{\mathcal{S}} [-(M_{\rho\delta,\rho} - Q_\delta')\, w_{,\delta} + pw + q_\rho\,\Phi_\rho]\, dS.$$

We have

$$I = \int_{\mathcal{S}} (pw + q_\rho\Phi_\rho)\, dS\,. \tag{$*$}$$

On the other hand, replacing the integrands by their expressions (5.3′), we can write the integral I as follows:

$$I = -\int_{\mathcal{S}} \{[M_{11} w_{,1} + M_{12}\, w_{,2} + Q_1 w + R_{11}\Phi_1 + R_{12}\Phi_2]_{,1}$$

$$M_{21}\, w_{,1} + M_{22} w_{,2} + Q_2 w + 8\alpha h w_{,2} + R_{21}\Phi_1 + R_{22}\Phi_2]_{,2}$$

$$- M_{11} w_{,11} - 2M_{12}\, w_{,12} - M_{22}\, w_{,22} + R_{11}\,\Phi_{1,1} + R_{12}\Phi_{2,1} + R_{21}\Phi_{1,2} +$$

$$R_{22}\Phi_{2,2} + 8\alpha h\, [(w_{,2} - \Phi_1)^2 + (w_{,1} + \Phi_2)^2]\}\, dS.$$

When we use the equalities

$$w_{,1} = w_{,n}\, n_1 - w_{,s} n_2\,,\qquad w_{,2} = w_{,n}\, n_2 + w_{,s}\, n_1\,,$$

the divergence theorem as well as (3.14)–(3.16), (4.3) and (4.9), we find

$$I = -\int_{\mathcal{C}} (-\hat{M}_n\, w_{,n} + \hat{V}_n w + \hat{R}_{\rho n}\,\Phi_\rho) + 2\int_{\mathcal{S}} \mathcal{V} dS.$$

As we have, ($*$), we conclude after making comparison with (5.3) that

$$\int_{\mathcal{S}} (pw + q_\rho\,\Phi_\rho)\, dS + \int_{\mathcal{C}} (-\hat{M}_n w_{,n} + \hat{V}_n w + \hat{R}_{\rho n}\,\Phi_\rho)\, ds = 2\int_{\mathcal{S}} \mathcal{V} dS,$$

i. e. the content of our theorem.

5.3. Betti's reciprocity relations. Let us introduce the column vectors $\mathbf{F} \equiv (p, q_1, q_2)$ and $\mathfrak{L}\mathbf{w} \equiv (\mathfrak{L}_1\mathbf{w}, \mathfrak{L}_2\mathbf{w}, \mathfrak{L}_3\mathbf{w})$ where

$$\mathfrak{L}_1\mathbf{w} \equiv D\Delta\Delta w - 8\alpha h\Delta w - 8\alpha h(\Phi_{2,1} - \Phi_{1,2}),$$

$$\mathfrak{L}_{\rho+1}\mathbf{w} \equiv -2h\left\{\left(\frac{\bar{E}}{2(1+\bar{\nu})} + \varepsilon\right)\Delta\Phi_\rho + \left(\frac{E}{2(1-\bar{\nu})} - \varepsilon\right)\Phi_{,\rho} + 4\alpha\,\varepsilon_{\rho\delta}\,w_{,\delta} - 4\alpha\Phi_\rho\right\}.$$

Then, the equilibrium Eqs. (4.8) can be written in vectorial form as

$$\mathfrak{L}\mathbf{w} = \mathbf{F} \tag{5.4}$$

We consider now an arbitrary vector function $v \equiv (v, \psi_1, \psi_2)$ with $v \in C^4(\mathscr{D}) \cap C^3(\mathscr{D} + \mathscr{C})$, $\psi_\rho \in C^2(\mathscr{D}) \cap C^1(\mathscr{D} + \mathscr{C})$ and define the bilinear form:

$$\begin{aligned} 2\mathscr{U}(\mathbf{v}, \mathbf{w}) = {} & D\{k_{11}(\mathbf{v})k_{11}(\mathbf{w}) + k_{22}(\mathbf{v})\,k_{22}(\mathbf{w}) + \nu[k_{11}(\mathbf{v})k_{22}(\mathbf{w}) + \\ & + k_{22}(\mathbf{v})\,k_{11}(\mathbf{w})] + \\ & 2(1-\nu)\,k_{12}(\mathbf{v})\,k_{12}(\mathbf{w})\} + D'\{\varkappa_{11}(\mathbf{v})\varkappa_{11}(\mathbf{w}) + \varkappa_{22}(\mathbf{v})\,\varkappa_{11}(\mathbf{w}) + \\ & \bar{\nu}[\varkappa_{11}(\mathbf{v})\varkappa_{22}(\mathbf{w}) + \varkappa_{22}(\mathbf{v})\,\varkappa_{11}(\mathbf{w})] + (1-\bar{\nu})\,\varkappa_{(12)}(\mathbf{v})\,\varkappa_{(12)}(\mathbf{w})\} + \\ & 8\alpha h[k_1(\mathbf{v})k_1(\mathbf{w}) + k_2(\mathbf{v})k_2(\mathbf{w})] + 8\varepsilon h\varkappa_{[12]}(\mathbf{v})\,\varkappa_{[12]}(\mathbf{w}) \end{aligned} \tag{5.5}$$

and the expression

$$N\mathbf{w}.P\mathbf{v} = -M_n(\mathbf{v})\,w_{,n} + V_n(\mathbf{v})w + R_{\rho n}(\mathbf{v})\Phi_\rho \tag{5.6}$$

where $M_n(\mathbf{v}), \ldots, R_{\rho n}(\mathbf{v})$ are given by (3.18) where $\mathbf{w}$ is substituted by $\mathbf{v}$.

We note that

$$\mathscr{U}(\mathbf{v}, \mathbf{w}) = \mathscr{U}(\mathbf{w}, \mathbf{v}) \tag{5.7}$$

and that $\mathscr{U}(\mathbf{w}, \mathbf{w})$ coincides with the density of internal energy of the plate, i. e., $\mathscr{U}(\mathbf{w}, \mathbf{w}) = \mathscr{U}(\mathbf{w})$ (see (5.2)).

Integrating by parts and using (5.4) — (5.6), we easily obtain the first Betti's formula:

$$\int_{\mathscr{D}} \mathbf{v}\mathfrak{L}\mathbf{w}\,dS = 2\int_{\mathscr{D}} \mathscr{U}(\mathbf{v}, \mathbf{w})\,dS - \int_{\mathscr{C}} N\mathbf{v}.P\mathbf{w}\,ds, \tag{5.8}$$

where

$$\mathbf{v}\mathfrak{L}\mathbf{w} = v\mathfrak{L}_1\mathbf{w} + \psi_\rho\mathfrak{L}_{\rho+1}\mathbf{w}.$$

Setting $\mathbf{v} = \mathbf{w}$ in (5.8), we deduce (the second Betti's formula)

$$\int_{\mathscr{D}} \mathbf{w}\mathfrak{L}\mathbf{w}\,dS = 2\int_{\mathscr{D}} \mathscr{U}(\mathbf{w})\,dS - \int_{\mathscr{C}} N\mathbf{w}.P\mathbf{w}\,ds. \tag{5.9}$$

By changing $\mathbf{v}$ with $\mathbf{w}$ in (5.4) and substracting the obtained results, together with (5.7), we get (the third Betti's formula):

$$\int_{\mathscr{D}} (\mathbf{v}\mathfrak{L}\mathbf{w} - \mathbf{w}\mathfrak{L}\mathbf{v})\,dS = \int_{\mathscr{C}} [N\mathbf{w}.P(\mathbf{v}) - N\mathbf{v}.P(\mathbf{w})]\,ds. \tag{5.10}$$

From this formula we can obtain the reciprocity relations:

To this end, we suppose that the plate is subjected to bending by two systems of loads:

a) the first system: p', q'_ρ, V'_n, M'_n, $R'_{\rho n}$ which produces throughout the plate the „stresses" M'_{11}, M'_{12}, $R'_{11}, \ldots, R'_{21}$ and the „displacement" $\mathbf{w}' \equiv (w', \Phi'_1, \Phi'_2)$ and

b) the second system: p'', q''_ρ, V''_n, M''_n, $R''_{\rho n}$ which gives rise to the „stresses" M''_{11}, M''_{12}, $R''_{11}, \ldots, R''_{21}$ and the „displacement" $\mathbf{w}'' \equiv (w'', \Phi''_1, \Phi''_2)$.

In both cases we consider the plate to occupy equilibrium configurations. Therefore, we have:

$$\mathfrak{L}_1\mathbf{w}' = p',\ \mathfrak{L}_{\rho+1}\mathbf{w} = q_\rho,\ \mathfrak{L}_1\mathbf{w}'' = p'',\ \mathfrak{L}_{\rho+1}\mathbf{w}'' = q''_\rho.$$

Inserting these equalities into the third Betti's formula (5.10) and using (5.6), where we take $\mathbf{v} = \mathbf{w}''$ and $\mathbf{w} = \mathbf{w}''$, we get

$$\int_{\mathscr{C}} (p''w' + q''_\rho\Phi'_\rho)\,dS + \int_{\mathscr{D}} (-M''_n w'_{,n} + V''_n w' + R''_{\rho n}\Phi'_{\rho n})\,ds =$$

$$= \int_{\mathscr{C}} (p'w'' + q'_\rho\,\Phi''_\rho)\,dS + \int_{\mathscr{D}} (-M'_n\,w''_{,n} + V'_n\,w'' + R'_{\rho n}\,\Phi_{\rho n})\,ds.$$

Thus, we have obtained the reciprocity relations in the present theory.

Theorem (Uniqueness). *Let the conditions (5.1) be satisfied in the domain $\mathscr{D}$ with the boundary $\mathscr{C}$. Then there exists at most one field M_{11}, M_{12}, $R_{11}, \ldots, R_{21}$ arising from the regular displacement and microrotation fields satisfying (5.4) and the boundary conditions:*

$$\begin{aligned} & w = \hat{w},\ w_{,n} = \hat{u},\ \Phi_\rho = \hat{\Phi}_\rho \quad (\rho = 1, 2) \quad \text{on } \mathscr{C}_1 \\ & V_n = \hat{V}_n,\ M_n = \hat{M}_n,\ R_{\rho n} = \hat{R}_{\rho n} \text{ on } \mathscr{C}_2,\ \mathscr{C}_1 \cup \mathscr{C}_2 = \mathscr{C},\ \mathscr{C}_1 \cap \mathscr{C}_2 = \varnothing \end{aligned} \tag{5.11}$$

Proof. Let $\mathbf{w}' \equiv (w', \Phi'_1, \Phi'_2)$ and $\mathbf{w}'' \equiv (w'', \Phi''_1, \Phi''_2)$ be two distinct solutions of (5.4) satisfying (5.11) on $\mathscr{C}$. Then we consider the vector $\mathbf{w}^0 =$

$= \mathbf{w}' - \mathbf{w}''$ which, in view of linearity of Eqs. (5.4), is a vector solution with vanishing external loads on the faces of the plate, i. e. $\mathbf{F} = 0$. In addition, $\mathbf{w}_0$ satisfies the conditions:

$$w^0 = 0, \quad w^0_{,n} = 0, \quad \Phi^0_\rho = 0 \quad \text{on } \mathscr{C}_1$$
$$V^0_n = 0, \quad M^0_n = 0, \quad R^0_{\rho n} = 0 \quad \text{on } \mathscr{C}_2 \tag{5.11'}$$

According to (5.8), where (5.11′) are inserted, we have

$$2 \int\limits_{\mathscr{D}} \mathscr{U}(\mathbf{w}^0)\, dS = 0$$

which reduces to

$$\mathscr{U}(\mathbf{w}^0) = 0. \tag{5.12}$$

Invoking now (5.1), we have (5.12) if and only if

$$k^0_{11} = k^0_{22} = k^0_{12} = k^0_{21} = \varkappa^0_{11} = \varkappa^0_{22} = \varkappa^0_{12} = \varkappa^0_{21} = 0,$$
$$k^0_1 = k^0_2 = 0 \tag{5.13}$$

The general solutions of the first eight equations are:

$$w^0 = A_1 x_1 + A_2 x_2 + A_0, \quad \Phi^0_1 = C_1, \quad \Phi^0_2 = C_2 \tag{5.13'}$$

where A_1, A_2, A_0, C_ρ, are arbitrary real constants.

When the last two equalities in (5.13) are taken into account, we obtain:

$$w^0_{,1} + \Phi^0_2 = 0, \qquad w^0_{,2} - \Phi^0_1 = 0,$$

or, in view of (5.13′):

$$A_1 = -C_2, \qquad A_2 = C_1.$$

Thus, we have:

a) If $\mathscr{C}_2$ is empty, then from $(5.11)_1$ we deduce $A_1 = A_2 = A_0 = C_1 = C_2 = 0$. But these equalities imply the uniqueness i. e.,

$$\mathbf{w}'' = \mathbf{w}'$$

b) If $\mathscr{C}_1$ is empty, the constants $A_1, \dots, C_2$ cannot be determined, but we have

$$\Phi^0_1 = \omega_1 = C_1 \qquad \text{and} \quad \Phi^0_2 = \omega_2 = C_2 \tag{5.14}$$

where ω_1, ω_2 are the components of the rigid rotation of the plate about the x_1, x_2-axes. By using (5.14) in (5.13′) we find:

$$w^0 = -\omega_2 x_1 + \omega_1 x_2 + A_0, \qquad \Phi^0_1 = \omega_1, \qquad \Phi^0_2 = \omega_2 \tag{5.15}$$

The equalities (5.14) and (5.15) indicate that $\mathbf{w}^0$ is determined up to a rigid rotation about the x_1, x_2-axes respectively, combined with a translation of magnitude A_0 along x_3-axis.

Therefore, in this case, we have:

$$\mathbf{w}'' = \mathbf{w}' + \mathbf{w}^0; \qquad \mathbf{w}_0 \equiv (-\omega_2 x_1 + \omega_1 x_2 + A_0, \ \omega_1, \ \omega_2)$$

c) In the case of mixed boundary conditions we obtain again the uniqueness of the solution i. e., $\mathbf{w}'' = \mathbf{w}'$.

Corollary. From the last two theorems we can deduce the necessary conditions which the forces and the couples specified on the surface of the plate must satisfy, so that the plate occupy an equilibrium configuration.

In this respect let $S \equiv (p, q_\rho, V_n, M_n, R_{\rho n})$ be a system of arbitrary external forces and couples given on the faces (p, q_ρ) and on the edge $(V_n, M_n, R_{\rho n})$ of the plate and let $\mathbf{w} \equiv (w, \Phi_1, \Phi_2)$ be the vector function which correspond to (S). Consider then the vectors: $\mathbf{w}_1 \equiv (C_0, 0, 0)$ and $\mathbf{w}_2 \equiv (-C_2 x_1 + C_1 x_2, C_1, C_2)$ where $C_0, C_1, C_2 \neq 0$ are arbitrary constants. As we have seen (Theorem 3) every $\mathbf{w}_\rho$, $(\rho = 1, 2)$, characterizes a rigid motion of the plate and, from the results of Section 3, it follows that the stresses and couple-stresses corresponding to $\mathbf{w}_\rho$ vanish. Consequently, the systems (S_ρ) are $S_\rho \equiv (0, 0, \dots, 0)$.

Consider now $\mathbf{w}' = \mathbf{w}$, $\mathbf{w}'' = \mathbf{w}$, $(S') = (S)$ and $(S'') = (S_1)$ in the reciprocity relations (5.10′). The integrals in the left hand disappear and one obtains

$$C_0\left(\int\limits_{\mathscr{D}} p\,dS + \int\limits_{\mathscr{C}} V_n\,ds\right) = 0.$$

In view of the arbitrarieness of the constant C_0 the previous equality leads to

$$\int\limits_{\mathscr{D}} p\,dS + \int\limits_{\mathscr{C}} V_n\,ds = 0. \tag{I}$$

Further, let us put $\mathbf{w}' = \mathbf{w}$, $\mathbf{w}'' = \mathbf{w}$, $(S') = (S)$ and $(S'') = (S_2)$ in (5.10′). we obtain a linear equation in C_1 and C_2 which can be satisfied only if the coefficients of the constants are zero i. e.,

$$\int\limits_{\mathscr{D}} (x_2 p + q_1)\, dS + \int\limits_{\mathscr{C}} (-M_n n_2 + x_2 V_n + R_{1n}) ds = 0,$$
$$\int\limits_{\mathscr{D}} (-x_1 p + q_2) dS + \int\limits_{\mathscr{C}} (M_n n_1 - x_1 V_n + R_{2n}) ds = 0. \tag{II}$$

The equalities (I) and (II) are the necessary conditions of equilibrium which we spoke above.

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ASUPRA ÎNCOVOIERII PLĂCILOR PLANE ÎN ELASTICITATEA ASIMETRICĂ

Rezumat

Unele aspecte ale teoriei plăcilor plane omogene de grosime constantă au fost studiate în [1, 3, 8]. În prezenta lucrare se stabilesc ecuaţiile unei noi teorii a încovoierii plăcilor plane izotrope, teorie care se dovedeşte a fi mai simplă decît cea dată în [1, 8] şi în acelaşi timp mai completă decît cea din [3].

În construcţia acestei teorii, rolul fundamental îl joacă grupul de ipoteze (a)—(e) (Secţ. 3). Trei dintre acestea: a), b), c_1) sînt similare ipotezelor lui Kirchhoff. Celelalte, c), d), c_2), caracterizează rotaţiile locale şi componentele μ_{23} şi μ_{33} ale tensorului micromomentelor. Autorul a introdus aceste ipoteze în scopul obţinerii unei teorii aproximative uşor de aplicat în rezolvarea problemelor concrete de încovoiere a plăcilor. Pe baza acestor ipoteze se arată că deformarea plăcii poate fi caracterizată cu ajutorul a trei funcţii de două variabile: deplasarea transversală a planului median $w(x_1, x_2)$ şi componentele $\Phi_\rho (x_1, x_2)$, $(\rho = 1, 2)$ ale vectorului rotaţiei locale în acest plan.

Se introduc momentele încovoietoare, forţele transversale generalizate şi rezultatele micromomentelor prin formulele (3.13) şi se stabileşte legătura acestora cu funcţiile de mai sus ((3.14)—(3.16)). Ecuaţiile de echilibru ale plăcii precum şi condiţiile naturale la limită pe care w şi Φ_ρ trebuie să le satisfacă ((4.8) şi (4.9) — Secţ. 4) se deduc din principiul variaţional al lui Lagrange. În ultima parte a lucrării (Secţ. 5) se demonstrează patru teoreme care prezintă importanţă pentru aplicaţii practice şi mai ales pentru rezolvarea aproximativă a problemelor la limită. Ultima dintre acestea este o teoremă de unicitate.

ASUPRA UNEI METODE DE REDUCERE A PERTURBAŢIEI APLICATĂ LA SISTEME CU PARAMETRU MIC

DE

CONSTANTIN SIMIRAD

1. Introducere. În [1] R. F. Arenstorf a dat o metodă de reducere a termenului perturbator $f(t, x)$ la sisteme de forma:

$$\dot{x} = A(x) + f(t, x)$$

valabilă pentru un interval finit de timp $0 \leq t \leq t_0$ şi $x \in D \subset C^n$, D fiind un compact, $f(t, x)$ fiind o perturbaţie convenabil de mică.

În lucrarea de faţă adaptăm această metodă la sisteme cu parametru mic de forma:

$$\dot{x} = A(x) + \varepsilon f(t, x, \varepsilon), \tag{1}$$

extinzînd intervalul de timp la semiaxa pozitivă R_+, $x \in D \subset C^n$, D fiind de asemenea un compact, iar $f(t, x, \varepsilon)$ mărginită în domeniul său de definiţie.

Extinderea intervalului de timp la semiaxă permite, în plus, unele rezultate referitoare la stabilitatea soluţiei banale a sistemului perturbat.

Metoda lui R. F. Arenstorf de reducere a perturbaţiei, extinsă la semiaxă, aplicată la sisteme autonome cu perturbaţii periodice care au proprietatea (n), (vezi mai jos), conduce la existenţa soluţiilor periodice pentru sistemul (1).

2. Reducerea termenului perturbator. Vom presupune că funcţia $A(x)$ are derivatele parţiale de ordinul întîi şi al doilea continue, iar $f(t,x,\varepsilon)$ satisface condiţiile obişnuite de regularitate.

Admitem că sistemul generator

$$\dot{x} = A(x) \tag{2}$$

are soluţia $x = \varphi(t, \zeta)$ cu $\varphi(0, \zeta) = \zeta$.

Considerăm schimbarea de variabilă dată de

$$x^* = x + \tau(t, x, \varepsilon), \tag{3}$$