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## DESPRE EXISTENŢA SOLUŢIILOR CONVERGENTE ÎN TEORIA ECUAŢIILOR INTEGRO-FUNCŢIONALE VOLTERRA

Rezumat

Această lucrare conţine cîteva rezultate privind existenţa soluţiilor unor ecuaţii integro-funcţionale Volterra în spaţiul $\mathcal{L}$ (spaţiul Banach al funcţiilor continue $x : [a, \infty) \to R^n$, cu limită la infinit şi cu norma $\sup_{t \geq a} \| x(t) \|$). Apartenenţa soluţiei la acest spaţiu îi conferă un anumit gen de comportare asimptotică.

Analele ştiinţifice ale Universităţii „Al. I. Cuza" — Iaşi
Secţia I a. Matematică, tom XX, 1974

# A FIXED POINT THEOREM ON METRIC SPACES

BY

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Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a mapping. In [4], S. R e i c h has established the following result:

**Theorem 1.** *Let $a, b, c$ be functions from $(0, +\infty)$ to $[0, +\infty)$ such that $f = a + b + c$ has the property:*

(P) $\forall t > 0\ \exists e(t) > 0,\ k(t) \in [0, 1)$ *such that* $u \in [t, t + e(t)) \Rightarrow f(u) \leqslant k(t)$.

*If $T : X \to X$ satisfies:*

$$d(Tx, Ty) \leqslant a(d(x, y))\, d(x, Tx) + b(d(x, y))\, d(y, Ty) + \\ + c(d(x, y))\, d(x, y), \qquad \forall x, y \in X,\ x \neq y, \tag{1}$$

*then $T$ has a unique fixed point.*

This result includes the fixed point theorems of B a n a c h [1], K a n n a n [2], S. R e i c h [3]. The purpose of the present paper is to extend this theorem.

**Definition.** *A function $f : [0, +\infty) \times [0, +\infty) \times (0, +\infty) \to [0, +\infty)$ will be said to have the property (Q) if it satisfies:*

(i) $$0 \leqslant f(u, v, w) \leqslant 1,\quad u, v \geqslant 0,\ w > 0,$$

(ii) $$\forall t > 0\ \exists e(t) > 0,\ k(t) \in [0, 1) \text{ such that}$$
$$(u, v) \in [0, e(t)) \times [0, e(t)) \cup [t, t + e(t)) \times [t, t + e(t)),$$
$$w \in [t, t + e(t)) \text{ implies } f(u, v, w) \leqslant k(t).$$

Our main result can be stated as follows:

**Theorem 2.** *Let $a, b, c$ be functions from $[0, +\infty) \times [0, +\infty) \times (0, +\infty) \to [0, +\infty)$ such that:*

(j) $a(u, v, w) = a(v, u, w),\ b(u, v, w) = b(v, u, w),\ c(u, v, w) = c(v, u, w),$

(jj) $a + b + c$ *has the property* $(Q)$.

*If* $T : X \to X$ *satisfies* :

$$(2)\qquad d(Tx,\ Ty) \leqslant a(d(x,\ Tx)\ ;\ d(y,\ Ty)\ ;\ d(x,\ y))\, d(x,\ Tx) + $$
$$+\ b(d(x,\ Tx)\ ;\ d(y,\ Ty)\ ;\ d(x,\ y))\, d(y,\ Ty) + $$
$$+\ c(d(x,\ Tx)\ ;\ d(y,\ Ty)\ ;\ d(x,\ y))\, d(x,\ y).$$

$\forall x,\ y \in X,\ x \neq y$, *then*, $T$ *has a unique fixed point*.

*Proof.* Since $a(u, v, w) + b(u, v, w) + c(u, v, w) \leqslant 1$ and $a, b, c$ satisfy (j), we may assume that in (2) $b(u, v, w) \leqslant 1/2$, $u, v \geqslant 0$, $w > 0$. Now, let $f$, $g$ be defined by :

$$f(u, v, w) = a(u, v, w) + b(u, v, w) + c(u, v, w),$$
$$g(u, v, w) = [a(u, v, w) + c(u, v, w)]/[1 - b(u, v, w)].$$

It is easy to see that $g(u, v, w) \leqslant f(u, v, w)$, $u, v \geqslant 0$, $w > 0$. Take $x \in X$ and consider the sequence $\{T^n x\ ;\ n = 0, 1, 2, \ldots\}$. Assuming that $T^n x \neq T^{n+1}x$, $n \geqslant 0$ (which is not a restriction) we have :

$$(2.1)\quad d(T^n x,\ T^{n+1}x) \leqslant a(d(T^{n-1}x,\ T^n x)\ ;\ d(T^n x,\ T^{n+1}x)\ ;\ d(T^{n-1}x, T^n x))$$
$$d(T^{n-1}x,\ T^n x) + b(d(T^{n-1}x,\ T^n x);\ d(T^n x,\ T^{n+1}x)\ ;\ d(T^{n-1}x,\ T^n x)).$$
$$d(T^n x,\ T^{n+1}x) + c(d(T^{n-1}x,\ T^n x)\ ;\ d(T^n x,\ T^{n+1}x)\ ;\ d(T^{n-1}x,\ T^n x)).$$
$$d(T^{n-1}x,\ T^n x),\quad n = 1, 2, 3, \ldots$$

which implies :

$$(2.2)\quad d(T^n x,\ T^{n+1}x) \leqslant g(d(T^{n-1}x,\ T^n x);\ d(T^n x,\ T^{n+1}x)\ ; d(T^{n-1}x,\ T^n x)).$$
$$d(T^{n-1}x,\ T^n x) \leqslant f(d(T^{n-1}x,\ T^n x)\ ;\ d(T^n x,\ T^{n+1}x)\ ;\ d(T^{n-1}x,\ T^n x)).$$
$$d(T^{n-1}x,\ T^n x),\quad n = 1, 2, 3, \ldots$$

Since $f(u, v, w) \leqslant 1$, we obtain $d(T^n x,\ T^{n+1}x) \leqslant d(T^{n-1}x,\ T^n x)$, $n = 1, 2, 3, \ldots$ i. e. the sequence $\{d(T^n x,\ T^{n+1}x)\ ;\ n = 0, 1, 2, \ldots\}$ decreases. Let $p = \lim\limits_{n \to \infty} d(T^n x,\ T^{n+1}x)$ and assume that $p > 0$. For $n \geqslant n(p)$ we have $d(T^{n-1}x,\ T^n x) \in [p,\ p + c(p))$ and using the property $(Q)$ of $f$ we get :

$$f(d(T^{n-1}x,\ T^n x)\ ;\ d(T^n x,\ T^{n+1}x)\ ;\ d(T^{n-1}x,\ T^n x)) \leqslant k(p) < 1.$$

The last inequality gives by (2.2) :

$$(2.3)\qquad d(T^n x,\ T^{n+1}x) \leqslant d(T^{n-1}x,\ T^n x)\, k(p),\ n \geqslant n(p).$$

Letting $n$ tend to infinity we have $p \leqslant p . k(p)$, that is, $1 \leqslant k(p)$, a contradiction. Therefore, $p = 0$.

If $\{T^n x\ ;\ n = 0, 1, 2, \ldots\}$ is not a Cauchy sequence, there exists



$\varepsilon > 0$ and two sequences of natural numbers, $m(j), n(j)$, $m(j) < n(j)$, $m(j) \to \infty$ as $j \to \infty$, such that $d(T^{m(j)}x,\ T_n{}^{(j)}x) \geqslant \varepsilon$, while $d(T^{m(j)}x,\ T^{n(j)-1}x) < \varepsilon$, $j = 1, 2, 3, \ldots$ For the sake of simplicity we shall write $m$, $n$, instead $m(j)$, $n(j)$, respectively. Since $\lim\limits_{k \to \infty} d(T^k x,\ T^{k+1}x) = 0$, we have for $j \geqslant j(\varepsilon)$

$$0 < d(T^{m-1}x,\ T^m x) \leqslant d(T^{n-1}x,\ T^n x) < c(\varepsilon), \text{ which implies :}$$

$$(2.4)\qquad \varepsilon \leqslant d(T^m x,\ T^n x) \leqslant d(T^m x,\ T^{n-1}x) + d(T^{n-1}x,\ T^n x) <$$
$$< \varepsilon + d(T^{n-1}x,\ T^n x) < \varepsilon + c(\varepsilon),\quad j \geqslant j(\varepsilon).$$

On the other hand, by (2.4) we obtain $\lim\limits_{j \to \infty} d(T^m x,\ T^n x) = \varepsilon$. Now we may write for $j \geqslant j(\varepsilon)$:

$$(2.5)\quad d(T^m x,\ T^n x) \leqslant d(T^m x,\ T^{m+1}x) + d(T^n x\ T^{n+1}x) + d(T^{m+1}x,\ T^{n+1}x) \leqslant$$
$$\leqslant d(T^m x,\ T^{m+1}x) + d(T^n x,\ T^{n+1}x) +$$
$$+\ a(d(T^m x,\ T^{m+1}x)\ ;\ d(T^n x, T^{n+1}x)\ ;\ d(T^m x,\ T^n x))\, d(T^m x,\ T^{m+1}x) +$$
$$+\ b(d(T^m x,\ T^{m+1}x);\ d(T^n x,\ T^{n+1}x);\ d(T^m x,\ T^n x))\ d(T^n x,\ T^{n+1}x) +$$
$$+\ c(d(T^m x,\ T^{m+1}x);\ d(T^n x,\ T^{n+1}x);\ d(T^m x,\ T^n x))\ d(T^m x,\ T^n x) \leqslant$$
$$\leqslant d(T^m x, T^{m+1}x) + d(T^n x, T^{n+1}x) + d(T^m x,\ T^{m+1}x) + d(T^n x,\ T^{n+1}x) +$$
$$+\ f(d(T^m x,\ T^{m+1}x);\ d(T^n x,\ T^{n+1}x) : d(T^m x, T^n x))\ d(T^m x,\ T^n x) \leqslant$$
$$\leqslant 2[d(T^m x,\ T^{m+1}x) + d(T^n x,\ T^{n+1}x)] + k(\varepsilon)\, d(T^m x,\ T^n x).$$

Letting $j$ tend to infinity, we have $\varepsilon \leqslant k(\varepsilon)\varepsilon$, that is, $1 \leqslant k(\varepsilon)$, a contradiction. Hence, $\{T^n x,\ n = 0, 1, 2, \ldots\}$ is a Cauchy sequence.

Let $z = \lim\limits_{n \to \infty} T^n x$; since $b(u, v, w) \leqslant 1/2$, we may write for $n \geqslant 0$:

$$(2.6)\qquad d(z,\ Tz) \leqslant d(z,\ T^{n+1}x) + d(T^{n+1}x,\ Tz) \leqslant d(z,\ T^{n+1}x) +$$
$$+\ a(d(T^n x,\ T^{n+1}x)\ ;\ d(z,\ Tz)\ ;\ d(T^n x,\ z))\, d(T^n x,\ T^{n+1}x) +$$
$$+\ b(d(T^n x,\ T^{n+1}x)\ ;\ d(z,\ Tz)\ ;\ d(T^n x,\ z))\, d(z,\ Tz) +$$
$$+\ c(d(T^n x,\ T^{n+1}x)\ ;\ d(z,\ Tz) : d(T^n x,\ z))\, d(T^n x,\ z) \leqslant$$
$$\leqslant d(z,\ T^{n+1}x) + d(T^n x,\ T^{n+1}x) + 1/2\, d(z,\ Tz) + d(T^n x,\ z).$$

Letting $n$ tend to infinity we get $d(z,\ Tz) \leqslant 1/2\, d(z,\ Tz)$, which implies $d(z,\ Tz) = 0$, i. e. $z$ is a fixed point of $T$.

Let $z_1, z_2, \in X$ be such that $z_1 = Tz_1$, $z_2 = Tz_2$. If $z_1 \neq z_2$, we may write :

$$d(z_1, z_2) = d(Tz_1, Tz_2) \leq a(0, 0, d(z_1, z_2)) \cdot 0 + \tag{2.7}$$
$$+ b(0, 0, d(z_1, z_2)) \cdot 0 + c(0, 0, d(z_1, z_2)) \cdot d(z_1, z_2) \leq$$
$$= f(0, 0, d(z_1, z_2))\, d(z_1, z_2) \leq k(d(z_1, z_2))\, d(z_1, z_2)$$

that is, $1 \leq k(d(z_1, z_2))$, a contradiction. Therefore, $z_1 = z_2$, which completes the proof. Q.E.D.

**Remark.** If $a(u, v, w) = a_1(w)$, $b(u, v, w) = b_1(w)$, $c(u, v, w) = c_1(w)$, we obtain Theorem 1. A symmetrical result with respect to Theorem 1 can be obtained by taking $a(u, v, w) = a_2(u, v)$, $b(u, v, w) = b_2(u, v)$, $c(u, v, w) = c_2(u, v)$. More precisely, we have the following:

**Theorem 3.** *Let $a, b, c$ be functions from $[0, +\infty) \times [0, +\infty)$ to $[0, +\infty)$ such that:*

(i) $a(u, v) = a(v, u)$, $b(u, v) = b(v, u)$, $c(u, v) = c(v, u)$, $u, v \geq 0$

(ii) $0 \leq a(u, v) + b(u, v) + c(u, v) \leq 1$, $u, v \geq 0$

(iii) $\forall t > 0\ \exists e(t) > 0$, $k(t) \in [0, 1)$ *such that* $u, v \in [t, t + e(t))$ *implies* $a(u, v) + b(u, v) + c(u, v) \leq k(t) < 1$.

*If $T : X \to X$ satisfies:*

$$d(Tx, Ty) \leq a(d(x, Tx) ; d(y, Ty))\, d(x, Tx) + \tag{3}$$
$$+ b(d(x, Tx) ; d(y, Ty))\, d(y, Ty) +$$
$$+ c(d(x, Tx) ; d(y, Ty))\, d(x, y), \quad x, y \in X$$

*then, $T$ has a unique fixed point.*

*Proof.* It is sufficient to verify that $f = a + b + c$ has the property $(Q)$. Let $t > 0$ be given. Define:

$$e_1(t) = \min\{e(t), e(0)\}, \; k_1(t) = \max\{k(t), k(0)\} \tag{3.1}$$

and let $(u, v) \in [0, e_1(t)) \times [0, e_1(t)) \cup [t, t + e_1(t)) \times [t, t + e_1(t))$.
If $(u, v) \in [0, e_1(t)) \times [0, e(t))$ then $u, v \in [0, e(0))$ and thus

$$f(u, v) \leq k(0) \leq k_1(t) < 1.$$

If $(u, v) \in [t, t + e(t)) \times [t, t + e(t))$ then $u, v \in [t, t + e(t))$ and thus

$$f(u, v) \leq k(t) \leq k_1(t) < 1.$$

Therefore, $f$ has the property $(Q)$, as asserted. Q.E.D.
It is possible to extend Theorem 2 in the following manner:

**Theorem 4.** *Let $a, b, c$ be functions from $[0, +\infty) \times [0, +\infty) \times (0, +\infty)$. to $[0, +\infty)$ such that:*

(j) $a(u, v, w) = a(v, u, w)$, $b(u, v, w) = b(v, u, w)$, $c(u, v, w) = c(v, u, w)$

(jj) $a + b + c$ *has the property* $(Q)$.



*If $T : X \to X$ satisfies:*

$$\exists p \geq 1 \text{ such that } T^p \text{ satisfies } (2) \tag{4}$$

*then, $T$ has a unique fixed point.*

*Proof.* Using Theorem 2, we find an element $z \in X$ such that $T^p z = z$ which implies $T(T^p z) = Tz$, that is, $T^p(Tz) = Tz$, i. e. $Tz$ is a fixed point of $T^p$. By the uniqueness of the fixed point of $T^p$ we get $z = Tz$. Let $z_1, z_2 \in X$, $z_1 = Tz_1$, $z_2 = Tz_2$. By iteration we have:

$$z_1 = T^p z_1, \quad z_2 = T^p z_2, \tag{4.1}$$

which implies (by the same argument as above) $z_1 = z_2$, which completes the proof. Q.E.D.

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## O TEOREMĂ DE PUNCT FIX ÎN SPAȚII METRICE


Rezumat

Se stabilește o teoremă de punct fix pe spații metrice abstracte, generalizîndu-se în acest fel un rezultat din [4].