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THE TENSOR-INTEGRAL IN SPACES WITH AFFINE CONNECTION

BY

J. GOTTLIEB, V. OPROIU, G. ZET

Introduction. In some previous works [2, 3, 5], which are a continuation of the researches of A. Móor [7, 8], the tensor-integral in parallelizable manifolds and product spaces was studied. In this work the problem of tensor-integral in spaces with affine connection is treated.

Let V_n be a differentiable manifold of necessary class C^k, with an affine connection Γ^i_{jk}, T_x the tangent space at $x \in V_n$ and T^*_x its dual. An element of the space $(\underset{\alpha}{\otimes} T_{x'}) \otimes (\underset{\beta}{\otimes} T^*_x) \otimes (\underset{\gamma}{\otimes} T_{x'}) \otimes (\underset{\delta}{\otimes} T^*_{x'})$ defined for every $(x, x') \in V_n \times V_n$ is called a bitensor. We shall use the convention that the indices $i, j, k, \ldots$ refer to the point x and the indices $i', j', k', \ldots$ to the point x'. Then, $T^i_{\cdot k'}$, are the components of a bitensor $T \in T_x \otimes T^*_{x'}$.

If $h^{k'}_i$ and $k^l_{j'}$ are the transport bitensors, [1, 6, 9] defined in $\Omega \subset V_n$, then it is possible to define the tensor-integral of a bitensor [4] over a p-dimensional manifold $S_p \subset \Omega$ and for this operation, the Stokes-type formulae are obtained. In this paper the property:

$$h^{l'}_i k^j_{l'} = \delta^j_i ; \quad h^{k'}_j k^j_{l'} = \delta^{k'}_{l'} \tag{0.1}$$

of the transport bitensors was used and we remark that if these bitensors admit covariant derivatives of order two, then:

$$\begin{aligned} \nabla_l h^{k'}_i (x, x') &= H_{li}^{\cdot\cdot t} (x, x') h^{k'}_t (x, x') \\ \nabla_{s'} k^l_{j'} (x, x') &= K_{s'j'}^{\cdot\cdot t'} (x, x') k^l_{t'} (x, x'), \end{aligned} \tag{0.2}$$

where the bitensors H (tensor of order three in x and scalar in x') an: K (tensor of order three in x' and scalar in x) satisfy the relationsd

$$(0.3)\qquad \begin{aligned} \nabla_{[j} H_{l]i}^{\ \ t} &= -\frac{1}{2} R_{jli}^{\ \ \ t} - S_{jl}^{\ \ s} H_{si}^{\ \ t} - H_{[j|s}^{\ \ \ t} H_{|l]i}^{\ \ \ s} \\ \nabla_{[m'} K_{s']j'}^{\ \ \ t'} &= -\frac{1}{2} R_{m's'j'}^{\ \ \ \ t'} - S_{m's'}^{\ \ \ k'} K_{k'j'}^{\ \ \ t'} - K_{[m'|u'}^{\ \ \ \ t'} K_{|s']j'}^{\ \ \ \ u'} . \end{aligned}$$

Here :

$$(0.4)\qquad R_{jli}^{\ \ \ t} = \frac{\partial \Gamma_{li}^t}{\partial x^j} - \frac{\partial \Gamma_{ji}^t}{\partial x^l} + \Gamma_{js}^t \Gamma_{li}^s - \Gamma_{ls}^t \Gamma_{ji}^s$$

is the curvature tensor and

$$(0.5)\qquad S_{ji}^{\ \ s} = \frac{1}{2}(\Gamma_{jl}^s - \Gamma_{lj}^s) = \Gamma_{[jl]}^s$$

is the torsion tensor.

To obtain relations (0.2) and (0.3), only the relation (0.1) was used. For this reason, we shall not use the transport bitensors and shall consider only the existence of the bitensors h and k which satisfy the relation (0.1).

§ **1. Tensor-integral and Stokes-type formulae.** Let Ω be a domain in V_n, homeomorfic with an open sphere in R^n and we suppose the existence of the bitensor $h_i^{k'}(x, x')$ defined over $\Omega \times \Omega$, with the property:

$$(1.1)\qquad \det | h_i^{k'}(x, x') | \neq 0,$$

for arbitrary x and x' in Ω. The relation (1.1) assures then the existence of bitensor $k_{i'}^j, (x, x')$ with the property (0.1). Let us suppose that $S_q \subset \Omega$ is a q dimensional domain and $x' \in S_q$. We shall define the tensor-integral of $T_{i'}^{\ j'}{}_{k_1' \ldots k_q'}(x') \equiv T_{i'}^{\ j'}{}_{[k_1' \ldots k_q']}(x')$ over the domain S_q in $x \in \Omega$, by

$$(1.2)\qquad \begin{aligned} V_i^j(x) &= I_{x(S_q)} T_{i'}^{\ j'}{}_{.k_1' \ldots k_q'}(x')\, dS^{k_1' \ldots k_q'} = \\ &= \int_{(S_q)} h_i^{m'}(x, x')\, k_{s'}^j(x, x')\, T_{m'}^{\ s'}{}_{.k_1' \ldots k_q'}(x')\, dS^{k_1' \ldots k_q'} , \end{aligned}$$

where $i = i', j = j'$ and $dS^{k_1' \ldots k_q'} = dx^{k'} \wedge \ldots \wedge dx^{k_q'}$. It is easily seen, using the definition (1.2), that $V_i^j(x)$ are the components of a tensor; therefore, the tensor-integral retains the tensorial character of the integrand.

Suppose now that the covariant derivatives of $h_i^{k'}$ and $k_{i'}^j$, exist and therefore the relations (0.2) are satisfied. Then, by a calculation similar to that used in [5], we obtain:

$$(1.3)\qquad I_{x(S_q)} T_{i'}^{\ j'}{}_{.k_1' \ldots k_q'}\, dS^{k_1' \ldots k_q'} = \int_{(S_{q+1})} h_i^{t'} k_{s'}^j \overset{q}{\nabla}{}^{*}_{[l'|} T_{t'}^{\ s'}{}_{.|k_1' \ldots k_q']}\, dS^{k_1' \ldots k_q'} ,$$

where (S_q) is the boundary of (S_{q+1}) and

$$(1.4)\qquad \begin{aligned} \overset{q}{\nabla}{}^{*}_{[l'|} T_{t'}^{\ s'}{}_{.|k_1' \ldots k_q']} &= \nabla_{[l'|} T_{t'}^{\ s'}{}_{.|k_1' \ldots k_q']} + q S_{[l' \ k_1'}^{\ \ r'} T_{t'}^{\ s'}{}_{.r'|k_2' \ldots k_q']} + \\ &+ K_{[l'|a'}^{\ \ \ \ s'} T_{t'}^{\ a'}{}_{.|k_1' \ldots k_q']} - K_{[l'|t'}^{\ \ \ \ b'} T_{b'}^{\ s'}{}_{.|k_1' \ldots k_q']} . \end{aligned}$$

The relation (1.3) is the generalization of Stokes formula, but on the right-hand side of (1.3) we have in general the tensor-integral of a bitensor [5], as it follows from (1.4) (because K is a bitensor). Namely, if $K_{i'j'}^{\ \ \ s'}(x, x') = K_{i'j'}^{\ \ \ s'}(x')$, that is if K is a tensor, then (1.3) becomes:

$$(1.5)\qquad I_{x(S_q)} T_{i'}^{\ j'}{}_{.k_1' \ldots k_q'}\, dS^{k_1' \ldots k_q'} = I_{x(S_{q+1})} \overset{q}{\nabla}{}^{*}_{[l'|} T_{i'}^{\ j'}{}_{.|k_1' \ldots k_q']}\, dS^{l' k_1' \ldots k_q'} ,$$

where we have used the definition (1.2). Note that (1.3) may be written generally in the form (1.5), but then $I_{x(S_{q+1})}$ is the tensor-integral of a bitensor [5]. With this hypothesis (that K is a tensor), (1.5) may be written:

$$(1.6)\qquad I_{(S_q)}\, \omega_{i'}^{\ k'} = I_{(S_{q+1})}\, d\omega_{i'}^{\ k'} ,$$

where:

$$(1.7)\qquad \omega_{i'}^{\ k'} = T_{i'}^{\ k'}{}_{.k_1' \ldots k_q'}\, dS^{k_1' \ldots k_q'}$$

is a tensorial form and:

$$(1.8)\qquad d\omega_{i'}^{\ k'} = \overset{q}{\nabla}{}^{*}_{[l'|} T_{i'}^{\ k'}{}_{.|k_1' \ldots k_q']}\, dS^{l' k_1' \ldots k_q'}$$

is the exterior derivative of $\omega_{i'}^{\ k'}$. The verification of the properties of the exterior derivative defined by (1.8) is analogous to that given in [3] and [5]. In the next section we shall examine the conditions under which the bitensor K can be reduced to a tensor, since this is an important case.

§**2. The analysis of the restrictions for the tensorial character of K**

Let us define a new connection $\overline{\Gamma}_{j'k'}^{t'}$ in Ω by:

$$(2.1)\qquad \overline{\Gamma}_{s'j'}^{t'} = \Gamma_{s'j'}^{t'} + K_{s'j'}^{\ \ \ t'},$$

where we have used the hypothesis that K is a tensor. The curvature tensor of the new connection is:

$$(2.2)\qquad \overline{R}_{m's'j'}^{\ \ \ \ \ t'} = R_{m's'j'}^{\ \ \ \ \ t'} + 2(\nabla_{[m'} K_{s']j'}^{\ \ \ \ t'} + S_{[m'|n'}^{\ \ \ \ \ t'} K_{n'j'}^{\ \ \ t'} + K_{[m'|n'}^{\ \ \ \ \ t'} K_{|s']j'}^{\ \ \ \ n'}).$$

But (2.2) and (0.3) yield :

$$\bar{R}_{m' s' j'}{}^{r'} = 0 ; \tag{2.3}$$

therefore, the new connection $\bar{\Gamma}$ has no curvature.

Thus, our consideration may be made only for a parallelizable neighbourhood of an arbitrary point $x \in V_n$. Locally, there is always a connection without curvature, since on a local chart we may consider the fields of those vectors which are tangent to the coordinates and thus the domain of the considered local chart becomes parallelizable.

It follows, in a manner similar to that of [4], that for the description of the conservation laws of the general relativity theory, this formalism may be used only on a parallelizable domain.

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TENSOR-INTEGRALĂ ÎN SPAŢII CU CONEXIUNE AFINĂ

Rezumat

Continuînd lucrarea [5], autorii definesc tensor-integrala în spaţii cu conexiune afină prin relaţia (1.2). Generalizarea formulelor de tip Stokes conduce, în general, la tensor-integrala bitensorilor, deoarece K din relaţia (1.4) este un bitensor. Restricţia pentru K de a avea un caracter tensorial (şi nu bitensorial) conduce la concluzia că domeniul în care efectuăm operaţia de integrare tensorială trebuie să fie paralelizabil.

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QUASI-CONNEXIONS COMPOSÉES D'ORDRE DEUX

PAR

A. M. NAVEIRA et L. M. HERVELLA

Les quasi-connexions sur une variété différentiable ont été définies et étudiées localement par Wong, [6]. Plus tard, Vamanu, [5], définit et étudie globalement ces quasi-connexions. Si la variété est riemannienne, elle détermine une unique quasi-connexion sans torsion et qui fait le tenseur métrique parallèle. Fava, [1], [2] introduit la notion de connexion composée d'ordre deux et en étudie quelques propriétés.

Dans cet article on définit les quasi-connexions composées d'ordre deux comme une généralisation simultanée des quasi-connexions et des connexions composées: on étudie leur compatibilité avec une métrique de Riemann, l'existence et l'unicité de l'unique quasi-connexion composée d'ordre deux, laquelle est compatible avec la métrique et qui est sans torsion. Finalement, on déduit quelques propriétés relatives au déplacement parallèle par rapport à ces quasi-connexions.

Soit V_n une variété différentiable de classe C^∞ avec une métrique de Riemann g, φ un champ tensoriel sur V_n du type (1, 1) et non-dégénéré. Dans la suite, nous ferons usage des notations suivantes.

$F(V_n)$: l'algèbre des fonctions sur V_n.

$L^1(V_n)$: F-module des champs vectoriels.

$L_1(V_n)$: F-module dual de $L^1(V_n)$.

$L_s(V_n)$: F-module des s-formes différentielles extérieures.

$L(V_n) = \bigcup_{s=0}^{n} L_s(V_n)$: L'algèbre des formes différentielles extérieures.

$L_s^r(V_n)$: F-module des champs tensoriels du type (r, s).

$\bar{L}(V_n) = \bigcup_{r, s, = 0}^{n} L_s^r(V_n)$: F-module des champs tensoriels.