Corollaire 3. *Etant donnée sur V_n une quasi-connexion composée d'ordre deux* $\overset{(c)}{{}_2\overline{\nabla}}$, *il existe une autre unique quasi-connexion* $\overset{(c)}{{}_2\nabla}$, *laquelle satisfait* a) *du Théorème 2 et est sans torsion.*

Preuve: On définit

$$\nabla_X Y = \overline{\nabla}_X Y - 1/2\, \overline{T}(X, Y),$$

$${}_2\nabla_{XY} Z = {}_2\overline{\nabla}_{XY} Z - 1/2\, \overline{T}(X, Y, Z),$$

et $(\nabla, {}_2\nabla)$ satisfait les propriétés requises.

Département de Géométrie et Topologie
Université de Santiago de Compostela
Espagne

BIBLIOGRAPHIE

1. Fava F. — *Connessioni composte e movimenti di ordine superiore.* Att. Ac. Sc. Torino, 98, (1963—64).
2. Fava F. — *Connessioni composte su varietà differenziabili.* Period. Mat., 46, (1968), 126—149.
3. Hicks N. J. — *Notes on differential Geometry.* Van Nostrand 1965.
4. Kobayashi S. - Nomizu K. — *Foundations of differential geometry.* Vol. I, II, Interse. Publ. (1963), (1969).
5. Vamanu E. — *Cuasi-conexiuni pe varietati diferentiable.* An. şt. Univ. Iaşi. Matematica, t. XVI. (1970), 383—388.
6. Wong Y. C. — *Linear connections and quasi connections on a differentiable manifold.* Tôhoku Math. Journ., 14, (1962), 48—63.

CVASI-CONEXIUNI COMPUSE DE ORDIN 2

Rezumat

Plecînd de la noţiunea de cvasi-conexiune şi conexiune compusă autorii definesc noţiunea de cvasi-conexiune compusă de ordin 2. Se studiază compatibilitatea unei astfel de cvasi-conexiuni cu o metrică riemanniană şi se arată că există o singură cvasi-conexiune compusă de ordin 2 compatibilă cu o metrică şi fără torsiune. În încheierea articolului se demonstrează cîteva proprietăţi referitoare la deplasările paralele în raport cu cvasi-conexiunile compuse de ordin 2.

Analele ştiinţifice ale Universităţii „Al. I. Cuza"—Iaşi
Secţia I a. Matematică, tom XX, 1971

ON FIBERING WITH A_2-STRUCTURE AND η-SEMI-SYMMETRIC CONNECTIONS

BY

PETRE STAVRE

Let V_n be a $n = 2m + 1$-dimensional differentiable manifold with an almost contact metric structure $A_2 = (\mathcal{J}, \eta, E, g)$. We denote by $\mathcal{X}(V_n)$ the Lie algebra of all vector fields on V_n. The structure A_2 is defined by a (1,1)-tensor field $\mathcal{J}$, a 1-form η, a (0, 2)—tensor field g(g is a Riemannian metric on V_n) and a vector field E satisfying the following conditions [1], [2]:

(1) $$\eta(E) = 1,$$

(2) $$\mathcal{J}E = 0,$$

(3) $$\eta(\mathcal{J}X) = 0 \quad \text{for all} \quad X \in \mathcal{X}(V_n),$$

(4) $$\mathcal{J}^2 = -I + \eta \otimes E,$$

(5) $$g(\mathcal{J}X, \mathcal{J}Y) = g(X, Y) - \eta(X)\,\eta(Y) \quad \text{for all } X, Y \in \mathcal{X}(V_n).$$

We define a 2—form F on V_n as follows:

(6) $$F(X, Y) = g(X, \mathcal{J}Y) \quad \text{for all } X, Y \in \mathcal{X}(V_n).$$

An almost contact metric structure A_2 is said to be invariant if $\mathcal{J}$, η and g are invariant under the action of G, the 1-parameter group generated by E. Moreover, if E is a (strictly) regular vector field, then A_2 is said to be an invariant strictly regular almost contact structure on V_n [2].

Let $V_r(r = 2m)$ be the orbit space defined by E; If A_2 is an invariant strictly regular almost contact structure on V_r, then;

(i) V_n is a principal G-bundle over V_r, and
(ii) η is a connection form on V_n[2].
We denote this principal fibre bundle by $V_n(V_r, G, \pi)$.
If,

(7) $$\pi : V_n \to V_r$$

is the natural projection of V_n onto V_r, we define a (1, 1) tensor field $\overline{\mathcal{J}}$ on V_r as follows:

(8) $$\overline{\mathcal{J}}_{\bar{x}}(X) = d\pi\, \mathcal{J}_x(X^L_x) ; \qquad \bar{x} = \pi(x) ; \qquad x \in V_n ; \ \bar{x} \in V_r,$$

where X^L_x is the lift of $X \in \mathcal{X}(V_r)$ at x with respect to the connection form η. Then $\overline{\mathcal{J}}$ is an almost complex structure on V_r, (see [2]).

If we define a tensor field $\bar{g}$ on V_r by

(9) $$\bar{g}(X, Y) = g(X^L, Y^L) \quad \text{for all } X, Y \in \mathcal{X}(V_r),$$

then the structure $\bar{A}_2 = (\overline{\mathcal{J}}, \bar{g})$ is an almost Hermitian structure on V_r [2]. If we denote,

(10) $$S = N + 2d\eta \otimes E,$$

where N is the Nijenhuis tensor field on V_r, then:

(11) $$d\pi S(X^L, Y^L) = d\pi N(X^L, Y^L) = \bar{N}(X, Y),$$

(12) $$E \text{ is vertical}; \ (\text{i. e. } d\pi\, E = 0),$$

($\bar{N}$ is the Nijenhuis tensor field on V_r).

We prove the following,

Theorem 1. *If D is an almost contact connection on $V_n(V_r, G, \pi)$ ($\mathcal{J}$—connection) (i. e. $D\mathcal{J} = 0$) then the induced structure $\bar{A}_2$ on V_r is integrable (i. e. $\bar{N} = 0$) if and only if the vector field, $B(X^L, Y^L)$ defined by,*

(13) $$-T(\mathcal{J}X^L, \mathcal{J}Y^L) + T(\mathcal{J}X^L, \mathcal{J}Y^L) + \mathcal{J}T(X^L, \mathcal{J}Y^L) + \\ + T(X^L, Y^L) = B(X^L, Y^L)$$

is vertical, for all $X, Y \in \mathcal{X}(V_n)$.

Proof. If $D\mathcal{J} = 0$, then from formula [4], it results:

(14) $$N(X^L, Y^L) = -\mathcal{J}T(\mathcal{J}X^L, Y^L) + \mathcal{J}T(\mathcal{J}X^L, Y^L) + \\ + \mathcal{J}T(X^L, \mathcal{J}Y^L) + T(X^L, Y^L) - \eta(T(X^L, Y^L))E,$$

for all $X, Y \in \mathcal{X}(V_r)$. Using (11) (12) and (14) we have, $\bar{N}(X, Y) = 0$ for all $X, Y \in \mathcal{X}(V_r)$ Q. E. D.

A linear connection $\tilde{D}$ on V_n defined by,

(15) $$\tilde{D}_{X'} Y' = D_{X'} Y' + g(V', Y')X' + g(V', \mathcal{J}Y')\mathcal{J}X' ; \eta(V') = 0,$$

for all $X', Y' \in \mathcal{X}(V_n)$ is called in [3] an η-coparallel connection with D of the first type. If $\tilde{D}$ is a symmetric connection, then D is called an η-semi-symmetric connection of the first type [3].

If we define a linear connection $\tilde{D}$ on V_n as follows,

(16) $$\tilde{D}_{X'} Y' = D_{X'} Y' + g(V', X')Y' + g(V', \mathcal{J}X')\mathcal{J}Y' ; \ \eta(V') = 0,$$

for all $X', Y' \in \mathcal{X}(V_n)$ then D is called η-coparallely with D of the second type [3]. If $\tilde{D}$ is a symmetric connection, then D is called an η-semi-symmetric connection of the second type. For an η-coparallel connection on V_n, the tensor field $\mathcal{T}$ defined by,

(16′) $$\mathcal{T}(X', Y') = T(X', Y') + \frac{1}{2m+1}(t(Y')X' - t(X')Y' + t(\mathcal{J}Y')\mathcal{J}X' - \\ - t(\mathcal{J}X')\mathcal{J}Y')$$

is an invariant, [3] (i. e. $\mathcal{T}(X', Y') = \tilde{\mathcal{T}}(X', Y')$), where t is the vector field of torsion T;

(17) $$t(X') = \text{trace of}: Y' \to T(X', Y').$$

If D is an η-semi-symmetric connection, then [3].

(18) $$\tilde{\mathcal{T}}(X', Y') = 0, \quad \text{for all } X', Y' \in \mathcal{X}(V_n).$$

Theorem 2: *If D is an η-semi-symmetric $\mathcal{J}$-connection on $V_n(V_r, G, \pi)$, then the induced structure on V_r is integrable.*

Proof. From (16) and (18) we have,

(19) $$T(X', Y') = \frac{1}{2m+1}(t(X')Y' - t(Y')X' + t(\mathcal{J}X')\mathcal{J}Y' - t(\mathcal{J}Y')\mathcal{J}X')$$

and hence [4],

(20) $$B(X^L, Y^L) = \eta(T(X^L, Y^L))E,$$

i. e. $B(X^L, Y^L)$ is vertical for all $X, Y \in \mathcal{X}(V_r)$. By Theorem 1, we have finally $\bar{N}(X, Y) = 0$, for all $X, Y \in \mathcal{X}(V)$.

On the other hand, from ii and (19) we obtain:

(21) $$\eta(T(X^L, Y^L)) = 0$$

i. e. $T(X^L, Y^L)$ is a horizontal vector field on V_n.

Theorem 3. *If D is an η-semi-symmetric $\mathcal{J}$-connection on $V_n(V_r, G, \pi)$, then the connection form η is involutive and the structure A_2 is integrable.*

Proof. When a connection is given, $d\pi$ is a isomorphism of Q_x (the horizontal subspace of $T_x(V_n)$) into $T_{\pi(x)}(V_r)$. Using this isomorphism, the condition,

$$\eta([\overline{X}, \overline{Y}]) = 0 \tag{23}$$

follows immediately from the definition of the Nijenhuis tensor field N and (22) (for all horizontal vector fields X, $\overline{Y}$. Because $\overline{X}$ is invariant under the action of G and η is involutive, we have

$$N(E, \overline{X}) = 0. \tag{24}$$

Since N is a tensor field, from (22) and (24) it results that A_2 is integrable (i. e. $N = 0$).

If we consider the 2-forms $F, \overline{F}$ defined by,

$$F(X', Y') = g(X', \mathcal{J}Y'), \tag{25}$$

$$\overline{F}(X, Y) = \overline{g}(X, \overline{\mathcal{J}}Y), \tag{26}$$

we obtain following relations :

$$\pi^*\overline{F} = F, \tag{27}$$

$$\pi^* d\overline{F} = dF. \tag{28}$$

Theorem 4. *If D is an η-semi-symmetric $\mathcal{J}$-connection on $V_n(V_r, G, H)$ and the 2-form F is closed, then*

(a) *$\overline{A}_2$ is a Kähler structure on V_r.*

(b) *The Riemannian connection $\overline{\nabla}$ defined by $\overline{g}$ is almost complex* ($\overline{\nabla}\,\overline{\mathcal{J}} = 0$ [5]).

(c) *For the Riemannian connection ∇ defined by g, we have* $\nabla_{X^L} Y^L = (\overline{\nabla}_X Y)^L$ *for all* $X, Y \in \mathcal{X}(V_r)$.

(d) *The bundle $U(V_r)$ of unitary frames admits a torsion-free connection (which is necessarily unique).*

Proof: (a) If F is closed, then $\overline{F}$ is a closed 1-form on V_r (i. e $d\overline{F} = 0$).

Thus we have, $\overline{N} = 0$; $d\overline{F} = 0$.

The condition (b) follows immediately from the condition (a) [5]. Since, [2]

$$\nabla_{X^L} Y^L = (\overline{\nabla}_X Y)^L + \frac{1}{2}\eta([X^L, Y^L])E, \tag{29}$$

the condition (c) is clear from the Theorem (3).

Finally d) results from her equivalence with a) proved in [5].

Fac. de studii economice
Univ. din Craiova

BIBLIOGRAPHY

1. Goldberg S. and Yano K. — *Integrability of almost cosimplectic structures*, Pacific J. of Math., vol. 31, nr. 2, 1969, p. 373—381.
2. Ogiue K. — *On Fiberings of almost contact manifolds*, Kodai Math. Sem. Rep., nr. 2 1965, p. 53.
3. Stavre P. — *Conexiuni η - coparalele*, Studia Univ. „Babeş-Bolyai", Cluj (sub tipar).
4. Stavre P. — *Asupra unor conexiuni semisimetrice aproape de contact sau quasi aproape de contact*, Bul. Inst. politehnic din Bucureşti (sub tipar).
5. Kobayashi S., Nomizu K. — *Foundations of diff. geometry*. New York Interscience Publ. vol. I (1963), vol. II (1969).

FIBRĂRI CU STRUCTURĂ A_2 ŞI CONEXIUNI η - SEMISIMETRICE

Rezumat

În lucrare se studiază structurile $A_2 = (\mathcal{J}, \eta, E, g)$ metrice aproape de contact, invariante, regulate pe o varietate V_n şi $\mathcal{J}$ - conexiunile η - semisimetrice pe aceasta.