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ON SOME PROPERTIES OF LINEAR BOOLEAN AUTOMATA

BY

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§1. Introduction. The class of linear Boolean automata (LBA) was introduced in [6] and studied in [7]. This sort of automata can be regarded as useful models for the theory of the communication networks, for the switching theory and also for the study of programming systems.

In this paper we give further properties of the LBA concerning the controllability (§2) and the stability (§3). We give also a simulation for some nondeterministic automata by means of LBA (§4).

We shall denote, following the usual notations [6], by B_2 the two-element Boolean algebra $\{0,1 ; \cup, \cap, ^-\}$ and with $B_2^{(p,q)}$ the matrix Boolean algebra, whose elements are matrices with p rows and q columns, with entries belonging to B_2.

A linear Boolean automata is a 5-tuple $< B_2^{(l,1)}, B_2^{(n,1)}, B_2^{(m,1)}, f, g >$ where

$$f : B_2^{(n,1)} \times B_2^{(l,1)} \to B_2^{(n,1)},$$
$$g : B_2^{(n,1)} \times B_2^{(l,1)} \to B_2^{(m,1)}.$$

The finite sets $B_2^{(l,1)}$, $B_2^{(n,1)}$, $B_2^{(m,1)}$ are respectively the set of inputs, the set of states and the set of outputs. The mapping f is the next-state fonction, g is the output function.

These functions are given by:

$$f(s(t), u(t)) = s(t+1) = A \times s(t) \cup B \times u(t),$$
$$g(s(t), u(t)) = y(t) = C \times s(t) \cup D \times u(t),$$

where $s(t), s(t+1) \in B_2^{(n,1)}$, $u(t) \in B_2^{(l,1)}$, $y(t) \in B_2^{(m,1)}$, and A, B, C, D, the characteristic matrices of the circuit, belong respectively to $B_2^{(n,n)}$, $B_2^{(n,l)}$, $B_2^{(m,n)}$ and $B_2^{(m,l)}$.

§2. Controllability of LBA. **Definition 1.** *The LBA is k-controlable in the state s_0, iff for $\forall s \in B_2^{(n,1)}$ there exists a history of inputs $\underset{\tau=0}{\overset{k-1}{\mathcal{J}}}[u] = (u(0), u(1) \dots u(k-1))$, such that $\underset{\tau=0}{\overset{k-1}{\mathcal{J}}}[u]$ transforms the state s_0 in s.*

Proposition 1. *If the LBA is k-controlable in the state s_0, then $s_0 \in \mathcal{N}_{A^k}$, where $\mathcal{N}_{A^k}$ is the null space of the matrix A^k.*

Proof. We have established [6] that the evolution of the states in a LBA is given by

$$s(k) = A^k \times s_0 \cup \left[\bigcup_{\nu=0}^{k-1} A^{k-\nu-1} \times B \times u(\nu)\right]. \tag{1}$$

The equality (1) can be written in the form

$$s(k) = A^k \times s_0 \cup (A^{k-1}B \,\vdots\, A^{k-2} \,\vdots\, B \,\vdots\, \dots \,\vdots\, B) \times \underset{\tau=0}{\overset{k-1}{\mathcal{J}}}[u], \tag{2}$$

hence

$$s(k) \geqslant A^k \times s_0 . \tag{3}$$

If $s_0 = 0$ then (3) is fulfiled. If $s_0 \neq 0$, then, taking $s(k) = 0$, we get $A^k \times s_0 = 0$, therefore $s_0 \in \mathcal{N}_{A^k}$.

Let us suppose now that $s_0 \in \mathcal{N}_{A^k}$, then taking into account (2), we get

$$s(k) = K \times \underset{\tau=0}{\overset{k-1}{\mathcal{J}}}[u], \tag{4}$$

where $K = (A^{k-1}B \,\vdots\, A^{k-2}B \,\vdots\, \dots \,\vdots\, B) \in B_2^{(n,kl)}$ is the k-control matrix of the LBA.

Remark 1. The LBA is k-controllable in the state s_0 if the equation (4) has a solution for every $s(k) \in B_2^{(n,1)}$.

Theorem 1. *The LBA is k-controllable in the state s_0 iff the following conditions are fulfiled:*

a) $s_0 \in \mathcal{N}_{A^k}$,

b) *the history of inputs* $\underset{\tau=0}{\overset{k-1}{\mathcal{J}}}[u] = \overline{K^T \times \bar{s}}$ *transforms the state s_0 in s.*

Here K^T is the transpose of the matrix K and $\bar{s}$ in the complement of s.

Proof. The necessity follows from proposition 1. The sufficiency follows from a result from [5, p. 56] and the previous remark.

Consequence 1. *If $A^k \geqslant E$ then the LBA can be k-controllable at most in the state 0.*

Indeed, from $A^k \geqslant E$ follows that $A^k s \geqslant s$ and if $A^k s = 0$, then $s = 0$, hence $\mathcal{N}_{A^k} = \{0\}$.

Consequence 2. *It is easy to see that*

$$K \times \overline{K^T \times \bar{s}} \leqslant s, \quad \forall s \in B_2^{(n,1)} . \tag{5}$$

The LBA is k-controllable in s_0 only if $s_0 \in \mathcal{N}_{A^k}$ and (5) becomes an equality.

§3. Stability in LBA. The deterministic automaton $\mathcal{A} = [I, S, O, f, g]$ is k-stable in the state s, [8, p. 85] iff $g(f(s, pr), i) = g(f(s, r), i)$, $\forall p, r \in I^*$, $l(r) = k - 1$, $\forall i \in I$. Here $l(r)$ is the length of the word r.

For the special case of LBA this comes to:

$$\begin{aligned} &C \times A^{l(p)+k} \times s \cup (C \times A^{l(p)+k-1} \times B \,\vdots\, \dots \,\vdots\, C \times B \,\vdots\, D)(u(0), u(1), \dots \\ &\dots, u(l(p)-1)), \dots, u(l(p)+k))^T = C \times A^k \times s \cup (C \times A^{k-1} \times B \dots \\ &\vdots\, C \times B \,\vdots\, D)(u(l(p)), u(l(p)+1), \dots, u(l(p)+k))^T. \end{aligned} \tag{6}$$

Now we can prove the following sufficient condition for k-stability.

Theorem 2. *Let a LBA with $A \geqslant E$. Then the automaton is k-stable in the state s, if $k \geqslant c$ and $s \geqslant (A^c \times B)^{\cup}$, where c is the order of the matrix A [3, 4] and $(A^c \times B)^{\cup}$ is the column obtained by the union of the columns of the matrix $(A^c \times B)$.*

Proof. By comparing the two members of the equation (6) we see that $A^{l(p)+k} = A^c$. To have the desired stability it is enough to have

$$s \geqslant A^{l(p)-1} \times B \times u(0) \cup \dots \cup B \times u(l(p)-1).$$

Therefore, it will be sufficient to have

$$s \geqslant (A^{l(p)-1} \times B \cup \dots \cup A \times B \cup B) \times (1, \dots, 1)^T,$$

or, moreover, it will suffice to have

$$s \geqslant (A^c \times B)^{\cup}.$$

§4. Simulation of nondeterministic automata by LBA. Let now $\mathcal{A}$ be a nondeterministic finite automaton, $\mathcal{A} = [I, S, O, f, g]$, where I, S, O are the set of inputs, the set of states and the set of outputs of the automaton $\mathcal{A}$, $f: S \times I \to \mathcal{P}^*(S)$ is the next-state function and $g: S \times I \to \mathcal{P}^*(O)$ is the output function. Here $|I| = m$, $|S| = n$.

We claim that, under certain conditions, the LBA are able to simulate [1, p. 95] the nondeterministic automata.

Let $S = \{s_1, \dots, s_n\}$ and $S_1 \subseteq S$. The characteristic vector of the set S_1, χ_{S_1} is defined by

$$\left(\chi_{S_1}\right)_j = \begin{cases} 1, & \text{if } s_j \in S_1 \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that χ_{S_1} depends on the numbering of the elements of the set S.

Now, to every input symbol $i \in I$ we shall associate a matrix A_i, according to:

Definition 2. *Let φ be the mapping $\varphi : I \to B_2^{(n,n)}$ where $\varphi(i) = A_i$:* the j-th column of the matrix A_i being $\chi_{f(s_j, i)}$, $j = \overline{1,n}$

Remark 2. We have

$$\chi_{f(S_1, i)} = A_i \times \chi_{S_1}. \tag{7}$$

Indeed $f(S_1, i) = \bigcup_{s_1 \in S_1} f(s_1, i)$ and according to the definition 2, we get (7).

A renumbering π of the elements of the state set S, can be described by a permutation matrix P [7]. Then $\chi'_{s_{\pi(j)}} = P \times \chi_{s_j}$, $\chi'_{S_1} = P \times \chi_{S_1}$ and $A'_i = P \times A_i \times P^T$, where the accent denotes the new form of the matrices after the renumbering π.

Regarding the outputs, the characteristic vector of the response-set involved by the state s_j and the input i, is given by

$$\chi_{g(s_j, i)} = B_i \times \chi_{s_j},$$

where the columns of the matrix B_i are the characteristic vectors of the sets $\{g(s_j, i)\}_{j=\overline{1,n}}$.

For the sake of simplicity we shall study here only the case of the simulation of nondeterministic semiautomata (that means $O = S$, $g = f$), asking for the existence of a triple of mappings (t, v, w), $t : I \to B_2^{(m,1)}$, $v : S \to B_2^{(n,1)}$, $w : B_2^{(n,1)} \to \mathcal{P}^*(S)$, where $w \circ v = 1_S$, which makes commutative the diagram:

$$\begin{tikzcd} I \arrow[r, "f_s"] \arrow[d, "t"'] & \mathcal{P}^*(S) \\ B_2^{(m,1)} \arrow[r, "F_{v(s)}"'] & B_2^{(n,1)} \arrow[u, "w"'] \end{tikzcd}$$

Here $F_{v(s)}(t(i)) = A \times v(s) \cup t(i)$, taking the characteristic matrices of the linear Boolean semiautomaton, A and E.

The commutativity of the diagram yields to

$$w(A \times v(s) \cup t(i)) = f_s(i).$$

Choosing w as a one-to-one correspondence, by $\chi_{f(s)} = \bigcup_{s \in w(S)} v(s)$, we obtain

$$A \times v(s) \cup t(i) = w^{-1}(f_s(i)) = A_i \times \chi_s, \quad \forall i \in I,$$

taking into account formula (7).

Hence

$$A \times \chi_s \cup t(i) = A_i \times \chi_s. \tag{8}$$

Proposition 2. *The form of the equations (8) is invariant to a renumbering of the set of states.*

Proof. When the set S is subjected to a renumbering, these equations become:

$$A \times P^T \times \chi_s \cup t(i) = A_i \times P^T \times \chi_s, \quad \forall s \in S, \ i \in I,$$

which comes to

$$A \times P^T \cup T_i = A_i \times P^T,$$

where T_i is the $n \times n$ matrix $T_i = (t(i), t(i), \ldots, t(i))$, or

$$A \cup T_i \times P = A_i,$$

hence

$$A \cup T_i = A_i. \tag{9}$$

The previous results yield to the

Theorem 3. *The nondeterministic semiautomaton $\mathcal{A} = [I, S, f]$ can be simulated by a linear Boolean semiautomaton iff the system*

$$A \cup T_i = A_i, \qquad \forall i \in I,$$

is solvable in $\{A, T_{i_1}, \ldots, T_{i_m}\}$. This linear Boolean semiautomaton will be given by the equation

$$s(t+1) = A \times s(t) \cup u(t).$$

Remark 3. For autonomous semiautomata ($|I| = 1$) the equation (9) is allways solvable taking

$$t(i) = \bigcap_{j=1}^{n} \chi_{f(s_j, i)}.$$

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ASUPRA UNOR PROPRIETĂŢI ALE AUTOMATELOR LINIARE BOOLEENE

Rezumat

Clasa automatelor liniare booleene a fost introdusă în [6] şi studiată în [7]. Acest tip de automate reprezintă un model util pentru teoria reţelelor de comunicaţie, în teoria comutaţiei şi în studiul sistemelor de programare.

În această Notă sînt studiate unele proprietăţi ale automatelor liniare booleene legate de controlabilitate şi stabilitate. Sînt studiate de asemenea condiţiile în care automatele liniare booleene pot simula automate nedeterministe.

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ON THE SEQUENTIAL RELATIONS COMPUTED BY SEQUENTIAL MACHINES WITH A TIME-VARIANT STRUCTURE

BY

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The sequential relations computed by sequential machines was studied by C. E l g o t and T. M e z e i [5], A. G i l l [6], D e u e l and G i l l [4], J. N. G r a y and Michael H a r r i s o n [7]. A g a s a n d j a n [2] and Arto S a l o m a a [3] solved some basic problems concerning time-variant automata.

In this paper we study sequential relations computed by sequential machines with a time-variant structure.

Definition 1. *A sequential machine with a time-variant structure* (for short t. v. m.) is:

$$M = < X, Y, S_n, f_n^{n+1}, g_n, n = 1, 2, \ldots >,$$

where

1. X *is a finite nonempty set (inputs)*;
2. Y *is a finite nonempty set (outputs)*;
3. S_n *is a finite nonempty set (states at the n-th time instant)*;
4. f_n^{n+1} *is a function from* $S_n \times X$ *into* S_{n+1} *(direct transition function at the n-th time instant)*;
5. g_n *is a function from* $S_n \times X$ *into* Y *(output function at the n-th time instant).*

Let p be a word over X. Then $p = x_1 x_2 \ldots x_n$, with $x_i \in X$ for $1 \leqslant i \leqslant n$. We denote $p_k^n = x_{k+1} \ldots x_n$ (Agasandjan). It follows that $p = x_1 x_2 \ldots x_n = p_0^n$. By definition $p_k^k = e$. Now we define the functions f_n^m $(n \leqslant m)$, which describe transitions from the n-th time instant into the m-th time instant, by means of the functions f_n^{n+1}:

$f_n^n(s, e) = s$ for every $s \in S_n$;

$f_n^m(s, p_n^m) = f_{n+1}^m[f_n^{n+1}(s, x_{n+1}), p_{n+1}^m]$, where $n < m$ and $s \in S_n$.