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ASUPRA UNOR PROPRIETĂŢI ALE AUTOMATELOR LINIARE BOOLEENE

Rezumat

Clasa automatelor liniare booleene a fost introdusă în [6] şi studiată în [7]. Acest tip de automate reprezintă un model util pentru teoria reţelelor de comunicaţie, în teoria comutaţiei şi în studiul sistemelor de programare.

În această Notă sînt studiate unele proprietăţi ale automatelor liniare booleene legate de controlabilitate şi stabilitate. Sînt studiate de asemenea condiţiile în care automatele liniare booleene pot simula automate nedeterministe.

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ON THE SEQUENTIAL RELATIONS COMPUTED BY SEQUENTIAL MACHINES WITH A TIME-VARIANT STRUCTURE

BY

V. FELEA

The sequential relations computed by sequential machines was studied by C. Elgot and T. Mezei [5], A. Gill [6], Deuel and Gill [4], J. N. Gray and Michael Harrison [7]. Agasandjan [2] and Arto Salomaa [3] solved some basic problems concerning time-variant automata.

In this paper we study sequential relations computed by sequential machines with a time-variant structure.

Definition 1. *A sequential machine with a time-variant structure* (for short t. v. m.) is:

$$M = < X, Y, S_n, f_n^{n+1}, g_n; n = 1, 2, \dots >,$$

where

1. X *is a finite nonempty set (inputs)*;
2. Y *is a finite nonempty set (outputs)*;
3. S_n *is a finite nonempty set (states at the n-th time instant)*;
4. f_n^{n+1} *is a function from* $S_n \times X$ *into* S_{n+1} *(direct transition function at the n-th time instant)*;
5. g_n *is a function from* $S_n \times X$ *into* Y *(output function at the n-th time instant).*

Let p be a word over X. Then $p = x_1 x_2 \dots x_n$, with $x_i \in X$ for $1 \leqslant i \leqslant n$. We denote $p_k^n = x_{k+1} \dots x_n$ (Agasandjan). It follows that $p = x_1 x_2 \dots x_n = p_0^n$. By definition $p_k^k = e$. Now we define the functions f_n^m $(n \leqslant m)$, which describe transitions from the n-th time instant into the m-th time instant, by means of the functions f_n^{n+1}:

$f_n^n(s, e) = s$ for every $s \in S_n$;

$f_n^m(s, p_n^m) = f_{n+1}^m [f_n^{n+1}(s, x_{n+1}), p_{n+1}^m]$, where $n < m$ and $s \in S_n$.

There are the relations:

$f_n^m(s, p_n^m) = f_{m-1}^m [f_n^{m-1}(s, p_n^{m-1}), x_m]$ $m = n+1, n+2, \ldots$ and $s \in S_n$.

Now it is necessary to define recursively the functions G_n by:

$G_n(s, e) = e$ for every $s \in S_n$

$G_n(s, p_n^m) = g_n(s, x_{n+1}) G_{n+1}(f_n^{n+1}(s, x_{n+1}), p_{n+1}^m)$, $n < m$ for every $s \in S_n$.

We have:

$G_n(s, p_n^m) = G_n(s, p_n^{m-1}) g_{m-1}(f_n^{m-1}(s, p_n^{m-1}), x_m)$, $n < m$ for every $s \in S_n$.

Let M be a t. v. m. as in definition 1. We denote two relations calculated by M denoted by $R_1(M)$ and $R_2(M)$ as:

$$R_1(M) = \bigcup_{n=1}^{\infty} \bigcup_{s_n \in S_n} \bigcup_{p \in X^*} \{(p, G_n(s_n, p))\}$$

$$R_2(M) = \bigcup_{s_1 \in S_1} \bigcup_{p \in X^*} \{(p, G_1(s_1, p))\}.$$

Remark. We have: $R_2(M) \subseteq R_1(M)$.

Definition 2. *A relation $R \subseteq (X \times Y)^*$ is a T_1 — sequential relation over $X \times Y$ (shortly $SRT_1/X \times Y$) if there is a t. v. m. M such that $R = R_1(M)$.*

The relation R is a T_2-sequential relation over $X \times Y$ (shortly $SRT_2/X \times Y$) if there is a t. v. m. M such that $R = R_2(M)$.

Definition 3. *Given a t. v. m. M and $s_n \in S_n$, we define:*

$$g_{s_n}(p_n^m) = G_n(s_n, p_n^m), \text{ for every } p_n^m \in X^*.$$

A function g from X^* into Y^* is a T_1 — sequential function (T_2-sequential function) if there is a t. v. m. M and $n \in N$, $s_n \in S_n$ ($s_1 \in S_1$), such that $g = g_{s_n}$ ($g = g_{s_1}$).

Remark. A T_1-sequential function is a T_2-sequential function and conversely.

Theorem 1. *A relation $R \subseteq (X \times Y)^*$ is $SRT_1/X \times Y$ iff there exist T_1-sequential functions g_{ij}, $i = 1, 2, \ldots$ $j = 1, 2, \ldots, n_i$, such that:*

(1) $$R = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{n_i} g_{ij};$$

(2) *For each $x \in X$, and each g_{ij}, there exist $g_{i+1,t}$ such that:*

$$g_{ij}(xp) = g_{ij}(x) g_{i+1,t}(p) \text{ for all } p \in X^*.$$

Proof. "Only if" follows from Definition 2:

$$R(M) = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{n_i} g_{ij}, \text{ where } g_{ij}(p) = G_i(s_{ij}, p)$$

$$S_i = \{s_{i1}, s_{i2}, \ldots s_{in_i}\}\ G_i(s_{ij}, xp) = g_i(s_{ij}, x) G_{i+1}(f_i^{i+1}(s_{ij}, x), p)$$

If we denote $f_i^{i+1}(s_{ij}, x) = s_{i+1,t}$, then $g_{ij}(xp) = g_{ij}(x) g_{i+1,t}(p)$.

Conversely, we assume that (1) and (2) yelds. We define $M = < X, Y, S_i, f_i^{i+1}, g_i, i = 1, 2, \ldots >$, where $S_i = \{s_{i1}, s_{i2}, \ldots, s_{in_i}\}$, for every $i = 1, 2, \ldots$ Let k be $\min_t \{g_{ij}(xp) = g_{ij}(x) g_{i+1,t}(p)$ for all $p \in X^*\}$.

We define $f_i^{i+1}(s_{ij}, x) = s_{i+1,k}$ and $g_i(s_{ij}, x) = g_{ij}(x)$. It follows that $g_i(s_{ij}, p) = g_{ij}(p)$ and

$$R(M) = \bigcup_{i=1}^{\infty} \bigcup_{s_i \in S_i} \bigcup_{p \in X^*} \{(p, g_i(s_i, p))\} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{n_i} \bigcup_{p \in X^*} \{p, g_{ij}(p)\} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{n_i} g_{ij} = R.$$

Theorem 2. *A relation $R \subseteq (X \times Y)^*$ is $SRT_2/X \times Y$ iff there are T_1-sequential functions $g_1, g_2, \ldots, g_k$ over $X \times Y$ such that:*

$$R = \bigcup_{i=1}^{k} g_i.$$

Proof. If R is $SRT_2/X \times Y$, then there is a t. v. m. M and $s \in S_1$, such that $R = \bigcup_{s \in S_1} \bigcup_{p \in X^*} \{(p, g_s(p))\} = \bigcup_{s \in S_1} g_s$.

If $R = \bigcup_{i=1}^{k} g_i$, where g_i are T_1-sequential functions, then for each $h \in \{1, 2, \ldots k\}$ there exists a t. v. m., $M_h = < X, Y, S_n^h, f_n^{n+1,h}, g_n^h, n = 1, 2, \ldots >$ and a state $s_1^h \in S_1^h$, such that $g_h = g_{s_1^h}$. Assume, without loss of generality, that $S_n^i \cap S_n^j = \varnothing$ for $i \neq j$ and $n = 1, 2, \ldots$ Construct a t. v. m. $M = < X, Y, S_n, f_n^{n+1}, g_n, n = 1, 2, \ldots >$ where $S_1 = \{s_1^1, s_1^2, \ldots, s_1^k\}$, $S_n = \bigcup_{h=1}^{k} \{s \in S_n^h \mid \exists p_1^n \in X^*, f_1^{n,h}(s_1^h, p_1^n) = s\}$, $n = 1, 2, \ldots$ Let s be a state from S_n. We define $f_n^{n+1}(s, x) = f_n^{n+1,h}(s, x)$ and $g_n(s, x) = g_n^h(s, x)$ iff $s \in \{s \in S_n^h \mid \exists p_1^n \in X^*, f_1^{n,h}(s_1^h, p_1^n) = s\}$.

Clearly f_n^{n+1}, g_n are functions for $n = 1, 2, \ldots$ and:

$$R_2(M) = \bigcup_{s \in S_1} \bigcup_{p \in X^*} \{(p, g_s(p))\} = \bigcup_{h=1}^{k} g_h = R.$$

Definition 4. *A t. v. m. M is said to be connected (for short c. t. v. m.) if for every $n \in \{1, 2, \ldots\}$ and $s \in S_n$ there exist $s_1 \in S_1$ and $p_1^n \in X^*$ such that $f_1^n(s_1, p_1^n) = s$.*

Definition 5. *A relation $R \subseteq (X \times Y)^*$ is a CT_1-sequential relation if there exist a c. t. v. m. M, such that $R = R_1(M)$.*

The following definitions appear in [7].

Definition 6. *A relation $R \subseteq X^*$ is prefix closed if for all $p, q \in X^*$, $pq \in R$ implies $p \in R$; R is suffix closed if $pq \in R$ implies $q \in R$ for all $p, q \in X^*$.*

Definition 7. *A relation $R \subseteq X^* \times Y^*$ is said to be length preserving if for each $(p, q) \in R$, $p = p_1^k$, $q = q_1^k$.*

Definition 8. *A length preserving relation R is prefix closed if $(p_1^k \bar{p}_1^h, q_1^k \bar{q}_1^h) \in R$ implies $(p_1^k, q_1^k) \in R$.*

Definition 9. *A relation $R \subseteq (X \times Y)^*$ is said to be strongly extendable over X if for each $(p, q) \in R$ and $x \in X$, there exists $y \in Y$ such that $(px, qy) \in R$.*

Theorem 3. *A relation $R \subseteq (X \times Y)^*$ is CT_1-sequential relation over $X \times Y$ iff there exist T_1-sequential functions $g_1, g_2, \ldots, g_k$ over $X \times Y$ such that:*

$$R = \mathrm{suf}\,(\bigcup_{h=1}^{k} g_h)^{1)}$$

Proof. If R is CT_1-sequential relation over $X \times Y$ there exist a c. t.v.m. M such that:

$$R = R_1(M) = \bigcup_{n=1}^{r} \bigcup_{s_n \in S_n} \bigcup_{p \in X^*} \{(p, G_n(s_n, p))\}.$$

Let s_n be from S_n; because M is connected, there exist $s_1 \in S_1$ and $p_1^n \in X^*$, such that $f_1^n(s_1, p_1^n) = s_n$. Let (p, q) be from R. Then $q = G_n(s_n, p)$ for any $s_n \in S_n$. But $(p_1^n p, G_1(s_1, p_1^n p)) = (p_1^n p, G_1(s_1, p_1^n) G_n(s_n, p)) \in g_{s_1}$. It follows that $(p, G_n(s_n, p)) = (p, q) \in \mathrm{suf}\,(g_{s_1})$, hence $R \subseteq \bigcup_{s_1 \in S_1} \mathrm{suf}\,(g_{s_1})$. Clearly $\bigcup_{s_1 \in S_1} \mathrm{suf}\,(g_{s_1}) \subseteq R$. If $R = \mathrm{suf}\,(\bigcup_{h=1}^{k} g_h)$, there are t.v.m. $M_1, M_2, \ldots, M_k$ where $M_i = < X, Y, S_n^i, f_n^{n+1,i}, g_n^i;\ n = 1, 2, \ldots >$, $i = 1, 2, \ldots k$, and $s_1^i \in S_1^i$, such that $g_i = g_{s_1}^i$. Assume, without loss of generality, that M_i are connected and $S_1^i = \{s_1^i\}$, for every $i \in \{1, 2, \ldots, k\}$. It is easy to show that $R_1(M_i) = \mathrm{suf}\,(g_i)$, $i = 1, 2, \ldots, k$. Hence $R = \mathrm{suf}\,(\bigcup_{h=1}^{k} g_h) = \bigcup_{h=1}^{k} \mathrm{suf}\,(g_h) = \bigcup_{h=1}^{k} R_1(M_h)$. Since if R and $S \subseteq (X \times Y)^*$ are CT_1-sequential relations over $X \times Y$, then their union $R \cup S$ is a CT_1-sequential relation over $X \times Y$, it follows that R is CT_1-sequential relation. This completes the proof of Theorem 3.

Remarks. Let M be a c.t.v.m. $R_2(M) = R_1(M)$ iff R_2 is suffix closed. A $SRT_2/X \times Y$ is CT_1-sequential relation iff it is suffix closed.

Definition 10. *A finite time-variant automaton (shortly t.v.a.) is defined by:*

$$A = < X, S_n, M_n^{n+1}, s_0, F_n;\ n = 0, 1, 2, \ldots >$$

where:

1. *X is a finite nonempty set (inputs).*

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1) For $R \subseteq (X \times Y)^*$ suf (R) is the set:

$\{q \in (X \times Y)^* \exists p \in (X \times Y)^*, pq \in R\}$.

2. *S_n is a finite nonempty set (states at the n-th time instant).*
3. *M_n^{n+1} is a function from $S_n \times X$ into S_{n+1} (direct function transition at the n-th time instant).*
4. *s_0 is the initial state, $s_0 \in S_0$.*
5. *F_n is the final state set at the moment n, $F_n \subseteq S_n$.*

Now, we define recursively the functions M_n^m, with $n \leqslant m$, by:

$$M_n^n(s, e) = e, \text{ for every } s \in S_n,\ n = 0, 1, 2, \ldots$$

$$M_n^m(s, p_n^m) = M_{n+1}^m [M_n^{n+1}(s, x_{n+1}), p_{n+1}^m]$$

for $s \in S_n$ and $p_n^m = x_{n+1} \ldots x_m \in X^*$.

Definition 11. *Let A be a t.v.a.*

$$T(A) = \{p_o^n \in X^* / M_o^n (s_o, p_o^n) \in F_n,\ n = 0, 1, 2, \ldots\}.$$

is called the behaviour of A, or the language accepted by A.

A language R from X^* is acceptable by a t.v.a. (shortly X — t.v.a.a.) iff there exist a t.v.a. A, such $R = T(A)$.

A g a s a n d j a n in [1,2] and S a l o m a a in [3] define a finite time-variant automaton with $S_n \subseteq S$, $n = 1, 2, \ldots$ The following theorem gives a number of properties of T_1-sequential relations, T_2-sequential relations and CT_1-sequential relations. (The most part of them are those of sequential relations).

Theorem 4. *If $R \subseteq X^* \times Y^*$ is $RT_2/X \times Y$, then:*

1. *R is $X \times Y$ t.v.a.a.*
2. *R is length preserving.*
3. *R is prefix closed.*
4. *R is strongly extendable over X;*
5. *If $Y' \supseteq Y$, then R is $SRT_2/X \times Y'$;*
6. *R is infinite (dom $(R) = \{x \in X^*,\ \exists y \in Y^*,\ (x, y) \in R\} = X^*$).*

If R is $SRT_1/X \times Y$ or R is CT_1-sequential relation over $X \times Y$, then 2), 3), 4), 5), 6) hold and

7. *R is suffix closed.*

The proof is immediately.

Theorem 5. *$R \subseteq X^* \times Y^*$ is a T_1-sequential function over X iff:*

1. *R is $X \times Y$ — t.v.a.a.;*
2. *R is a function;*
3. dom $(R) = X^*$;
4. *R is prefix closed.*

Proof. If R is T_1-sequential function then by Theorems 2 and 4, 1), 3), 4) follow. Conversely, if R is $X \times Y$ — t.v.a.a., then there exists

$A = < X \times Y, S_n, M_n^{n+1}, s_o, F_n,\ n = 0, 1, 2, \ldots >$ such that

$R = T(A) = \{(p_o^n, q_o^n) \in (X \times Y)^* / M_o^n (s_o, (p_o^n, q_o^n)) \in F_n,\ n = 0, 1, 2, \ldots\}$

Since R is a function and dom $(R) = X^*$, for every $p_o^n \in X^*$, there is a unique $q_o^n \in Y^*$, such that $M_o^n(s_o, (p_o^n, q_o^n)) \in F_n$. Construct $M = \; = < X, Y, S_n, f_n^{n+1}, g_n, \; n = 1, 2, \ldots >$, where $S_1 = \{s_0\}$, $S_n = \{M_o^{n-1}(s_o, (p_o^{n-1}, q_o^{n-1})) / (p_o^{n-1}, q_o^{n-1}) \in R\}$

$f_n^{n+1}(s, x) = M_{n-1}^n(s, (x, y))$, $g_n(s, x) = y$ for every $s \in S_n$, and $n = 1, 2, \ldots$ y is chosen such that $M_{n-1}^n(s, (x, y)) \in F_n$. We show now that y is unique; let s be from S_n and $x \in X$; there is $(p_o^{n-1}, q_o^{n-1}) \in (X \times Y)^*$ such that $M_o^{n-1}(s_o, (p_o^{n-1}, q_o^{n-1})) = s$ and $(p_o^{n-1}, q_o^{n-1}) \in R$. For $p_o^{n-1}x$ there exists a unique q' with $(p_o^{n-1}x, q') \in R$, but R is prefix closed, hence $(p_o^{n-1}, q'') \in R$, then $q'' = q_o^{n-1}$, $q' = q''y = q_o^{n-1}y$, $(p_o^{n-1}x, q_o^{n-1}y) \in R$ and y is unique with this property. $M_o^n(s_o, (p_o^{n-1}x, q_o^{n-1}y)) \in F_n$ $M_{n-1}^n(M_o^{n-1}(s_o, (p_o^{n-1}, q_o^{n-1})), (x, y)) \in F_n$ $M_{n-1}^n(s, (x, y)) \in F_n$. It follows that for every $s \in S_n$ and $p_n^m \in X^*$, there exists a unique $q_n^m \in Y^*$, such that $M_{n-1}^{m-1}(s, (p_n^m, q_n^m)) \in F_{m-1}$. Then we have:

$$f_n^m(s, p_n^m) = M_{n-1}^{m-1}(s, (p_n^m, q_n^m)); \; g_n(s, p_n^m) = q_n^m .$$

$\bigcup_{p \in X^*} \{(p, g_{s_o}(p))\} = R$, hence R is T_1-sequential function.

Definition 12. *Let A be a t.v.a. A is said to be connected if for each $n \in \{1, 2, \ldots\}$ and $s \in S_n$ there is $p_o^n \in X^*$, such that $M_o^n(s_o, p_o^n) = s$.*

Theorem 6. *Let A be a connected t.v.a. The following conditions are equivalent:*

1. *$T(A)$ is prefix closed.*
2. *for each $(s, x) \in (S_n - F_n) \times X$, $M_n^{n+1}(s, x) \in F_{n+1}$, $n = 1, 2, \ldots$*

Proof. Suppose that condition (1) is satisfied. We assume that there exists $(s, x) \in (S_n - F_n) \times X$, such that $M_n^{n+1}(s, x \in F_{n+1}$. There is $p_o^n \in X^*$, such that $M_o^n(s_o, p_o^n) = s$, then $M_o^{n+1}(s_o, p_o^n x) \in F_{n+1}$, that is $p_o^n x \in T(A)$, which is prefix closed, hence $p_o^n \in T(A)$, $M_o^n(s_o, p_o^n) = s \in F_n$. This contradicts that $s \in S_n - F_n$.

Now, suppose that (2) is satisfied. We assume that $T(A)$ is not prefix closed. There is $p_o^h \bar{p}_o^k \in T(A)$ and $p_o^h \notin T(A)$, where $h, k \neq 0$. We denote by i the minimum number, such that $p_o^h x_1 x_2 \ldots x_i \notin T(A)$ and $p_o^h x_1 x_2 \ldots \ldots x_{i+1} \in T(A)$, $0 \leqslant i \leqslant k - 1$. We have:

$$M_o^{h+i}(s_o, p_o^h x_1 x_2 \ldots x_i) = t \notin F_{h+i}, \; t \in S_{h+i} - F_{h+i};$$

$$M_{h+i}^{h+i+1}(t, x_{i+1}) = M_o^{h+i+1}(s_o, p_o^h x_1, \ldots x_{i+1}) \in F_{h+i+1}$$ this is a contradiction.

Corollary 1. *Let A be a connected t.v.a. The following conditions are equivalent:*

1. *$T(A)$ is prefix closed.*
2. *for every $s \in S_n - F_n$ and $q_o^k \in X^*$ $M_n^{n+k}(s, q_o^k) \notin F_{n+k}$, $n = 1, 2, \ldots,$ $k = 1, 2, \ldots$*

Definition 13. *Let A be a t.v.a. Two states $s, t \in S_n$, $n \in \{1, 2, \ldots\}$, are said to be equivalent, if for each $p_o^k \in X^*$, $M_n^{n+k}(s, p_o^k) \in F_{n+k}$ iff $M_n^{n+k}(t, p_o^k) \in F_{n+k}$.*

A is said to be minimal if no pair of distinct states from $S_n (n \in \{1, 2, \ldots\})$ are equivalent.

Corollary 2. *Let A be a t.v.a. connected and minimal, for every $s \in (S_n - F_n)$ and $q_o^k \in X^*$, $M_n^{n+k}(s, q_o^k) \notin F_{n+k}$, $n = 1, 2, \ldots$ Then the following conditions are equivalent:*

1. $|S_n - F_n| = 1$, $n = 1, 2, \ldots$
2. *$T(A)$ is prefix closed.*

Proof. If $T(A)$ is prefix closed, we assume that there exist n and $s, t \in S_n - F_n$, where $s \neq t$. Then we have $M_n^{n+k}(s, q_o^k) \notin F_{n+k}$ and $M_n^{n+k}(t, q_o^k) \notin F_{n+k}$ for every $q_o^k \in X^*$, i.e. t and s are equivalent, which contradicts that A is minimal.

If $|S_n - F_n| = 1$, $n = 1, 2, \ldots$, then $S_n - F_n = \{s_n\}$, we assume that $T(A)$ is not prefix closed: there exists $p_o^k q_o^h \in T(A)$ and $p_o^k \notin T(A)$.

It follows that:

$M_o^k(s_o, p_o^k) \in S_k - F_k$, $M_o^k(s_o, p_o^k) = s_k$, $M_o^{k+h}(s_o, p_o^k q_o^h) = M_k^{k+h}(s_k, q_o^h) \in F_{k+h}$.

Then we have $s_k \in S_k - F_k$ and $M_k^{k+h}(s_k, q_o^h) \in F_{k+h}$, which contradicts the hypotesis.

Theorem 7. *Let A be a t.v.a. over $X \times Y$. $T(A)$ is T_1-sequential function iff:*

1. $s_0 \in F_0$.
2. *$T(A)$ is prefix closed.*
3. *for each $(s, x) \in F_n \times X$, there is a unique $y \in Y$, such that $M_n^{n+1}(s, (x, y)) \in F_{n+1}$, $n = 0, 1, 2, \ldots$*

Proof. If $T(A)$ is T_1-sequential function, since $M_o^o(s_o, (e, e)) = s_o$ and $(e, e) \in T(A)$ it follows $s_0 \in F_0$. By theorem 5, $T(A)$ is prefix closed. Suppose that (3) is false. Then we have:

4. There exist $n \in N$, and $(s, x) \in F_n \times X$, such that $M_n^{n+1}(s, (x, y)) \notin F_{n+1}$ for every $y \in Y$, or
5. There exists $n \in N$, $y_1 \neq y_2$ and $(s, x) \in F_n \times X$, such that $M_n^{n+1}(s, (x, y_1)) \in F_{n+1}$ and $M_n^{n+1}(s, (x, y_2)) \in F_{n+1}$. If 4) yelds, there exists $(p_o^n, q_o^n) \in (X \times Y)^*$, such that $M_o^n(s_o, (p_o^n, q_o^n)) = s$, $M_o^{n+1}(s_o, (p_o^n x, q_o^n y)) \notin F_{n+1}$ for every $y \in Y$, hence $(p_o^n x, q_o^n y) \notin T(A)$ for every $y \in Y$. We want to show that $(p_o^n x, \bar{q}_o^n y) \in T(A)$ for

$$q_o^n \neq \bar{q}_o^n . \tag{6}$$

Suppose that $(p_o^n x, \bar{q}_o^n y_1) \in T(A)$ for $\bar{q}_o^n \neq q_o^n$, because $T(A)$ is prefix closed it follows $(p_o^n, \bar{q}_o^n) \in T(A)$, which contradicts that $T(A)$ is function. Relation 6) means that dom $(T(A)) \neq X^*$, which contradicts Theorem 5. Now we suppose that 5) holds. We know that there exists (p_o^n, q_o^n) such that

$M_o^n(s_o, (p_o^n, q_o^n)) = s$.

$M_o^{n+1}(s_o, (p_o^n x, q_o^n y_1)) \in F_{n+1}$ and $M_o^{n+1}(s_o, (p_o^n x, q_o^n y_2)) \in F_{n+1}$ that is

$(p_o^n x, q_o^n y_1) \in T(A)$ and $(p_o^n x, q_o^n y_2) \in T(A)$ with $y_1 \neq y_2$, which contradicts that $T(A)$ is function. Suppose that 1), 2), 3) hold. Because $T(A)$ is $X \times Y$ — t.v.a.a. and prefix closed it sufficies to show that dom $(T(A)) = X^*$ and $T(A)$ is a function (by Theorem 5). From 1), 2), 3) it follows that for every $p_o^n \in X^*$, there exists a unique $q_o^n \in Y^*$ such that $(p_o^n, q_o^n) \in T(A)$. This completes the proof.

Let $R(X, Y)$, $RT_1(X, Y)$, $RT_2(X, Y)$, $CRT_1(X, Y)$ be the class of sequential relations over $X \times Y$, the class of $SRT_1/X \times Y$, the class of $SRT_2/X \times Y$ and the class of CT_1-sequential relations, respectively.

Theorem 8. *There exist the following relations*:
$R(X, Y) \subset RT_2(X, Y)$, $R(X, Y) \subset CRT_1(X, Y) \subset RT_1(X, Y)$.

Proof. Every sequential machine $M = \langle X, Y, S, \delta, \lambda \rangle$ may be interpreted as a t.v.m., for which
$S_n = S$, $f_n^{n+1}(s, x) = \delta(s, x)$, $g_n(s, x) = \lambda(s, x)$, $n \in N$, $x \in X$ and as a c.t.v.m. for which $S_n = \{s/s \in S,\ \exists s_1 \in S\ \exists p_1^n \in X^*,\ \delta(s_1, p_1^n) = s\}.f_n^{n+1}(s, x) = \delta(s, x)$, $g_n(s, x) = \lambda(s, x)$, $n \in N$.

It is easy to show that there is a t.v.m. M, for which the relation $R_2(M)$ is not suffix closed, hence $R_2(M)$ is not a sequential relation over $X \times Y$. There is a c.t.v.m. M for which $R_1(M)$ is not a sequential relation.

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ASUPRA RELAȚIILOR SECVENȚIALE CALCULATE DE MAȘINI SECVENȚIALE CU STRUCTURĂ VARIABILĂ

Rezumat

Se definesc cîteva relații calculate de mașini secvențiale cu structură variabilă, dîndu-se caracterizări ale lor, precum și legătura cu limbajele acceptate de automate cu structură variabilă. Se stabilesc relațiile între ele și relațiile secvențiale.

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SAINT-VENANT PROBLEM FOR CYLINDRICAL BEAMS WITH GENERAL ANISOTROPY

BY

C. I. BORȘ

1. General equations and formulation of the problem. Let us consider a beam limited by a cylindrical surface $\mathfrak{S}$ and by two planes $x_3 = 0$, $x_3 = h$ $(h > 0)$. We shall denote by $\mathfrak{V}$ the region occupied by the beam and by S the domain of a cross-section. We denote by Γ the boundary of S.

We suppose that there are no body forces so that the equilibrium equations are given by

$$\sigma_{ij,j} = 0 \quad \text{in } \mathfrak{V}\ ^{*)}. \tag{1}$$

The lateral surface is supposed to be free from tractions so that we have the boundary conditions

$$\sigma_{ij} n_j = 0 \quad \text{on } \mathfrak{S}, \tag{2}$$

where n_i are the direction cosines of the exterior normal to the surface $\mathfrak{S}$.

The tractions applied at the end $x_3 = 0$ are equivalent with a resultant $\bar{R}$ and a moment resultant $\bar{\mathfrak{M}}$:

$$\bar{R} = (R_1, R_2, R_3), \quad \bar{\mathfrak{M}} = (\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3) \quad \text{on } x_3 = 0, \tag{3}$$

where R_i and $\mathfrak{M}_i$ are given.

On the end $x_3 = h$ there are applied tractions so that the beam is in equilibrium.

The material of the beam is supposed to be with general anisotropy so that the relations between the components of the stress σ_{ij} and the strain components γ_{ij} are given by [1, 3]

*) We use the summation convention over the repeated indices. The index j after comma indicates partial differentiation with respect to x_j.