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SOME PROPERTIES OF THE WEAK RING CATEGORY

BY

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§1. The concept of weak ring was introduced by Al. Climescu [1], [2], as a generalization — in the direction of the distributive law — of the classic ring concept. Some problems on this new algebraic system were studied by T. Luchian in several papers, among them [4], [5].

A weak ring $R^{\cdot}$ is defined, [2], as an abelian additive group R_+ with a multiplication „·" related to the addition by the weakened distributive laws

$$x \cdot (y + z) + x \cdot 0 = x \cdot y + x \cdot z, \tag{1}$$

$$(y + z) \cdot x + 0 \cdot x = y \cdot x + z \cdot x, \tag{2}$$

which imply

(3) $\quad x \cdot 0 = \varphi(x) + c, \qquad$ (4) $\quad 0 \cdot x = \psi(x) + c,$

where φ, ψ are group endomorphisms of R_+ and c a fixed element. We shall denote it by

$$R^{\cdot} = (R_+ ;\ \varphi,\ \psi,\ c).$$

Remark 1. Obviously, the class of ordinary rings is obtained from the above definition, for $c = 0$, $\varphi = \psi = \odot$ — the null—endomorphism.

If $R^{\times}$ is an ordinary ring (with the multiplication not necessarily associative), then its additive group R_+ with a new multiplication „·" defined by

$$x . y = x \times y + \varphi(x) + \psi(y) + c, \tag{5}$$

— with given group endomorphisms φ, ψ and $c \in R_+$ — is a weak ring $R^{\cdot}$, [2] and conversely :

Remark 2. To a weak ring $R^{\cdot}$ it corresponds an unique ordinary ring $R^{\times}$, with the same additive group R_+ and the multiplication „$\times$" defined by

$$x \times y = x \cdot y - x \cdot 0 - 0 \cdot y + 0 \cdot 0. \tag{6}$$

We shall call $R^{\times}$ *the associated ordinary ring of the weak ring* $R^{\cdot}$.

§2. Weak ring homomorphisms are defined as usually and the following three propositions hold, [4], [5]:

Proposition 1. *If $f^{\cdot}: R_1^{\cdot} \to R_2^{\cdot}$ is a weak ring homomorphism with*

$$R_i^{\cdot} = (R_{i,+}\,;\,\psi_i\,,\varphi_i,\ c_i), \qquad i = 1, 2,$$

then we have necessarily

$$(7)\quad f_+ \circ \varphi_1 = \varphi_2 \circ f_+\,, \qquad (7')\quad f_+ \circ \psi_1 = \psi_2 \circ f_+ \qquad (7'')\quad f(c_1) = c_2.$$

Proposition 2. *If $f^{\cdot} : R_1^{\cdot} \to R_2^{\cdot}$ is a weak ring homomorphism, then f induces an ordinary ring homomorphism $f^{\times} : R_1^{\times} \to R_2^{\times}$, where $R_i^{\times}$, $i = 1, 2$, is the associated ordinary ring of $R_i^{\cdot}$, such that $f^{\cdot}$ and $f^{\times}$ have the same restriction in the additive groups,*

$$f_+^{\times} = f_+^{\cdot} : R_{1,+} \to R_{2,+}$$

and the map $\alpha : \mathrm{Hom}\,(R_1^{\cdot}, R_2^{\cdot}) \to \mathrm{Hom}\,(R_1^{\times}, R_2^{\times})$, $\alpha\,(f^{\cdot}) = f^{\times}$ *is injective.*

A somehow converse assertion is the following:

Proposition 3. *If $f^{\times} : R_1 \to R_2$ is an ordinary ring homomorphism and $\varphi_i\,, \psi_i \in \mathrm{End}\,(R_{i,+})$, $i = 1, 2$, such that (7), (7′), (7″) hold — with f_+ the restriction of $f^{\times}$ to the additive groups and $c_1 \in R_1$ an arbitrary fixed element — then $f^{\times}$ determines a weak ring homomorphism $f^{\cdot}$*

$$f^{\cdot} : R_1^{\cdot} \to R_2^{\cdot}, \qquad R_i^{\cdot} = (R_{i,+}\,;\,\varphi_i\,,\ \psi_i,\ c_i).$$

From Propositions 2, 3, one can easily deduce:

Proposition 4. *Let R_1 be an ordinary ring, $R_2^{\cdot} = (R_{2,+}\,;\,\varphi_2,\ \psi_2\,, c_2)$ a weak ring, $R_2^{\times}$ its associated ordinary ring, $f : R_1 \to R_2^{\times}$ an ordinary ring homomorphism; then f induces a weak ring homomorphism $f^{\cdot}$*

$$f^{\cdot} : R_1^{\cdot} = R_1 \to R_2^{\cdot}$$

if and only if

$$\mathrm{Im}\, f_+ \subseteq \ker \varphi_2 \cap \ker \psi_2 \ \text{and}\ c_2 = 0. \tag{8}$$

In fact, by Remark 1, R_1 can be seen as a weak ring $R_1 = R_1^{\cdot} = (R_+\,;\,\varphi = 0,\ \psi = 0,\ c_1 = 0)$; then (7), (7′), (7″) are equivalent to (8).

Obviously we have:

Proposition 5. *The class of weak rings, together with their homomorphisms is a category $\mathcal{S}$ and the category $\mathcal{A}^*$ of ordinary rings is a full subcategory of $\mathcal{S}$.* It is easy to verify:

Proposition 5′. *The class $\widehat{\mathcal{S}}$ of weak rings of the form $R^{\cdot} = (R_+\,;\,\varphi,\,\psi,\,0)$, together with their homomorphisms, are a category and a full subcategory of $\mathcal{S}$; clearly $\mathcal{A}^* \subset \widehat{\mathcal{S}}$ and the null ring 0 is the zero object of $\widehat{\mathcal{S}}$.* Using Remark 2 and Proposition 2, we have:

Proposition 6. *There exists a faithful covariant functor $\alpha : \mathcal{S} \to \mathcal{A}^*$ defined by the scheme*

$$\begin{array}{ccc} \mathcal{S} \ni R_1^{\cdot} & \xrightarrow{\ \alpha\ } & \alpha\,(R_1^{\cdot}) = R_1^{\times} \in \mathcal{A}^* \\ \downarrow f^{\cdot} & \longrightarrow & f^{\times} \downarrow \\ R_2^{\cdot} & \longrightarrow & \alpha\,(R_2^{\cdot}) = R_2^{\times} \end{array}$$

where $R_i^{\times}$ is the associated ordinary ring of the weak ring $R_i^{\cdot}$, $i = 1, 2$, and $f^{\times}$ the ordinary ring homomorphism induced by $f^{\cdot}$.

§3. **Theorem 7.** *In the weak ring category $\mathcal{S}$ there exists the pull-back $E^{\cdot} \underset{R^{\cdot}}{\times} F^{\cdot}$ for any weak rings $E^{\cdot} = (E_+\,;\,\varphi_1,\,\psi_1,\,c_1)$, $F^{\cdot} = (F_+\,;\,\varphi_2\,,\,\psi_2\,,\,c_2)$, $R^{\cdot} = (R_+\,;\,\varphi,\,\psi,\,c)$ and any weak ring homomorphisms*

$$f_1 : E^{\cdot} \to R^{\cdot},\ f_2 : F^{\cdot} \to R^{\cdot},$$

namely this pull-back is the weak ring defined by

$$E^{\cdot} \underset{R^{\cdot}}{\times} F^{\cdot} = G^{\cdot} = (G_+\,;\,(\varphi_1\,,\,\varphi_2),\ (\psi_1\,,\,\psi_2),\ (c_1\,,\,c_2)).$$

where $G_+ = E_+ \underset{R_+}{\times} F_+$ is the pull-back of the additive abelian groups of the given weak rings.

Proof. It is known that there exists the pull-back diagram of abelian groups

$$\begin{tikzcd} E_+ \underset{R_+}{\times} F_+ \arrow[r, "p_F"] \arrow[d, "p_E"'] & F_+ \arrow[d, "f_{2,+}"] \\ E_+ \arrow[r, "f_{1,+}"] & R_+ \end{tikzcd} \tag{9}$$

$$E_+ \underset{R_+}{\times} F_+ = \{(x_1\,,\,x_2) \mid x_1 \in E_+\,,\,x_2 \in F_+\,,\,f_1 x_1 = f_2 x_2\} = G_+\,,$$

and for any other commutative square

$$\begin{tikzcd} X_+ \arrow[r, "h_{2,+}"] \arrow[d, "h_{1,+}"'] & F_+ \arrow[d, "f_{2,+}"] \\ E_+ \arrow[r, "f_{1,+}"] & R_+ \end{tikzcd} \tag{10}$$

there is a unique group homomorphism q_+

(11) $$q_+ : X_+ \to E_+ \underset{R_+}{\times} F_+ \qquad q_+(x) = (h_{1,+}\, x, h_{2,+}\, x).$$

such that

(12) $$h_{1,+} = p_E \circ q_+ , \; h_{2,+} = p_F \circ q_+.$$

Let us define the maps

(13) $$(\varphi_1, \varphi_2) : G_+ \to G_+ \qquad (\varphi_1, \varphi_2)(x_1, x_2) = (\varphi_1(x_1), \varphi_2(x_2))$$

(13') $$(\psi_1, \psi_2) : G_+ \to G_+ \qquad (\psi_1, \psi_2)(x_1, x_2) = (\psi_1(x_1), \psi_2(x_2))$$

and let $(c_1, c_2) \in G_+$ be the fixed element (from (7'') $\Rightarrow f_1(c_1) = c = f_2(c_2)$ in R). The maps (13), (13') are evidently group homomorphisms and we have

$$G_+ \ni (x_1, x_2) \to (\varphi_1(x_1), \varphi_2(x_2)) \in G_+,$$

that is

$$f_1[\varphi_1(x_1)] = f_2[\varphi_2(x_2)]$$

since, applying (7), (7') from Proposition 1,

$$\varphi \circ f_i = f_i \circ \varphi_i , \; \psi \circ f_i = f_i \circ \psi_i , \; i = 1, 2;$$

hence

$$f_1[\varphi_1(x_1)] = \varphi[f_1(x_1)] = \varphi[f_2(x_2)] = f_2[\varphi_2(x_2)].$$

Defining in G_+ a second composition law, the multiplication „$\cdot$" by

(14) $$(x_1, x_2) \cdot (y_1, y_2) = (x_1 \cdot y_1, x_2 \cdot y_2),$$

one can easily verify the weak distributive laws (1), (2) and

(3') $$(x_1, x_2) \cdot (0, 0) = (\varphi_1, \varphi_2)(x_1, x_2) + (c_1, c_2),$$

(4') $$(0, 0) \cdot (x_1, x_2) = (\psi_1, \psi_2)(x_1, x_2) + (c_1, c_2).$$

So we have obtained the weak ring

$$G^{\cdot} = (G_+ ; (\varphi_1, \varphi_2), (\psi_1, \psi_2), (c_1, c_2)).$$

The group projections p_E, p_F from (9) obviously preserve the multiplication (14) and the conditions (7), (7'), (7'') hold, for instance

$$p_E \circ (\psi_1, \psi_2) = \psi_1 \circ p_E, \quad p_E(c_1, c_2) = c_1.$$

So the diagram

(9₁) $$\begin{array}{ccc} G^{\cdot} & \xrightarrow{p_F} & F^{\cdot} \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle f_2} \\ E^{\cdot} & \xrightarrow{f_1} & R^{\cdot} \end{array}$$

is a commutative square in the category $\mathcal{S}$. Given in $\mathcal{S}$ another commutative square of the form

(9₂) $$\begin{array}{ccc} X^{\cdot} & \xrightarrow{h_2} & F^{\cdot} \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ E^{\cdot} & \xrightarrow{f_1} & R^{\cdot} \end{array}, \qquad \begin{array}{c} X^{\cdot} = (X_+ ; \varphi_0, \psi_0, c_0) \\ h_i \circ \varphi_0 = \varphi_i \circ h_i \quad h_i \circ \psi_0 = \psi_i \circ h_i \\ {\scriptstyle i=1,\;2} \end{array}$$

this diagram induces the commutative group square (10) and the unique group homomorphism q_+, which can be extended to a weak ring homomorphism q, for q_+ preserves the multiplication (14) and we have

(7₁) $$q_+ \circ \varphi_0 = (\varphi_1, \varphi_2) \circ q_+ , \qquad (7'_1) \quad q_+ \circ \psi_0 = (\psi_1, \psi_2) \circ q_+$$

since, by (11),

$$(q_+ \circ \varphi_0)\, b = ((h_1 \circ \varphi_0)\, b, (h_2 \circ \varphi_0)\, b) = ((\varphi_1 \circ h_1)\, b, (\varphi_2 \circ h_2)\, b) =$$
$$= [(\varphi_1, \varphi_2) \circ q]\, b$$

and q satisfies the properties (12).

So we have proved that (9₁) is a pull-back diagram in the category $\mathcal{S}$. As a consequence we have:

Remark 3. In the full subcategory $\hat{\mathcal{S}}$, for any weak ring homomorphism $f : R_1^{\cdot} \to R_2^{\cdot}$, there exists ker f, namely it is given by the pull-back

$$\begin{array}{ccc} \ker f & \longrightarrow & 0 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle 0} \\ R_1^{\cdot} & \xrightarrow{f} & R_2^{\cdot} \end{array}, \qquad \ker f = R_1^{\cdot} \underset{R_2^{\cdot}}{\times} 0$$

§ 4. **Theorem 8.** *Let be*

$$R_i^{\cdot} = (R_{i,+} ; \varphi_i, \psi_i, c_i), \quad i \in I$$

any set of weak rings; then, in the category $\mathcal{S}$ there exists the direct product

$$R^{\cdot} = \prod_{i \in I} R_i^{\cdot}$$

of the above set.

Proof. In the direct product of abelian groups $R_{i,+}$

$$R_+ = \prod_{i \in I} R_{i,+},$$

we define the multiplication „$\cdot$" of the elements $a = (a_i)_{i \in I}$, $b = (b_i)_{i \in I}$, $a, b \in R_+$ by

(15) $$a \cdot b = (a_i \cdot b_i)_{i\in I}.$$

Then (1), (2) are easily verified. If we note

(16) $$\varphi = (\varphi_i)_{i\in I}, \quad \psi = (\psi_i)_{i\in I}, \quad c = (c_i)_{i\in I}.$$

with $\varphi(a) = (\varphi_i(a_i))_{i\in I}$, $\psi(a) = (\psi_i(a_i))_{i\in I}$ the maps φ, ψ are endomorphisms of R_+ and — by definitions (15), (16) and (3), (4) for $R^\cdot$ — we obtain the properties

$$a\cdot 0 = (a_i)_{i\in I}\cdot(0_i)_{i\in I} = (a_i\cdot 0_i)_{i\in I} = (\varphi_i(a_i)+c_i)_{i\in I} = (\varphi_i)_{i\in I}(a)+c;$$

similarly $0\cdot a = (\psi_i)_{i\in I}(a)+c$, $0\cdot 0 = c$.
The group projections

$$p_{i,+} : R_+ \to R_{i,+}, \quad i\in I$$

preserve the multiplication (15), hence p_i, $i\in I$, are weak ring homomorphisms. So we obtained the weak ring

(17) $$R^\cdot = (R_+;\ (\varphi_i)_{i\in I},\ (\psi_i)_{i\in I},\ (c_i)_{i\in I})$$

and the projection diagram

(17′) $$\{p_i : R^\cdot \to R_i^\cdot\}_{i\in I}.$$

Let us prove that (17′) is a couniversal diagram: given any diagram of the form

(18) $$\begin{tikzcd} B^\cdot \arrow[r, "u_i"] \arrow[d, dotted, "u"'] & R_i^\cdot \\ R^\cdot \arrow[ur, "p_i"'] & \end{tikzcd} \qquad i \in I$$

there exists a unique weak ring homomorphism $u : B^\cdot \to R^\cdot$ such that (18) is a commutative diagram, that is

(19) $$u_i = p_i \circ u, \quad i \in I.$$

Indeed, the assertion holds for the additive groups diagram

(18′) $$\begin{tikzcd} B_+ \arrow[r, "u_{i,+}"] \arrow[d, "u_+"'] & R_{i,+} \\ R_+ \arrow[ur, "p_{i,+}"'] & \end{tikzcd} \quad , \qquad (19') \quad U_{i,+} = p_{i,+}\circ u_+, \quad i\in I$$

and the group homomorphism u_+ preserves the multiplication of $B^\cdot$: since u_i and p_i are weak ring homomorphisms and (19′) hold, we have

$$u_i(b_1\cdot b_2) = u_i(b_1)\cdot u_i(b_2) = (p_i\circ u)b_1\cdot(p_i\circ u)b_2 = p_i[u(b_1)\cdot u(b_2)],$$

but also $u_i(b_1\cdot b_2) = (p_i\circ u)(b_1\cdot b_2) = p_i[u(b_1\cdot b_2)]$. So we obtain

$$p_i[u(b_1)\cdot u(b_2)] = p_i[u(b_1\cdot b_2)],\ i\in I \Rightarrow u(b_1\cdot b_2) = u(b_1)\cdot u(b_2).$$

In conclusion, (17) is a weak ring which is the direct product of the weak ring set $(R_i)_{i\in I}$

$$R^\cdot = (R_+;\ (\varphi_i)_{i\in I},\ (\psi_i)_{i\in I},\ (c_i)_{i\in I}) = \prod_{i\in I} R_i^\cdot.$$

Remark 4. Since $\mathcal{S}$ is a category with pull-back and direct product, it follows that $\mathcal{S}$ has projective limits; particularly in $\mathcal{S}$ there exists ker (u, v), for every weak-ring homomorphism pair (u, v).
Ker (u, v) is constructed as follows:
Let be a diagram in $\mathcal{S}$

$$\begin{tikzcd} A^\cdot \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B^\cdot \end{tikzcd}$$

The universality of the direct product in $\mathcal{S}$ implies that there exists two weak ring homomorphisms

$$u' : A^\cdot \to A^\cdot \times B^\cdot, \quad v' : A^\cdot \to A^\cdot \times B^\cdot$$

such that the diagrams

$$\begin{tikzcd} & A^\cdot & \\ A^\cdot\times B^\cdot \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & A^\cdot \arrow[ul, "1_{A^\cdot}"'] \arrow[ll, dashed, "u'"'] \arrow[dl, "u"] \\ & B^\cdot & \end{tikzcd} \qquad \begin{tikzcd} & A^\cdot & \\ A^\cdot\times B^\cdot \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & A^\cdot \arrow[ul, "1_{A^\cdot}"'] \arrow[ll, dashed, "v'"'] \arrow[dl, "v"] \\ & B^\cdot & \end{tikzcd}$$

are commutative,

$$p_1\circ u' = 1_{A^\cdot}, \quad p_2\circ u' = u \quad , \quad p_1\circ v' = 1_{A^\cdot}, \quad p_2\circ v' = v.$$

Then ker (u, v) is given by the cartesian diagram

$$\begin{tikzcd} K^\cdot \arrow[r, "p"] \arrow[d, "p"'] \arrow[dr, phantom, "\sim"] & A^\cdot \arrow[d, "v'"] \\ A^\cdot \arrow[r, "u'"] & A^\cdot\times B^\cdot \end{tikzcd} \quad , \qquad \ker(u, v) = (K^\cdot, p),$$

where p satisfies the properties

a) $$u \circ p = v \circ p.$$

b) For every $h : X' \to A$ with $h = v \circ h$, there exists a unique weak ring homomorphism $q : X' \to K$ such that $p \circ q = h$.

§5. In this paragraph we shall recall some known results which we shall use in § 6.

A category I is called a pseudo-filter category (catégorie pseudo-filtrante, [6]) if the following conditions hold

PS 1. Every diagram

$$\begin{tikzcd} & j \\ i \arrow[ur, "u"] \arrow[dr, "v"'] & \\ & j' \end{tikzcd} \quad , \text{ with } i, j, j' \in \operatorname{Ob} I$$

can be completed to a commutative one

$$\begin{tikzcd} & j \arrow[dr, "s"] & \\ i \arrow[ur, "u"] \arrow[dr, "v"'] & & k \\ & j' \arrow[ur, "t"'] & \end{tikzcd}$$

PS 2. If $i \underset{v}{\overset{u}{\rightrightarrows}} j$ is a diagram in I, then in I there exists a morphism $w : j \to k$ such that $w \circ u = w \circ v$. A category I is called *a connected category* (catégorie connexe, [6]) if it is nonvoid and for every objects i, j from I there exists in I a diagram of the form

$$i \nearrow i_1 \nwarrow i_2 \cdots\cdots i_{n-1} \nearrow i_n \nwarrow j$$

The following theorems are true:

Theorem 9. *A pseudo-filter category I is a connected one if and only if for every pair (i, j) of* Ob I *there exists an object k and two morphisms $u : i \to k$, $v : j \to k$.*

Theorem 10. *Let I be a small connected pseudo-filter category and $F : I \to$ Ens a covariant functor; let*

$$X = \coprod_{i \in \operatorname{Ob}(I)} F(i)$$

be the direct sum of the set family $(F(i))_{i \in \operatorname{Ob} I}$ and let us define in X a binary relation ρ as follows: if $x_j \in F(j)$, $x_i \in F(i)$, then $x_i \rho x_j$, if there exist $u, v \in \operatorname{Hom} I$, $u : i \to k$, $v : j \to k$ such that $F(u)(x_i) = F(v)(x_j)$; Then we have:

a) ρ is an equivalence relation in X and X/ρ together with the canonic morphisms σ_i

$$\sigma_i : F(i) \to X/\rho \quad \sigma_i(x_i) = x_i^\rho = \bar{x}_i ,$$

define the inductive limit *of* F

$$\varinjlim_I F = X/\rho.$$

b) If $x, y \in \varinjlim_I F$, then there exists $k \in \operatorname{Ob} I$ such that the classes x, y have representatives x_k, y_k in the same set $F(k)$.

c) The element $x_i, x'_i \subset F(i)$ are the representatives of the same class in $\varinjlim_I F$ iff there exist $j \in \operatorname{Ob} I$ and $u : i \to j$ such that

$$F(u)(x_i) = F(u)(x'_i).$$

Using this theorem, one can deduce:

Remark 5. Let $F : I \to$ Ab be a covariant functor from the small category I to the abelian group category Ab and let us consider $\widetilde{F}_i$, the support sets of the abelian groups $F(i)$, $\widetilde{F} : I \to$ Ens the induced functor, $(X/\rho, \sigma_i)$ the inductive limit of $\widetilde{F}$; then X/ρ can be endowed with an abelian group structure, such that σ_i become group homomorphisms and $(X/\rho, \sigma_i)$ the inductive limit of F. Namely, the composition law in X/ρ is defined as follows: if $x, y \in X/\rho$, by Th. 10, b), there exists $i \in \operatorname{Ob} I$ such that $x_i, y_i \in F(i)$; then $x + y = z$ is defined as the class with the representative $x_i + y_i$. This addition is well defined, that is it does not depend upon the choice of the representatives. In fact, if $x_j, y_j \in F(j)$ are other representatives, with the same index, for x and y, then by Th. 10 – the definition of ρ – there exists diagrams

$$\begin{tikzcd} i \arrow[dr, "u'"] & \\ & k' \\ j \arrow[ur, "v'"'] & \end{tikzcd} \quad , \quad \begin{tikzcd} i \arrow[dr, "u''"] & \\ & k'' \\ j \arrow[ur, "v''"'] & \end{tikzcd}$$

such that

$$F(u')(x_i) = F(v')(x_j)$$
(20)
$$F(u'')(y_i) = F(v'')(y_j).$$

Hence, we have the diagram

$$\begin{tikzcd}[row sep=small] & k' \\ i \arrow[ur, "u'"] \arrow[dr, "u''"'] & \\ & k'' \end{tikzcd}$$

which, by *PS 1*, can be completed to the commutative one, namely

$$\begin{tikzcd}[row sep=small] & k' \arrow[dr, "\alpha'"] & \\ i \arrow[ur, "u'"] \arrow[dr, "u''"'] & & l \\ & k'' \arrow[ur, "\alpha''"'] & \end{tikzcd} \qquad (21) \quad \alpha' \circ u' = \alpha'' \circ u'',$$

so, we have also the diagram

$$j \overset{\alpha' \circ v'}{\underset{\alpha'' \circ v''}{\rightrightarrows}} l.$$

Hence, by *PS 2*, there exists $\beta : l \to k$ such that

(22) $$\beta \circ \alpha' \circ v' = \beta \circ \alpha'' \circ v'' = v, \quad v : j \to k.$$

From (21) we deduce

(22') $$\beta \circ \alpha' \circ u' = \beta \circ \alpha'' \circ u'' = u, \quad u : i \to k.$$

By (22) and (22') and applying (20), we obtain

$$F(u)(x_i) = [F(\beta) \circ F(\alpha) \circ F(u')](x_i) = [F(\beta) \circ F(\alpha') \circ F(v')](x_j) =$$
$$= F(\beta \circ \alpha'' \circ v'')(x_j) = F(v)((x_j)) \Rightarrow$$

(23) $$F(u)(x_i) = F(v)(x_j)$$

and similarly

(23') $$F(u)(y_i) = F(v)(y_j).$$

Since $F(u)$, $F(v)$ are additive group homomorphisms, this implies

$$F(u)(x_i + y_i) = F(v)(x_j + y_j)$$

hence, by the definition of the équivalence ρ, $x_i + y_i$, $x_j + y_j$ are in the same class $z = x + y$. The null element in $\varinjlim_I F$, $F : I \to \mathrm{Ab}$, is the class represented by the null elements of the groups $F(i)$ since if 0_i, 0_j are the null elements of $F(i), F(j)$, then $0_i \ \rho \ 0_j$, because, I being a connected pseudo-filter category, there exists u, v $u : i \to k$, $v : j \to k$ with $F(u)(0_i) = F(v)(0_j) = 0_k$, for $F(u)$, $F(v)$ are abelian group homomorphisms.

§ 6. Now we can prove, for the weak ring category $\mathcal{S}$:

Theorem 11. *Let I be a connected pseudo-filter small category and $F : I \to \mathcal{S}$ a covariant functor; then, in the weak ring category $\mathcal{S}$, there exists the inductive limit of the functor F: in other words, the weak ring category $\mathcal{S}$ is a category with connected pseudo-filter inductive limits.*

Proof. If $i \in \mathrm{Ob}\, I$, we shall denote

$$F(i) = (R_{i,+}, \varphi_i, \psi_i, c_i) = R_i$$

the corresponding weak ring, $R_{i,+}$ being its additive abelian group. By Remark 5, in Ab there exists

$$\lim_{i \in \mathrm{Ob} I} R_{i,+} = \mathcal{R}_+.$$

Let

(24) $$\sigma_i : R_{i,+} \to \lim_{i \in \mathrm{Ob} I} R_{i,+} = \mathcal{R}_+, \ i \in \mathrm{Ob} I$$

be the canonic homomorphisms of the inductive limit ; σ_i are abelian group homomorphisms. In the abelian group $\mathcal{R}_+$, we define a second operation „·" as follows : if $x, y \in \mathcal{R}_+$, by Th. 10, b), there exists $i \in \mathrm{Ob} I$ with x_i, $y_i \in F(i)$, respectively representatives of x, y ; then we put $x.y = z$, where z is the class represented by $x_i.y_i \in F(i) = R_i$. This operation is well defined ; the proof is the same as that given in Remark 5 for the addition, observing that — in the present case — $F(u)$, $F(v)$ are weak ring homomorphisms and hence they conserve the multiplication „ . " The weak distributivity (1), (2) holds in $\mathcal{R}_+$: for every three element $x, y, z \in \mathcal{R}_+$, there exist their representatives with the same index i, let they be respectively $x_i, y_i, z_i \in R_{i,+}$, so that $\overline{x_i} = x$, $\overline{y_i} = y$, $\overline{z_i} = z$; then we have

$$x \cdot (y + z) + x \cdot 0 = \overline{x_i \cdot (y_i + z_i)} + \overline{x_i \cdot 0_i} = \overline{x_i \cdot (y_i + z_i) + x_i \cdot 0_i} = \overline{x_i \cdot y_i +}$$
$$\overline{+ x_i \cdot z_i} \neq \overline{x_i \cdot y_i} + \overline{x_i \cdot z_i} = \overline{x_i} \cdot \overline{y_i} + \overline{x_i} \cdot \overline{z_i} = x \cdot y + x \cdot z ;$$

similarly for the proof of (2).

So we have proved that $\varinjlim_{i \in I} R_{+,i}$, became a weak ring $\mathcal{R}$. Now, let us point out for the weak ring $\mathcal{R}$ the endomorphisms φ, ψ and the fixed element c from (3), (4).

From the universal property of the inductive limit in Ab, there exist two group endomorphisms

$$\varphi, \psi : \varinjlim_I R_{i,+} \to \varinjlim_I R_{i,+}$$

making commutative the diagram

$$\begin{tikzcd} F(i) \arrow[rr, shift left, "\varphi_i"] \arrow[rr, shift right, "\psi_i"'] \arrow[d, "\sigma_i"'] & & F(i) \arrow[d, "\sigma_i"] \\ \varinjlim_I R_{i,+} \arrow[rr, dotted, shift left, "\varphi"] \arrow[rr, dotted, shift right, "\psi"'] & & \varinjlim_I R_{i,+} \end{tikzcd}$$

that is

$$(25) \qquad \varphi \circ \sigma_i = \sigma_i \circ \varphi_i, \quad \psi \circ \sigma_i = \sigma_i \circ \psi_i, \ \forall i \in \mathrm{Ob}\, I.$$

If $x \in \varinjlim R_{i,+}$ and $x_i \in F(i)$ is a representative for x, then by (25), we define $\varphi(x)$ as the class represented by $\varphi_i(x_i)$, $\varphi(x) = \overline{\varphi_i(x_i)}$. Similarly $\psi(x) = \overline{\psi_i(x_i)}$.

If $(c_i)_{i \in I}$ is the family of the fixed elements c_i from the weak rings $F_i = R_i$, $i \in \mathrm{Ob}\, I$, then this family form a single class c modulo ρ. In fact, since I is a connected pseudo-filter category, by Th. 9, for every $i, j \in \mathrm{Ob} I$, there exist $k \in \mathrm{Ob} I$ and two morphisms $u : i \to k$, $v : j \to k$. Then there exist the weak ring homomorphisms $F(u) : F(i) \to F(k)$, $F(v) : F(j) \to F(k)$ with $F(u)(c_i) = F(v)(c_j) = c_k$ (from (7″) of Proposition 1) and thus the above assertion is proved.

Now, one can easily prove that (3) and (4) hold, for φ, ψ, c defined above. For instance

$$x \cdot 0 = \overline{x_i \cdot 0_i} = \overline{\varphi_i(x_i) + c_i} = \overline{\varphi_i(x_i)} + \overline{c_i} = \varphi(x) + c.$$

The group homomorphisms σ_i, (24), become weak ring homomorphisms, since they conserve the multiplication,

$$\forall x_i, y_i \in F_i = R_i, \ \sigma_i(x_i \cdot y_i) = \overline{x_i \cdot y_i} = \overline{x_i} \cdot \overline{y_i} = \sigma_i(x_i) \cdot \sigma_i(y_i).$$

So (24) became a weak ring diagram

$$(26) \qquad \{\sigma_i : F(i) = R_i \to \mathcal{R}\}_{i \in \mathrm{Ob} I}.$$

Let us prove now the universal property of this diagram. Let

$$\mathcal{X} = (X_+, \lambda, \mu, d)$$

be any weak ring and

$$\{f_i : R_i \to \mathcal{X}\}_{i \in \mathrm{Ob} I}$$

a family of weak ring homomorphisms such that, for any $u : i \to j$, we have $f_j \circ F(u) = f_i$. This implies that, considering the corresponding family of abelian group homomorphisms

$$\{f_{i,+} : R_{i,+} \to X_+\}_{i \in \mathrm{Ob} I} \text{ with } f_{j,+} \circ F(u)_+ = f_{i,+}$$

by the universal property of $\mathcal{R}_+$, there exists a single abelian group homomorphisms

$$f_+ : \mathcal{R}_+ \to X_+.$$

which makes commutative the diagram

$$\begin{tikzcd} F_{i,+} = R_{i,+} \arrow[d, "\sigma_{i,+}"'] \arrow[dr, "f_{i,+}"] & \\ \mathcal{R}_+ \arrow[r, dashed, "f_+"] & X_+ \end{tikzcd} \qquad (27)\ f_+ \circ \sigma_{i,+} = f_{i,+}.$$

Moreover, f_+ is a weak ring homomorphism, since we have, for every $x, y \in \mathcal{R}$ (by Th. 10, b), $x = \overline{x_i} = \sigma_i(x_i)$, $y = \overline{y_i} = \sigma_i(y_i)$ and by (27)

$$f(x) = f_i(x_i), \ f(y) = f_i(y_i)$$

so that, f_i being a weak ring homomorphism, we obtain $f(x \cdot y) = f_i(x_i \cdot y_i) = f_i(x_i) \cdot f_i(y_i) = f(x) \cdot f(y)$. In conclusion, we have proved that the weak ring

$$\mathcal{R} = (\varinjlim_I R_{i,+} ; \varphi, \psi, c)$$

together with the weak ring homomorphisms (26) define the inductive limit of the functor $F : I \to \mathcal{S}$ in the weak ring category $\mathcal{S}$.

BIBLIOGRAPHY

1. Climescu Al. — *Anneaux faibles*, Bul. Inst. polit. Iaşi, T. VII(XI) 1961, p. 1.
2. Climescu Al. — *O nouă clasă de inele slabe*, Bul. Inst. polit. Iaşi, T. X(XIV). 1964, p. 29.
3. Jurchescu M. şi colab. — *Categorii (Topologic, Categorii, Suprafeţe riemanniene)* Ed. Acad. R. S. R., Bucureşti, 1966.
4. Luchian Tudora — *Homomorfisme în inele slabe*, An. şt. Univ. Iaşi, Matematica, T. XIII, 1967, fasc. 1, p. 17.
5. Luchian Tudora — *The weak field of fractions of a weak integral domain*, An. şt. Univ. Iaşi, Matematica, T. XVII (1971), fasc. 2, p. 245.
6. Verdier J. L. — *Topologies et faisceaux*. Sém. de Géom. alg. de I'I. H. E. F, 1963—1964.

CÎTEVA PROPRIETĂŢI ALE CATEGORIEI INELELOR SLABE

Rezumat

Inelele slabe, introduse de Al. Climescu în [1], [2] au fost studiate de T. Luchian în cîteva note, printre care [4], [5]. În prezenta Notă autorii demonstrează cîteva proprietăţi ale categoriei $\mathcal{S}$ a inelelor slabe, dintre care cităm :

$\mathcal{S}$ are produse fibrate (Th. 7). În $\mathcal{S}$ există produsul direct pentru orice familie de obiecte (Th. 8). Categoria $\mathcal{S}$ are limite proiective. $\mathcal{S}$ este o categorie cu limite inductive pseudofiltrante conexe (Th. 11).

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LES R-ESPACES SIMPLES

PAR

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En surprenant les propriétés les plus générales des espaces pré-topologiques, topologiques, de proximité faible, de proximité, quasi-uniformes et uniformes, M. Hacque introduit dans [1] comme une généralisation de ces espaces la notion de T-espace. Un T-espace simple est un ensemble X, structuré à l'aide d'une application $\rho : \mathfrak{P}(X) \to \mathfrak{F}(X)$ (où $\mathfrak{F}(X)$ est l'ensemble des filtres sur X), vérifiant les conditions :

T_1) $\rho(\emptyset) = \mathfrak{P}(X)$ (le filtre impropre sur X)

T_2) $Z \in \rho(A)$ implique $A \subset Z$.

T_3) $\rho(A_1 \cup A_2) = \rho(A_1) \cap \rho(A_2)$.

La modélation mathématique de divers phénomènes physiques, chimiques, biologiques, linguistiques nous a amenés à considérer des structures typologiques pareilles à celles rappelées ci-dessus, mais pour lesquelles cette association ensemble-filtre est considérée dans un cadre plus large. Notamment, il s'agit de la définition d'une structure plus générale que les structures pré-uniformes de J. Chacron [3] et de l'étude des propriétés topologiques qui peuvent être posées dans ce cadre : séparation, continuité, compacité.

Dans le premier paragraphe nous avons introduit la notion de R-espace simple et nous avons donné quelques définitions équivalentes (le lemme 1.11, le lemme 1.17).

Dans le deuxième paragraphe nous avons introduit la notion de R-espace simple ponctuel, la notion de (Y, X) quasi-adhérence et nous avons étudié la liaison entre ces deux notions (le lemme 2.4).

Les propriétés de séparation T_1 et T_2 sont étudiées dans le troisième paragraphe et la quasi-compacité à droite, dans le quatrième paragraphe. Dans le cinquième paragraphe nous avons introduit et étudié les morphismes continus entre deux R-espaces simples.