$(p_o^n x, q_o^n y_1) \in T(A)$ and $(p_o^n x, q_o^n y_2) \in T(A)$ with $y_1 \neq y_2$, which contradicts that $T(A)$ is function. Suppose that 1), 2), 3) hold. Because $T(A)$ is $X \times Y$ — t.v.a.a. and prefix closed it sufficies to show that dom $(T(A)) = X^*$ and $T(A)$ is a function (by Theorem 5). From 1), 2), 3) it follows that for every $p_o^n \in X^*$, there exists a unique $q_o^n \in Y^*$ such that $(p_o^n, q_o^n) \in T(A)$. This completes the proof.

Let $R(X, Y)$, $RT_1(X, Y)$, $RT_2(X, Y)$, $CRT_1(X, Y)$ be the class of sequential relations over $X \times Y$, the class of $SRT_1/X \times Y$, the class of $SRT_2/X \times Y$ and the class of CT_1-sequential relations, respectively.

Theorem 8. *There exist the following relations:*
$R(X, Y) \subset RT_2(X, Y)$, $R(X, Y) \subset CRT_1(X, Y) \subset RT_1(X, Y)$.

Proof. Every sequential machine $M = < X, Y, S, \delta, \lambda >$ may be interpreted as a t.v.m., for which
$S_n = S$, $f_n^{n+1}(s, x) = \delta(s, x)$, $g_n(s, x) = \lambda(s, x)$, $n \in N$, $x \in X$ and as a c.t.v.m. for which $S_n = \{s/s \in S,\ \exists s_1 \in S\ \exists p_1^n \in X^*,\ \delta(s_1, p_1^n) = s\}.f_n^{n+1}(s, x) = \delta(s, x)$, $g_n(s, x) = \lambda(s, x)$, $n \in N$.

It is easy to show that there is a t.v.m. M, for which the relation $R_2(M)$ is not suffix closed, hence $R_2(M)$ is not a sequential relation over $X \times Y$. There is a c.t.v.m. M for which $R_1(M)$ is not a sequential relation.

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ASUPRA RELAȚIILOR SECVENȚIALE CALCULATE DE MAȘINI SECVENȚIALE CU STRUCTURĂ VARIABILĂ

Rezumat

Se definesc cîteva relații calculate de mașini secvențiale cu structură variabilă, dîndu-se caracterizări ale lor, precum și legătura cu limbajele acceptate de automate cu structură variabilă. Se stabilesc relațiile între ele și relațiile secvențiale.

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SAINT-VENANT PROBLEM FOR CYLINDRICAL BEAMS WITH GENERAL ANISOTROPY

BY

C. I. BORȘ

1. General equations and formulation of the problem. Let us consider a beam limited by a cylindrical surface $\mathfrak{S}$ and by two planes $x_3 = 0$, $x_3 = h$ $(h > 0)$. We shall denote by $\mathfrak{V}$ the region occupied by the beam and by S the domain of a cross-section. We denote by Γ the boundary of S.

We suppose that there are no body forces so that the equilibrium equations are given by

$$\sigma_{ij,j} = 0 \quad \text{in } \mathfrak{V}\ ^{*}). \tag{1}$$

The lateral surface is supposed to be free from tractions so that we have the boundary conditions

$$\sigma_{ij} n_j = 0 \quad \text{on } \mathfrak{S}, \tag{2}$$

where n_i are the direction cosines of the exterior normal to the surface $\mathfrak{S}$.

The tractions applied at the end $x_3 = 0$ are equivalent with a resultant $\bar{R}$ and a moment resultant $\bar{\mathfrak{M}}$:

$$\bar{R} = (R_1, R_2, R_3), \quad \bar{\mathfrak{M}} = (\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3) \quad \text{on } x_3 = 0, \tag{8}$$

where R_i and $\mathfrak{M}_i$ are given.

On the end $x_3 = h$ there are applied tractions so that the beam is in equilibrium.

The material of the beam is supposed to be with general anisotropy so that the relations between the components of the stress σ_{ij} and the strain components γ_{ij} are given by [1, 3]

*) We use the summation convention over the repeated indices. The index j after comma indicates partial differentiation with respect to x_j.

$$
(4)\qquad \| \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12} \|^{*} = \left\| \begin{matrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{matrix} \right\| \left\| \begin{matrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{matrix} \right\|
$$

where c_{ij} are constants which characterize the elastic qualities of the material of the beam.

The inverse relations are taken in the form

$$
(5)\qquad \| \gamma \| = \| s_{ij} \| \, \| \sigma \| ,
$$

where γ is the vector of the strain components γ_{ij} and σ is the vector of the stress components σ_{ij}.

The strain components γ_{ij} are connected with the components of the displacement u_i by

$$
\begin{aligned} \gamma_{ii} &= u_{i,i} && \text{(not summed)}, \\ \gamma_{ij} &= u_{i,j} + u_{j,i} && (i \neq j). \end{aligned}
$$

2. Auxiliary plane problems. The solution of the Saint-Venant problem may be expressed in terms of four plane strain problems which we shall define below.

Let $t_{ij}^{(k)}$ $(k = 1, 2, 3, 4)$ be the stresses of the mentioned plane problems. Let a, b, c be arbitrary constants.

The first three plane problems are defined by the equations

$$
(6)\qquad \begin{cases} \dfrac{\partial t_{11}}{\partial x_1} + \dfrac{\partial t_{12}}{\partial x_2} + c_{13}\, a + c_{33}\, b + (c_{14} + c_{56})\, Q + (c_{14} - c_{56})\, A + \\ \qquad + (c_{14}\, \omega_{,2} + c_{15}\, \omega_{,1})_{,1} + (c_{46}\, \omega_{,2} + c_{56}\, \omega_{,1})_{,2} = 0, \\ \dfrac{\partial t_{12}}{\partial x_1} + \dfrac{\partial t_{22}}{\partial x_2} + c_{36}\, a + c_{23}\, b + (c_{46} + c_{25})\, Q + (c_{46} - c_{25}) A + \\ \qquad + (c_{46}\, \omega_{,2} + c_{56}\, \omega_{,1})_{,1} + (c_{24}\, \omega_{,2} + c_{25}\, \omega_{,1})_{,2} = 0, \\ \dfrac{\partial t_{31}}{\partial x_1} + \dfrac{\partial t_{23}}{\partial x_2} = 0 \quad \text{in } S \end{cases}
$$

and

$$
(7)\qquad \begin{cases} t_{11}\, n_1 + t_{12}\, n_2 = -\{[c_{13}(a x_1 + b x_2 + c) + c_{14}(\omega_{,2} + Q x_1 + A x_1) + \\ \qquad + c_{15}(\omega_{,1} + Q x_2 - A x_2)]\, n_1 + [c_{36}(a x_1 + b x_2 + c) + \\ \qquad + c_{46}(\omega_{,2} + Q x_1 + A x_1) + c_{56}(\omega_{,1} + Q x_2 - A x_2)]\, n_2\}, \\ t_{12}\, n_1 + t_{22}\, n_2 = -\{[c_{36}(a x_1 + b x_2 + c) + c_{46}(\omega_{,2} + Q x_1 + A x_1) + \\ \qquad + c_{56}(\omega_{,1} + Q x_2 - A x_2)]\, n_1 + [c_{23}(a x_1 + b x_2 + c) + \\ \qquad + c_{24}(\omega_{,2} + Q x_1 + A x_1) + c_{25}(\omega_{,1} + Q x_2 - A x_2)]\, n_2\}, \\ t_{31}\, n_1 + t_{23}\, n_2 = 0 \quad \text{on } \Gamma. \end{cases}
$$

The function ω is defined by

$$
(8)\qquad \begin{cases} \Delta\omega + c_{35}\, a + c_{34}\, b + 2 c_{45}\, Q = 0 \quad \text{in } S, \\ \mathfrak{D}\omega = -\{[c_{35}(a x_1 + b x_2 + c) + c_{45}(Q + A)\, x_1 + c_{55}(Q - A)\, x_2]\, n_1 + \\ \qquad + [c_{34}(a x_1 + b x_2 + c) + c_{44}(Q + A)\, x_1 + c_{45}(Q - A)\, x_2]\, n_2\} \quad \text{on } \Gamma \end{cases}
$$

where

$$
(9)\qquad \begin{cases} \Delta = c_{55} \dfrac{\partial^2}{\partial x_1^2} + 2 c_{45} \dfrac{\partial^2}{\partial x_1\, \partial x_2} + c_{44} \dfrac{\partial^2}{\partial x_2^2}, \\ \mathfrak{D} = n_1 \left(c_{55} \dfrac{\partial}{\partial x_1} + c_{45} \dfrac{\partial}{\partial x_2} \right) + n_2 \left(c_{45} \dfrac{\partial}{\partial x_1} + c_{44} \dfrac{\partial}{\partial x_2} \right), \\ \omega = a \omega_1 + b \omega_2 + c \omega_3, \\ t_{ij} = a t_{ij}^{(1)} + b t_{ij}^{(2)} + c t_{ij}^{(3)}. \end{cases}
$$

Now, let us consider a plane problem with body forces defined by

$$
(10)\qquad \begin{cases} \sigma_{i\alpha,\alpha} + X_i = 0 & \text{in } S \\ \sigma_{i\alpha}\, n_\alpha = p_i & \text{on } \Gamma \quad (i = 1, 2, 3;\ \alpha = 1, 2;\ X_3 = p_3 = 0). \end{cases}
$$

The necessary and sufficient conditions for the existence of a solution for the problem (10) are given by

$$
(11)\qquad \begin{cases} \displaystyle\int_\Gamma p_\alpha\, ds + \iint_S X_\alpha\, d\sigma = 0 \quad (\alpha = 1, 2), \\ \displaystyle\int_\Gamma (x_1 p_2 - x_2 p_1)\, ds + \iint_S (x_1 X_2 - x_2 X_1)\, d\sigma = 0. \end{cases}
$$

We can prove that the conditions (11) are satisfied in the case of the problem (6), (7).

It is easy to prove that the stresses

$$
(12\,a)\quad
\begin{cases}
t_{11} = -\,[c_{13}(ax_1 + bx_2 + c) + c_{14}(\omega_{,2} + Qx_1 + Ax_1) + \\
\quad + c_{15}(\omega_{,1} + Qx_2 - Ax_2)], \\
t_{22} = -\,[c_{23}(ax_1 + bx_2 + c) + c_{24}(\omega_{,2} + Qx_1 + Ax_1) + \\
\quad + c_{25}(\omega_{,1} + Qx_2 - Ax_2)], \\
t_{31} = -\,[c_{35}(ax_1 + bx_2 + c) + c_{45}(\omega_{,2} + Qx_1 + Ax_1) + \\
\quad + c_{55}(\omega_{,1} + Qx_2 - Ax_2)], \\
t_{23} = -\,[c_{34}(ax_1 + bx_2 + c) + c_{44}(\omega_{,2} + Qx_1 + Ax_1) + \\
\quad + c_{45}(\omega_{,1} + Qx_2 - Ax_2)], \\
t_{12} = -\,[c_{36}(ax_1 + bx_2 + c) + c_{46}(\omega_{,2} + Qx_1 + Ax_1) + \\
\quad + c_{56}(\omega_{,1} + Qx_2 - Ax_2)], \\
t_{33} = -\,[c_{33}(ax_1 + bx_2 + c) + c_{34}(\omega_{,2} + Qx_1 + Ax_1) + \\
\quad + c_{35}(\omega_{,1} + Qx_2 - Ax_2)] + \dfrac{1}{s_{33}}(ax_1 + bx_2 + c)
\end{cases}
$$

satisfy the equations (6) and (7).

The corresponding strains are given by

$$
(12\,b)\quad
\begin{cases}
\varepsilon_{11} = \dfrac{s_{13}}{s_{33}}(ax_1 + bx_2 + c), \qquad \varepsilon_{22} = \dfrac{s_{23}}{s_{33}}(ax_1 + bx_2 + c), \qquad \varepsilon_{33} = 0, \\
\varepsilon_{23} = \dfrac{s_{34}}{s_{33}}(ax_1 + bx_2 + c) - (\omega_{,2} + Qx_1 + Ax_1), \\
\gamma_{31} = \dfrac{s_{35}}{s_{33}}(ax_1 + bx_2 + c) - (\omega_{,1} + Qx_2 - Ax_2), \\
\gamma_{12} = \dfrac{s_{36}}{s_{33}}(ax_1 + bx_2 + c)
\end{cases}
$$

and they will satisfy the compatibility conditions if

$$
(13)\qquad Q = \frac{1}{2s_{33}}(s_{35}\,b + s_{34}\,a), \qquad A = -\frac{1}{2s_{33}}(s_{35}\,b - s_{34}\,a).
$$

The fourth plane problem is defined by the equations

$$
(14)\quad
\begin{cases}
\dfrac{\partial t_{11}^{(4)}}{\partial x_1} + \dfrac{\partial t_{12}^{(4)}}{\partial x_2} + c_{14} - c_{56} + (c_{14}\,\varphi_{,2} + c_{15}\varphi_{,1})_{,1} + (c_{46}\,\varphi_{,2} + c_{56}\,\varphi_{,1})_{,2} = 0, \\
\dfrac{\partial t_{12}^{(4)}}{\partial x_1} + \dfrac{\partial t_{22}^{(4)}}{\partial x_2} + c_{46} - c_{25} + (c_{46}\,\varphi_{,2} + c_{56}\,\varphi_{,1})_{,1} + (c_{24}\,\varphi_{,2} + c_{25}\,\varphi_{,1})_{,2} = 0, \\
\dfrac{\partial t_{13}^{(4)}}{\partial x_1} + \dfrac{\partial t_{23}^{(4)}}{\partial x_2} = 0 \quad \text{in } S
\end{cases}
$$

and

$$
(15)\quad
\begin{cases}
t_{11}^{(4)}\,n_1 + t_{12}^{(4)}\,n_2 = -\,\{[c_{14}(\varphi_{,2} + x_1) + c_{15}(\varphi_{,1} - x_2)]\,n_1 + [c_{46}(\varphi_{,2} + x_1) + \\
\quad + c_{56}(\varphi_{,1} - x_2)]\,n_2\}, \\
t_{12}^{(4)}\,n_1 + t_{22}^{(4)}\,n_2 = -\,\{[c_{46}(\varphi_{,2} + x_1) + c_{56}(\varphi_{,1} - x_2)]\,n_1 + [c_{24}(\varphi_{,2} + x_1) + \\
\quad + c_{25}(\varphi_{,1} - x_2)]\}, \\
t_{13}^{(4)}\,n_1 + t_{23}^{(4)}\,n_2 = 0 \quad \text{on } \Gamma.
\end{cases}
$$

The function φ is defined by

$$
(16)\quad
\begin{cases}
\Delta\varphi = 0 \quad \text{in } S, \\
\mathfrak{D}\varphi = (c_{55}\,x_2 - c_{45}\,x_1)\,n_1 + (c_{45}\,x_2 - c_{44}\,x_1)\,n_2 \quad \text{on } \Gamma.
\end{cases}
$$

We shall call φ the function of torsion.

From the equations (6), (7) we can finde the equations for the first three plane problems.

All our boundary value problems, defined above, have solutions.

3. Saint-Venant problem. In order to solve the problem defined by (1), (2), (3) let us suppose that the displacements are of the form

$$
(17)\quad
\begin{cases}
u_1 = \left(\psi_1^* - \dfrac{1}{2}A_1\,x_2\,x_3 - \dfrac{1}{2}\tau_1\,x_2\,x_3 - \dfrac{1}{6}a_1\,x_3^2\right)x_3 + \left(\psi_1^0 - A_0\,x_2\,x_3 - \right. \\
\quad \left. -\,\tau_0\,x_2\,x_3 - \dfrac{1}{2}a_0\,x_3^2\right) + U_1, \\
u_2 = \left(\psi_2^* + \dfrac{1}{2}A_1\,x_1\,x_3 + \dfrac{1}{2}\tau_1\,x_1\,x_3 - \dfrac{1}{6}b_1\,x_3^2\right)x_3 + \left(\psi_2^0 + A_0\,x_1\,x_3 + \right. \\
\quad \left. +\,\tau_0\,x_1\,x_3 - \dfrac{1}{2}b_0\,x_3^2\right) + U_2, \\
u_3 = (\psi_3^* + \omega^* + Q_1\,x_1\,x_2)\,x_3 + (\psi_3^0 + \omega_0 + Q_0\,x_1\,x_2) + \\
\quad + \dfrac{1}{2}(a_1\,x_1 + b_1\,x_2 + c_1)\,x_3^2 + (a_0\,x_1 + b_0\,x_2 + c_0)\,x_3 + U_3 + \Omega,
\end{cases}
$$

where

$$
(18)\quad
\begin{cases}
\psi_i^* = a_1\,\psi_i^{(1)} + b_1\,\psi_i^{(2)} + c_1\,\psi_i^{(3)} + \tau_1\,\psi_i^{(4)}\,, \\
\psi_i^0 = a_0\,\psi_i^{(1)} + b_0\,\psi_i^{(2)} + c_0\,\psi_i^{(3)} + \tau_0\,\psi_i^{(4)}\,, \\
\omega^* = a_1\,\omega_1 + b_1\,\omega_2 + c_1\,\omega_3 + \tau_1\,\varphi, \\
\omega_0 = a_0\,\omega_1 + b_0\,\omega_2 + c_0\,\omega_3 + \tau_0\,\varphi,
\end{cases}
$$

the functions $\psi_i^{(k)}$ $(i = 1, 2, 3\,;\ k = 1, 2, 3, 4)$ being the displacements of the auxiliary plane problems defined by (6), (7) and (14), (15); the functions ω_i and φ are defined by (8) and (16); $U_i\,(x_1\,, x_2)$ and $\Omega\,(x_1\,, x_2)$ are unknown functions and $a_\alpha\,, b_\alpha\,, c_\alpha\,, \tau_\alpha$ $(\alpha = 0{,}1)$ are unknown constants (τ_1 can be taken as zero).

The constants $A_\alpha\,, Q_\alpha$ are given by (13).

We shall choose the function Ω such that the problem which will be obtained for the determination of the displacements $U_1\,, U_2\,, U_3$ be a plane strain problem. On that account we shall call Ω function of correction. It is easy to see that the functions ω^* and ω^0 have the same role.

The strain components of the displacements (17) are given by

$$
(19)\quad
\begin{cases}
\gamma_{11} = \gamma_{11}^*\,x_3 + \gamma_{11}^0 + e_{11}, \\
\gamma_{22} = \gamma_{22}^*\,x_3 + \gamma_{22}^0 + e_{22}, \\
\gamma_{33} = (a_1\,x_1 + b_1\,x_2 + c_1)\,x_3 + (a_0\,x_1 + \\
\qquad + b_0\,x_2 + c_0) + (\psi_3^* + \omega^* + Q_1\,x_1\,x_2), \\
\gamma_{23} = \gamma_{23}^*\,x_3 + \gamma_{23}^0 + e_{23} + (\omega_{,2}^* + Q_1\,x_1 + A_1\,x_1)\,x_3 + \\
\qquad + (\omega_{0,2} + Q_0\,x_1 + A_0\,x_1 + \tau_0\,x_1) + (\psi_2^* + \Omega_{,2}), \\
\gamma_{31} = \gamma_{31}^*\,x_3 + \gamma_{31}^0 + e_{31} + (\omega_{,1}^* + Q_1\,x_2 - A_1\,x_2)\,x_3 + \\
\qquad (\omega_{0,1} + Q_0\,x_2 - A_0\,x_2 - \tau_0\,x_2) + (\psi_1^* + \Omega_{,1}), \\
\gamma_{12} = \gamma_{12}^*\,x_3 + \gamma_{12}^0 + e_{12}\,,
\end{cases}
$$

where γ_{ij}^* are the strains of displacements $u_i^* = \psi_i^*$, γ_{ij}^0 are the strains of the displacements $u_i^* = \psi_i^0$ and e_{ij} the strains of the displacements U_i.

Taking into account the solution (12) of the auxiliary plane problems, at the last, we find for the stress components the following formulae

$$
(20)\quad
\begin{cases}
\sigma_{11} = \tau_{11} + \tau_0\,[t_{11}^{(4)} + c_{14}\,(\varphi_{,2} + x_1) + c_{15}\,(\varphi_{,1} - x_2)] + \\
\qquad + c_{13}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{14}\,(\psi_2^* + \Omega_{,2}) + c_{15}\,(\psi_1^* + \Omega_{,1}), \\
\sigma_{22} = \tau_{22} + \tau_0\,[t_{22}^{(4)} + c_{24}\,(\varphi_{,2} + x_1) + c_{25}\,(\varphi_{,1} - x_2)] + \\
\qquad + c_{23}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{24}\,(\psi_2^* + \Omega_{,2}) + c_{25}\,(\psi_1^* + \Omega_{,1}),
\end{cases}
$$

$$
(20)\quad
\begin{cases}
\sigma_{33} = \tau_{33} + \tau_0\,[t_{33}^{(4)} + c_{34}\,(\varphi_{,2} + x_1) + c_{35}\,(\varphi_{,1} - x_2)] + \\
\qquad + c_{33}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{34}\,(\psi_2^* + \Omega_{,2}) + c_{35}\,(\psi_1^* + \Omega_{,1}) + \\
\qquad + \dfrac{1}{s_{33}}\,(a_1\,x_1 + b_1\,x_2 + c_1)\,x_3 + \dfrac{1}{s_{33}}\,(a_0\,x_1 + b_0\,x_2 + c_0), \\
\sigma_{23} = \tau_{23} + \tau_0\,[t_{23}^{(4)} + c_{44}\,(\varphi_{,2} + x_1) + c_{45}\,(\varphi_{,1} - x_2)] + \\
\qquad + c_{34}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{44}\,(\psi_2^* + \Omega_{,2}) + c_{45}\,(\psi_1^* + \Omega_{,1}), \\
\sigma_{31} = \tau_{31} + \tau_0\,[t_{31}^{(4)} + c_{45}\,(\varphi_{,2} + x_1) + c_{55}\,(\varphi_{,1} - x_2)] + \\
\qquad + c_{35}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{45}\,(\psi_2^* + \Omega_{,2}) + c_{55}\,(\psi_1^* + \Omega_{,1}), \\
\sigma_{12} = \tau_{12} + \tau_0\,[t_{12}^{(4)} + c_{46}\,(\varphi_{,2} + x_1) + c_{56}\,(\varphi_{,1} - x_2)] + \\
\qquad + c_{36}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{46}\,(\psi_2^* + \Omega_{,2}) + c_{56}\,(\psi_1^* + \Omega_{,1}),
\end{cases}
$$

where τ_{ij} are the stresses corresponding to the displacements U_i.

Taking into account the equations (14) and (15), finally, we find for the stresses τ_{ij} the equations

$$
(21)\quad
\begin{cases}
\dfrac{\partial\tau_{11}}{\partial x_1} + \dfrac{\partial\tau_{12}}{\partial x_2} + [c_{13}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{14}\,(\psi_2^* + \Omega_{,2}) + c_{15}\,(\psi_1^* + \\
\qquad + \Omega_{,1})]_{,1} + [c_{36}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{46}\,(\psi_2^* + \Omega_{,2}) + \\
\qquad + c_{56}\,(\psi_1^* + \Omega_{,1})]_{,2} = 0, \\
\dfrac{\partial\tau_{12}}{\partial x_1} + \dfrac{\partial\tau_{22}}{\partial x_2} + [c_{36}(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{46}\,(\psi_2^* + \Omega_{,2}) + c_{56}(\psi_1^* + \\
\qquad + \Omega_{,1})]_{,1} + [c_{23}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{24}\,(\psi_2^* + \Omega_{,2}) + \\
\qquad + c_{25}\,(\psi_1^* + \Omega_{,1})]_{,2} = 0, \\
\dfrac{\partial\tau_{31}}{\partial x_1} + \dfrac{\partial\tau_{23}}{\partial x_2} = 0 \quad \text{in } S
\end{cases}
$$

and

$$
(22)\quad
\begin{cases}
\tau_{11}\,n_1 + \tau_{12}\,n_2 = -\,\{[c_{13}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{14}\,(\psi_2^* + \Omega_{,2}) + \\
\qquad + c_{15}\,(\psi_1^* + \Omega_{,1})]\,n_1 + [c_{36}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{46}\,(\psi_2^* + \\
\qquad + \Omega_{,2}) + c_{56}\,(\psi_1^* + \Omega_{,1})]\,n_2\}, \\
\tau_{12}\,n_1 + \tau_{22}\,n_2 = -\,\{[c_{36}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{46}\,(\psi_2^* + \Omega_{,2}) + \\
\qquad + c_{56}(\psi_1^* + \Omega_{,1})]\,n_1 + [c_{23}\,(\psi_3^* + \omega^* + Q_1\,x_1\,x_2) + c_{24}\,(\psi_2^* + \\
\qquad + \Omega_{,2}) + c_{25}\,(\psi_1^* + \Omega_{,1})]\,n_2\}, \\
\tau_{31}\,n_1 + \tau_{23}\,n_2 = 0 \quad \text{on } \Gamma.
\end{cases}
$$

The equations (21) and (22) are valid if the function Ω is taken so that

$$\Delta\Omega + [c_{35}(\psi_3^* + \omega^* + Q_1 x_1 x_2) + c_{45}\psi_2^* + c_{55}\psi_1^*]_{,1} + [c_{34}(\psi_3^* + \omega^* + Q_1 x_1 x_2) + c_{44}\psi_2^* + c_{45}\psi_1^*]_{,2} + \frac{1}{s_{33}}(a_1 x_1 + b_1 x_2 + c_1) = 0 \text{ in } S \tag{23}$$

and

$$\mathfrak{D}\Omega = -\{[c_{35}(\psi_3^* + \omega^* + Q_1 x_1 x_2) + c_{45}\psi_2^* + c_{55}\psi_1^*]\, n_1 + [c_{34}(\psi_3^* + \omega^* + Q_1 x_1 x_2) + c_{44}\psi_2^* + c_{45}\psi_1^*]\, n_2\} \quad \text{on } \Gamma. \tag{24}$$

The equations (21) and (22) define a plane strain problem and for it the conditions (11) are fulfilled.

Now, let us show that we can calculate the constants a_1, b_1, c_1 before to solve the problems defined by (21), (22) and by (23), (24).

Firstly, the problem (23), (24) is of the Neumann type. The existence condition of the function Ω requires that

$$\frac{1}{s_{33}}\iint_S (a_1 x_1 + b_1 x_2 + c_1)\, d\sigma = 0 \tag{25 a}$$

and from two of the end conditions we obtain

$$\frac{1}{s_{33}}\iint_S (a_1 x_1 + b_1 x_2 + c_1)\, x_1\, d\sigma = -R_1, \tag{25 b}$$

$$\frac{1}{s_{33}}\iint_S (a_1 x_1 + b_1 x_2 + c_1)\, x_2\, d\sigma = -R_2. \tag{25 c}$$

From the other end conditions we get the following equations:

$$\left\{\begin{aligned}
&\frac{1}{s_{33}}\iint_S (a_0 x_1 + b_0 x_2 + c_0)\, d\sigma = -R_3 - \iint_S H\, d\sigma,\\
&\frac{1}{s_{33}}\iint_S (a_0 x_1 + b_0 x_2 + c_0)\, x_2\, d\sigma = -\mathfrak{M}_1 - \iint_S H x_2\, d\sigma,\\
&\frac{1}{s_{33}}\iint_S (a_0 x_1 + b_0 x_2 + c_0)\, x_1\, d\sigma = \mathfrak{M}_2 - \iint_S H x_1\, d\sigma,\\
&D_t \tau_0 = -\mathfrak{M}_3 - \iint_S \{[\tau_{23} + c_{34}(\psi_3^* + \omega^* + Q_1 x_1 x_2) + c_{44}(\psi_2^* + \Omega_{,2}) +\\
&\quad + c_{45}(\psi_1^* + \Omega_{,1})]\, x_1 - [\tau_{31} + c_{35}(\psi_3^* + \omega^* + Q_1 x_1 x_2) +\\
&\quad + c_{45}(\psi_2^* + \Omega_{,2}) + c_{55}(\psi_1^* + \Omega_{,1})]\, x_2\}\, d\sigma,
\end{aligned}\right. \tag{26}$$

where

$$H(a_1, b_1, c_1, \tau_0) = \tau_{33} + \tau_0[t_{33}^{(4)} + c_{34}(\varphi_{,2} + x_1) + c_{35}(\varphi_{,1} - x_2)] + c_{33}(\psi_3^* + \omega^* + Q_1 x_1 x_2) + c_{34}(\psi_2^* + \Omega_{,2}) + c_{35}(\psi_1^* + \Omega_{,1}),$$

$$D_t = \iint_S \{[t_{23}^{(4)} + c_{44}(\varphi_{,2} + x_1) + c_{45}(\varphi_{,1} - x_2)]\, x_1 - [t_{31}^{(4)} + c_{45}(\varphi_{,2} + x_1) + c_{55}(\varphi_{,1} - x_2)]\, x_2\}\, d\sigma.$$

We can easily prove that the generalized rigidity D_t is different form zero and we have $D_t > 0$.

Some remarks:

— The given solution is adequate by solving the general problem from the beams [2], namely when the conditions (12) are replaced by

$$\sigma_{ij}\, n_j = \sum_{k=0}^{n} \tau_i^{(k)}(x_1, x_2)\, x_3^k \text{ on } \mathcal{F}.$$

— It seems that the used method suits to any material with linear constitutive equations.

— Voigt has given a representation of the displacements in order to studie the problem of bending by a transverse load [4, p. 663]. If we calculate the displacements by using (12 b) and we particularize the results for the case of bending by a transverse load we find the formulae proposed by Voigt.

— In our monograph [1, p. 252 and p. 263] and in [3, p. 96 and p. 213] are given results concerning the torsion and bending by couples and respectively results concerning the bending by a terminal load. It remains to decide if the formulae (25), (26) are or not are in total concordance with the mentioned results.

— In the formulae (17) the terms

$$-\left(\tau_0 x_2 + \frac{1}{2} a_0 x_3\right) x_3 - \frac{1}{2}\left(\tau_1 x_2 + \frac{1}{3} a_1 x_3\right) x_3^2,$$

$$\left(\tau_0 x_1 - \frac{1}{2} b_0 x_3\right) x_3 + \frac{1}{2}\left(\tau_1 x_1 - \frac{1}{3} b_1 x_3\right) x_3^2,$$

$$(a_0 x_1 + b_0 x_2 + c_0)\, x_3 + \frac{1}{2}(a_1 x_1 + b_1 x_2 + c_1)\, x_3^2$$

are closely connected with the displacement of a rigid body.

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PROBLEMA LUI SAINT-VENANT PENTRU BARE CILINDRICE CU ANIZOTROPIE GENERALĂ

Rezumat

Se consideră o bară cilindrică confecţionată dintr-un material anizotrop pentru care legea lui Hooke este dată de (4).

Pentru o astfel de bară se rezolvă problema lui Saint-Venant definită de (1), (2), (3).

Pentru a rezolva problema se presupune că deplasările sînt de forma 17).

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THE TORSION PROBLEM FOR ISOTROPIC MIXTURES OF TWO LINEAR ELASTIC SOLIDS

BY

MARIAN ARON

1. Introduction. G r e e n and S t e e l [1] have developed a general non-linear thermodynamic theory of two interacting continua. The linearized theory of mixtures of two elastic solids, as summarized below, was obtained by S t e e l [2]. In this paper the torsion problem of cylindrical beams, in the context of linear theory is considered.

2. Statement of the problem. The linearized constitutive equations of isotropic mixtures of two elastic solids, as given by S t e e l [2] are*)

$$\sigma_{(ij)} = \{-\alpha_2 + \lambda_1 e_{pp} + \lambda_3 g_{pp}\}\,\delta_{ij} + 2\mu_1 e_{ij} + 2\mu_3 g_{ij}\,, \tag{2.1}$$

$$\pi_{(ij)} = \{\alpha_2 + \lambda_4 e_{pp} + \lambda_2 g_{pp}\}\,\delta_{ij} + 2\mu_3 e_{ij} + 2\mu_2 g_{ij}\,, \tag{2.2}$$

$$\sigma_{[ij]} = -\pi_{[ij]} = -2\lambda_5 h_{[ij]}\,, \tag{2.3}$$

$$\omega_i = \frac{\rho_2}{\rho}\alpha_2 e_{pp,i} + \frac{\rho_1}{\rho}\alpha_2 g_{pp,i}\,. \tag{2.4}$$

In the above $\sigma_{(ij)}$ and $\pi_{(ij)}$, $\sigma_{[ij]}$ and $\pi_{[ij]}$ represent, respectively, the symmetric and skew-symmetric parts of the partial stresses σ_{ij} and π_{ij}, ω_i are the components of the diffusive force, α_2, λ_1, λ_2,... etc. are material constants and ρ_1 and ρ_2 are the mass-densities of the two elastic solids. We denoted also

$$\rho = \rho_1 + \rho_2, \tag{2.5}$$

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*) Throughout this paper, all suffixes run over the values 1, 2, 3 and the usual convention of summing over repeated indices is adopted. A comma denotes the partial derivative.