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PROBLEMA LUI SAINT-VENANT PENTRU BARE CILINDRICE CU ANIZOTROPIE GENERALĂ

Rezumat

Se consideră o bară cilindrică confecţionată dintr-un material anizotrop pentru care legea lui Hooke este dată de (4).

Pentru o astfel de bară se rezolvă problema lui Saint-Venant definită de (1), (2), (3).

Pentru a rezolva problema se presupune că deplasările sînt de forma 17).

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THE TORSION PROBLEM FOR ISOTROPIC MIXTURES OF TWO LINEAR ELASTIC SOLIDS

BY

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1. Introduction. Green and Steel [1] have developed a general non-linear thermodynamic theory of two interacting continua. The linearized theory of mixtures of two elastic solids, as summarized below, was obtained by Steel [2]. In this paper the torsion problem of cylindrical beams, in the context of linear theory is considered.

2. Statement of the problem. The linearized constitutive equations of isotropic mixtures of two elastic solids, as given by Steel [2] are*)

$$\sigma_{(ij)} = \{-\alpha_2 + \lambda_1 e_{pp} + \lambda_3 g_{pp}\}\delta_{ij} + 2\mu_1 e_{ij} + 2\mu_3 g_{ij}, \tag{2.1}$$

$$\pi_{(ij)} = \{\alpha_2 + \lambda_4 e_{pp} + \lambda_2 g_{pp}\}\delta_{ij} + 2\mu_3 e_{ij} + 2\mu_2 g_{ij}, \tag{2.2}$$

$$\sigma_{[ij]} = -\pi_{[ij]} = -2\lambda_5 h_{[ij]}, \tag{2.3}$$

$$\omega_i = \frac{\rho_2}{\rho}\alpha_2 e_{pp,i} + \frac{\rho_1}{\rho}\alpha_2 g_{pp,i}. \tag{2.4}$$

In the above $\sigma_{(ij)}$ and $\pi_{(ij)}$, $\sigma_{[ij]}$ and $\pi_{[ij]}$ represent, respectively, the symmetric and skew-symmetric parts of the partial stresses σ_{ij} and π_{ij}, ω_i are the components of the diffusive force, α_2, λ_1, λ_2,... etc. are material constants and ρ_1 and ρ_2 are the mass-densities of the two elastic solids. We denoted also

$$\rho = \rho_1 + \rho_2, \tag{2.5}$$

*) Throughout this paper, all suffixes run over the values 1, 2, 3 and the usual convention of summing over repeated indices is adopted. A comma denotes the partial derivative.

$$e_{ij} = \frac{1}{2}(w_{i,j} + w_{j,i}),\quad g_{ij} = \frac{1}{2}(\eta_{i,j} + \eta_{j,i}),$$

(2.6)

$$h_{[ij]} = \frac{1}{2}(w_{j,i} - w_{i,j} + \eta_{i,j} - \eta_{j,i}),$$

where w_i and η_i are the components of the two displacement vectors.

We consider a cylindrical beam consisting of an isotropic mixture of two homogeneous elastic solids bounded by plane ends perpendicular to the generators. The cross-section Σ is assumed to be a simply conected region bounded by a closed Liapunov curve L. The beam is initially in equilibrium under no deformation, with no total applied force and we are assuming throughout, that there are no extraneous body forces. We choose rectangular Cartesian axes in the initial body such that one of its bases lie in the $x_1 O x_2$-plane, while the other is in the plane $x_3 = l$, l being the lenght of the beam. The lateral surface is supposed to be free of applied force.

In the torsion problem the beam is kept in equilibrium when the end $x_3 = l$ is twisted by a couple of magnitude M.

In order to solve this problem we must find the partial stresses σ_{ij} and π_{ij} and the displacements w_i and η_i which in the domain Ω occupied by the beam satisfy the equilibrium equations

(2.7) $$\sigma_{pi,p} - \omega_i = 0,\quad \pi_{pi,p} + \omega_i = 0,$$

the boundary conditions

(2.8) $$n_k\,\sigma_{ki} = 0,\quad n_k\,\pi_{ki} = 0,\quad \text{on } L,$$

(where n_k are the components of the outward normal to the boundary $\partial\Omega$ of the domain Ω), the constitutive equations (2.1) — (2.4) and the following conditions on the plane $x_3 = l$:

(2.9) $$\int_\Sigma (\sigma_{33} + \pi_{33})\,d\sigma = 0,\quad \int_\Sigma x_2(\sigma_{33} + \pi_{33})\,d\pi = 0,\quad \int_\Sigma x_1(\sigma_{33} + \pi_{33})\,d\tau = 0,$$

(2.10) $$\int_\Sigma (\sigma_{31} + \pi_{31})\,d\sigma = 0,\quad \int_\Sigma (\sigma_{32} + \pi_{32})\,d\sigma = 0,$$

(2.11) $$\int_\Sigma [x_1(\sigma_{32} + \pi_{32}) - x_2(\sigma_{31} + \pi_{31})]\,d\sigma = M.$$

3. Solution of the problem. We try to solve the problem by assuming *á priori* that the displacement components are given by

(3.1) $$w_1 = -\alpha x_2 x_3 + k_1 x_1,\quad w_2 = \alpha x_1 x_3 + k_1 x_2,\quad w_3 = \alpha\varphi(x_1, x_2) + k_1 x_3,$$

and

(3.2) $$\eta_1 = -\alpha x_2 x_3 + k_2 x_1,\quad \eta_2 = \alpha x_1 x_3 + k_2 x_2,\quad \eta_3 = \alpha\varphi(x_1, x_2) + k_2 x_3,$$

where the function φ and the constants k_1, k_2, α will be determined in the following.

From (2.6) we get

$$e_{11} = e_{22} = e_{33} = k_1,\quad g_{11} = g_{22} = g_{33} = k_2,\quad e_{12} = g_{12} = 0,$$

(3.3)

$$e_{13} = g_{13} = \frac{\alpha}{2}\left(-x_2 + \frac{\partial\varphi}{\partial x_1}\right),\quad e_{23} = g_{23} = \frac{\alpha}{2}\left(x_1 + \frac{\partial\varphi}{\partial x_2}\right),$$

(3.4) $$h_{[12]} = h_{[13]} = h_{[23]} = 0.$$

The substitution of (3.3) and (3.4) into (2.1) — (2.4) yields

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -\alpha_2 + k_1(3\lambda_1 + 2\mu_1) + k_2(3\lambda_3 + 2\mu_3),$$

$$\sigma_{31} = \sigma_{13} = \alpha(\mu_1 + \mu_3)\left(-x_2 + \frac{\partial\varphi}{\partial x_1}\right),$$

(3.5)

$$\sigma_{32} = \sigma_{23} = \alpha(\mu_1 + \mu_3)\left(x_1 + \frac{\partial\varphi}{\partial x_2}\right),$$

$$\sigma_{21} = \sigma_{12} = 0,$$

$$\pi_{11} = \pi_{22} = \pi_{33} = \alpha_2 + k_1(3\lambda_4 + 2\mu_3) + k_2(3\lambda_2 + 2\mu_2),$$

$$\pi_{31} = \pi_{13} = \alpha(\mu_2 + \mu_3)\left(-x_2 + \frac{\partial\varphi}{\partial x_1}\right),$$

(3.6)

$$\pi_{32} = \pi_{23} = \alpha(\mu_2 + \mu_3)\left(x_1 + \frac{\partial\varphi}{\partial x_2}\right),\quad \pi_{21} = \pi_{12} = 0,$$

and

(3.7) $$\omega_i = 0.$$

The equilibrium equations (2.8) are reduced to

(3.8) $$(\mu_1 + \mu_3)\Delta\varphi = 0,\quad (\mu_2 + \mu_3)\Delta\varphi = 0.$$

From (2.9), (3.5) and (3.6) we obtain the following boundary conditions:

$$[-\alpha_2 + k_1(3\lambda_1 + 2\mu_1) + k_2(3\lambda_3 + 2\mu_3)]\, n_1 = 0,$$
$$[\alpha_2 + k_1(3\lambda_4 + 2\mu_3) + k_2(3\lambda_2 + 2\mu_2)]\, n_2 = 0,$$

$$(3.9)\qquad (\mu_1 + \mu_3)\left(n_1 x_2 - n_2 x_1 + \frac{d\varphi}{dn}\right) = 0,$$
$$(\mu_2 + \mu_3)\left(n_1 x_2 - n_2 x_1 + \frac{d\varphi}{dn}\right) = 0.$$

It is clear that $(3.9)_{1,2}$ are satisfied if and only if

$$(3.10)\qquad \begin{aligned} k_1(3\lambda_1 + 2\mu_1) + k_2(3\lambda_3 + 2\mu_3) &= \alpha_2\,,\\ k_1(3\lambda_4 + 2\mu_3) + k_2(3\lambda_2 + 2\mu_2) &= -\alpha_2\,, \end{aligned}$$

so that we can infer

$$(3.11)\qquad k_1 = \frac{\alpha_2}{\Delta}[2(\mu_2 + \mu_3) + 3(\lambda_2 + \lambda_3)],\ k_2 = -\frac{\alpha_2}{\Delta}[2(\mu_1 + \mu_3) + 3(\lambda_1 + \lambda_4)],$$

provided that

$$(3.12)\qquad \Delta = \begin{vmatrix} 3\lambda_1 + 2\mu_1 & 3\lambda_3 + 2\mu_3 \\ 3\lambda_4 + 2\mu_3 & 3\lambda_2 + 2\mu_2 \end{vmatrix} \neq 0.$$

We notice that if there are no interactions between solids whatsoever, and the diffusive force is zero, then putting $\alpha_2 = \lambda_2 = \lambda_4 = \lambda_5 = \mu_3 = 0$, we obtain from (2.1) — (2.4) the constitutive equations of two individual solids so that, as usualy, we can suppose (see also [2])

$$(3.13)\qquad \mu_1 > 0,\ \mu_2 > 0,\ 3\lambda_1 + 2\mu_1 > 0,\ 3\lambda_2 + 2\mu_2 > 0.$$

An analogous situation appears if we assume that

$$(3.15)\qquad e_{ij} = g_{ij} \text{ and } \alpha_2 = \lambda_5 = 0.$$

Taking into account (2.1) — (2.4), (3.14) and (3.13) we can write

$$(3.16)\qquad \begin{aligned} &\mu_1 + \mu_3 > 0, \quad \mu_2 + \mu_3 > 0,\\ &3\lambda_3 + 2\mu_3 > 0, \quad 3\lambda_4 + 2\mu_3 > 0. \end{aligned}$$

Now, from (3.8) and (3.9), we obtain the following boundary — value problem of Neumann's type

$$(3.16)\qquad \Delta\varphi = 0,\quad \frac{d\varphi}{dn} = -n_1 x_2 + n_2 x_1\,,\ \text{on } L.$$

The existence of solution results from the fact that:

$$(3.17)\qquad \int_{\partial\Omega}(-n_1 x_2 + n_2 x_1)\,d\sigma = \int_L x_2\,dx_2 + x_1\,dx_1 = 0.$$

The conditions (3.10) represent the vanishing of normal tractions σ_{ii} and π_{ii} (do'nt sum). Hence the conditions (2.9) are automatically satisfied. As in the classical case, we can prove (see [3], p. 498) that the conditions (2.10) are also verified. From (2.11) we obtain

$$(3.18)\qquad \alpha D = M,$$

where

$$(3.19)\qquad D = (\mu_1 + \mu_2 + 2\mu_3)\int_\Sigma\left(x_1^2 + x_2^2 + x_1\frac{\partial\varphi}{\partial x_2} - x_2\frac{\partial\varphi}{\partial x_1}\right)d\sigma.$$

The integral in (3.19) is different from zero (see [3], p. 500) so that the constant α is also well-determined.

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PROBLEMA TORSIUNII PENTRU MIXTURILE IZOTROPE A DOUĂ SOLIDE ELASTICE LINIARE

Rezumat

Se rezolvă problema torsiunii barelor omogene şi izotrope în teoria liniară a mixturii a două solide elastice prin metoda semi-inversă.