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ON ALMANSI-MICHELL'S PROBLEM FOR VISCOELASTIC BEAMS

BY

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1. Let be an isotropic homogeneous and linear viscoelastic material with the constitutive equations [7].

$$\sigma_{ij}(x, t) = \lambda(t)\,\theta(x, t)\,\delta_{ij} + 2\mu(t)\,\varepsilon_{ij}(x, t) + \int_0^t [l(t, \tau)\,\theta(x, \tau)\,\delta_{ij} + 2m(t, \tau)\,\varepsilon_{ij}(x, \tau)]\,d\tau, \tag{1.1}$$

with respect to a rectangular coordinate system $Ox_1\,x_2\,x_3$. In (1.1) we have used the following notations: $\sigma_{ij}(x, t)$ and $\varepsilon_{ij}(x, t)$ are the stress and the strain tensors in the point $x\,(x_1, x_2, x_3)$ at the moment t, $\theta(x, t) = \varepsilon_{ii}(x, t)$ and δ_{ij} are Kronecker's delta.

Latin subscripts have the range 1, 2, 3 and the usual convention of summation over repeated indices is adopted.

If we denote by $u_i(x, t)$ the components of the displacement vector at the point x at the instant t then

$$2\varepsilon_{ij}(x, t) = u_{i,j}(x, t) + u_{j,i}(x, t). \tag{1.2}$$

The index i after comma indicates partial differentiation with respect to x_i.

We have assumed that the viscoelastic material is originally undisturbed — i.e.

$$u_i(x, t) = \varepsilon_{ij}(x, t) = \sigma_{ij}(x, t) = 0$$

for $t < 0$.

When $\lambda(t)$, $\mu(t)$ are constants, (1.1) leads to the constitutive equations suggested by V o l t e r r a [9]. Moreover, if $l(t, \tau) = l(t - \tau)$ and $m(t, \tau) = m(t - \tau)$, (1.1) is identical in form to the accepted general linear viscoelastic constitutive equations [6].

We assume the elastic moduli $\lambda(t)$, $\mu(t)$ and relaxation moduli $l(t,\tau)$, $m(t,\tau)$ are continuous and bounded functions on $[0, \infty)$ and $[0, \infty) \times [0, \infty)$, respectively. Moreover, we suppose the existence of two positive constants λ_0, μ_0 so that $\lambda(t) \geqslant \lambda_0$, $\mu(t) \geqslant \mu_0$ on $[0, \infty)$.

When there are no body forces, the basic equations for the quasi-static theory of homogenous and isotropic viscoelastic materials are: the quasi-static equilibrium equations

$$\sigma_{ij,j}(x, t) = 0, \quad t \in [0, \infty), \tag{1.3}$$

the constitutive law (1.1) and the geometrical equations (1.2).

2. Let us consider a beam of a homogeneous and isotropic linear viscoelastic material bounded by plane ends perpendicular to the generators and by a cylindrical surface $\sum$. We will chose the rectangular coordinate system so that the Ox_3-axis be directed paralel to the generators of the beam. The domain occupied by the beam will be denoted by V. Let D be a cross-section domain occupied by the beam will be denoted by V. Let D be a cross-section in the beam and S its area. The cross-section in the beam is assumed to be a simply-connected region bounded by a closed regular curve Γ. The unit outward normal to Γ will be denoted by $n(n_1, n_2, 0)$.

By $\tilde{x}$ we denote, sometimes, an arbitrary point (x_1, x_2) from D.

We assume that the body forces are absent and that the beam is subjected to the surface loading prescribed over its ends ond lateral surface$\sum$. We suppose that on the lateral surface $\sum$ of the beam, the surface tractions are

$$\sigma_{i\alpha}(x, t)\, n_\alpha = p_i(x_1, x_2, t), \quad t \in [0, \infty), \tag{2.1}$$

where $p_i(\tilde{x}, t)$ are given functions of x_1, x_2 and t.

In (2.1) and subsequently, Greek indices take values 1,2.

If we determine a solution of the equations (1.3) which satisfies the condition (2.1) on $\sum$, then the problem can be reduced to the problem of the quasi-static equilibrium of considered visco-elastic beam when it is subjected to surface tractions prescribed over its ends and is free on $\sum$. This new problem is the Saint-Venant's problem for viscoelastic beam and it was studied in [2], [3], [4].

Here, we will find a solution of the equations (1.3), $x \in V$, which satisfies the conditions (2.1) on the lateral surface $\sum$.

We will call this problem — the Almansi-Michell's problem for viscoelastic beam. Almansi-Michell's problem for elastic cylinder were considered in various papers (see e. g. [1], [8]).

We suppose that $p_i(\tilde{x}, t)$ are continuous functions of $\tilde{x} \in \Gamma$ for every $t \in [0, \infty)$ and continuous of $t \in [0, \infty)$, uniformly with respect to $\tilde{x} \in \Gamma$.

3. By using the form of the solution for Almansi-Michell's problem in the elastic case [8], we intend to solve this problem for viscoelastic beam by seeking its solution under the following form

$$\begin{aligned}
u_1(x, t) = {} & -\frac{1}{2}\left\{A_1(t)\left[x_1^2 x_2^2 - \frac{1}{3} x_1^4 + (x_1^2 - x_2^2)x_3^2\right] - \frac{1}{6} a_1(t)(x_1^4 - x_3^4)\right\} - \\
& - A_2(t)\left(x_3^2 - \frac{1}{3} x_2^2\right) x_1 x_2 - A_3(t) x_1 x_3^2 + A_4(t) x_1 x_3 + k(t)\left(\frac{1}{3} x_2^2 - x_3^2\right) x_2 + v_1(\tilde{x}, t), \\
u_2(x, t) = {} & - A_1(t)\left(x_3^2 - \frac{1}{2} x_1^2\right) x_1 x_2 - \frac{1}{2}\left\{A_2(t)\left[x_1^2 x_2^2 - \frac{1}{3} x_2^4 + \right.\right. \\
& \left.\left. + (x_2^2 - x_1^2) x_3^2\right] - \frac{1}{6} a_2(t)(x_2^4 - x_3^4)\right\} - A_3(t) x_2 x_3^2 + A_4(t) x_2 x_3 + \\
& + k(t)\left(x_3^2 - \frac{1}{3} x_1^2\right) x_1 + v_2(\tilde{x}, t), \\
u_3(x, t) = {} & \left[A_1(t)\left(x_2^2 - \frac{1}{3} x_1^2\right) - \frac{1}{3} a_1(t)(2x_1^2 - x_3^2)\right] x_1 x_3 + \\
& + \left[A_2(t)\left(x_1^2 - \frac{1}{3} x_2^2\right) - \frac{1}{3} a_2(t)(2x_2^2 - x_3^2)\right] x_2 x_3 + \frac{1}{3} a_3(t) x_3^2 + \\
& + \frac{1}{4} A_4(t)(x_1^2 + x_2^2) + a_4(t)\left(x_1^2 + x_2^2 - \frac{1}{2} x_3^2\right) + \Phi(\tilde{x}, t) + x_3 \psi(\tilde{x}, t).
\end{aligned} \tag{3.1}$$

In (3.1) $A_j(t)$, $a_j(t)$, $j = 1, 2, 3, 4$, $k(t)$ are unknown continuous functions on $[0, \infty)$ and $\Phi(\tilde{x}, t)$, $\psi(\tilde{x}, t)$, $V_\alpha(\tilde{x}, t)$ are unknown functions which belong to $C^2(D) \cap C^1(\bar{D})$ for every $t \in [0, \infty)$, continuous of t, uniformly with respect to $\tilde{x} \in \bar{D}$. These functions will be determined by the condition that (3.1) is a solution of the problem (1.3), (2.1).

From (3.1) and (1.2) we get

$$\begin{aligned}
\varepsilon_{11}(x, t) = {} & -\left[A_1(t)\left(x_2^2 + x_3^2 - \frac{2}{3} x_1^2\right) - \frac{1}{3} a_1(t) x_1^2\right] x_1 - A_2(t)\left(x_3^2 - \right. \\
& \left. - \frac{1}{3} x_2^2\right) x_2 - A_3(t) x_3^2 + A_4(t) x_3 + \tilde{\varepsilon}_{11}(\tilde{x}, t), \\
\varepsilon_{22}(x, t) = {} & - A_1(t)\left(x_3^2 - \frac{1}{3} x_1^2\right) x_1 - \left[A_2(t)\left(x_1^2 + x_3^2 - \frac{2}{3} x_2^2\right) - \right.
\end{aligned}$$

$$-\frac{1}{3}a_2(t)x_2^2\Big]\,x_2 - A_3(t)\,x_3^2 + A_4(t)\,x_3 + \tilde{\varepsilon}_{22}(\tilde{x}, t),$$

$$\varepsilon_{33}(x,t) = \Big[A_1(t)\Big(x_2^2 - \frac{1}{3}x_1^2\Big) - \frac{1}{3}a_1(t)\,(2x_1^2 - 3x_3^2)\Big]x_1 + \Big[A_2(t)\Big(x_1^2 -$$

(3.2)
$$-\frac{1}{3}x_2^2\Big) - \frac{1}{3}a_2(t)\,(2x_2^2 - 3x_3^2)\Big]x_2 + a_3(t)\,x_3^2 - a_4(t)\,x_3 + \psi(\tilde{x}, t),$$

$$\varepsilon_{23}(x,t) = [A_2(t)\,(x_1^2 - x_2^2) - a_2(t)\,x_2^2]\,x_3 - A_3(t)\,x_2x_3 + (A_4(t) +$$

$$+ a_4(t))\,x_2 + k(t)\,x_1x_3 + \frac{1}{2}[\Phi_{,2}(\tilde{x}, t) + x_3\,\psi_{,2}(\tilde{x}, t)],$$

$$\varepsilon_{31}(x,t) = [A_1(t)\,(x_2^2 - x_1^2) - a_1(t)\,x_1^2]\,x_3 - A_3(t)\,x_1x_3 + (A_4(t) +$$

$$+ a_4(t))\,x_1 - k(t)\,x_2x_3 + \frac{1}{2}[\Phi_{,1}(\tilde{x}, t) + x_3\,\psi_{,1}(\tilde{x}, t),$$

$$\varepsilon_{12}(x,t) = \frac{1}{2}k(t)\,(x_2^2 - x_1^2) + \tilde{\varepsilon}_{12}(\tilde{x}, t)],$$

where

(3.3)
$$2\tilde{\varepsilon}_{\alpha\beta}(\tilde{x}, t) = V_{\alpha,\beta}(\tilde{x}, t) + V_{\beta,\alpha}(\tilde{x}, t)$$
$$\tilde{\theta}(x, t) = \tilde{\varepsilon}_{\alpha\alpha}(\tilde{x}, t).$$

We use the notations:

(3.4)
$$A_j^*(t) = L\,[2A_j(t) - a_j(t)] + 2M\,[A_j(t)],$$
$$E_j(t) = L\,[a_j(t) - 2A_j(t)] + 2M\,[a_j(t)],$$

$j = 1, 2, 3, 4$, where $L\,[\,.\,]$ and $M\,[\,.\,]$ are linear Volterra operators

(3.5)
$$L\,[f(t)] \equiv \lambda(t)f(t) + \int_0^t l(t,\tau)f(\tau)\,d\tau,$$
$$M\,[f(t)] \equiv \mu(t)f(t) + \int_0^t m(t,\tau)f(\tau)\,d\tau.$$

By (1.1) and (3.2) — (3.5) it follows

$$\sigma_{11}(x,t) = -2M\Big[A_1(t)\,x_2^2 - \frac{1}{3}(A_1(t) + a_1(t))\,x_1^2\Big]x_1 + A_1^*(t)\Big(\frac{1}{3}x_1^2 -$$

$$- x_3^2\Big)x_1 + A_2^*(t)\Big(\frac{1}{3}x_2^2 - x_3^2\Big)x_2 - A_3^*(t)x_3^2 + A_4^*(t)x_3 + \tilde{\sigma}_{11}(\tilde{x}, t) + L\,[\psi(\tilde{x}, t)],$$

$$\sigma_{22}(x,t) = -2M\Big[A_2(t)\,x_1^2 - \frac{1}{3}(A_2(t) + a_2(t))\,x_2^2\Big]x_2 + A_1^*(t)\Big(\frac{1}{3}x_1^2 -$$

$$- x_3^2\Big)x_1 + A_2^*(t)\Big(\frac{1}{3}x_2^2 - x_3^2\Big)x_2 - A_3^*(t)x_3^2 + A_4^*(t)x_3 + \tilde{\sigma}_{22}(\tilde{x}, t) + L\,[\psi(\tilde{x}, t)],$$

(3.6)
$$\sigma_{33}(x,t) = \Big\{2M\Big[(A_1(t)\,x_2^2 - \frac{1}{3}(A_1(t) + a_1(t))x_1^2\Big] + E_1(t)\Big(x_3^2 - \frac{1}{3}x_1^2\Big)\Big\}x_1 +$$

$$+ \Big\{2M\Big[A_2(t)\,x_1^2 - \frac{1}{3}(A_2(t) + a_2(t))\,x_2^2\Big] + E_2(t)\Big(x_3^2 - \frac{1}{3}x_2^2\Big)\Big\}x_2 +$$

$$+ E_3(t)\,x_3^2 + E_4(t)\,x_3 + \tilde{\sigma}_{33}(\tilde{x}, t) + L\,[\psi(\tilde{x}, t)] + 2M\,[\psi(\tilde{x}, t)],$$

$$\sigma_{23}(x,t) = 2M\,[A_2(t)\,(x_1^2 - x_2^2) - a_2(t)\,x_2^2]\,x_3 - 2M\,[A_3(t)]\,x_2x_3 +$$

$$+ 2M\,[A_4(t) + a_4(t)]\,x_2 + 2M\,[k(t)]\,x_1x_3 + M\,[\Phi_{,2}(\tilde{x}, t)] + x_3\,M\,[\psi_{,2}(\tilde{x}, t)],$$

$$\sigma_{31}(x,t) = 2M\,[A_1(t)\,(x_1^2 - x_1^2) - a_1(t)\,x_1^2]\,x_3 - 2M\,[A_3(t)]\,x_1x_3 +$$

$$+ 2M\,[A_4(t) + a_4(t)]\,x_1 - 2M\,[k(t)]\,x_2x_3 + M\,[\Phi_{,1}(\tilde{x}, t)] + x_3\,M\,[\psi_{,1}(\tilde{x}, t)],$$

$$\sigma_{12}(x,t) = M\,[k(t)]\,(x_2^2 - x_1^2) + \tilde{\sigma}_{12}(\tilde{x}, t),$$

where

(3.7)
$$\tilde{\sigma}_{\alpha\beta}(\tilde{x}, t) = L\,[\tilde{\theta}(\tilde{x}, t)]\,\delta_{\alpha\beta} + 2M\,[\tilde{\varepsilon}_{\alpha\beta}(\tilde{x}, t)],$$
$$\tilde{\sigma}_{33}(\tilde{x}, t) = L\,[\tilde{\theta}(\tilde{x}, t)].$$

4. From (3.6) and the equilibrium equations (1.3) we obtain

(4.1)
$$\tilde{\sigma}_{1\beta,\beta}(\tilde{x}, t) + F_1(\tilde{x}, t) + A_1^*(t)\,(x_1^2 - x_3^2) = 0$$
$$\tilde{\sigma}_{2\beta,\beta}(\tilde{x}, t) + F_2(\tilde{x}, t) + A_2^*(t)\,(x_2^2 - x_3^2) = 0$$
$$\{[2E_\alpha(t) - 4M\,[A_\alpha(t) + a_\alpha(t)]]\,x_\alpha + 2E_3(t) - 4M\,[A_3(t)] +$$
$$+ M\,[\Delta\psi(\tilde{x}, t)]\}\,x_3 + 4M\,[A_4(t) + a_4(t)] - E_4(t) + M\,(\Delta\Phi(\tilde{x}, t)] = 0,$$

for $x \in V$ and $t \in [0, \infty)$, where

(4.2)
$$F_\alpha(\tilde{x}, t) = -2M\,[A_3(t)]\,x_\alpha + (L + M)\,[\psi_{,\alpha}(x, t)].$$

The first two equations (4.1) lead to

(4.3)
$$A_1^*(t) = A_2^*(t) = 0,$$

(4.4)
$$\tilde{\sigma}_{\alpha\beta,\beta}(\tilde{x}, t) + F_\alpha(\tilde{x}, t) = 0$$

and by $(4.1)_3$ it follows

$$M[\Delta\psi(\widetilde{x}, t) = 4M[A_3(t)] - 2E_3(t), \tag{4.5}$$

$$M[\Delta\Phi(\widetilde{x}, t)] = E_4(t) - 4M[A_4(t) + a_4(t)] \tag{4.6}$$

for $\widetilde{x} \in D$, $t \in [0, \infty)$, since (4.3) implies that $E_\alpha(t) = 2M[A_\alpha(t) + a_\alpha(t)]$.

Using (3.6) and (4.3) from the conditions (2.1) on $\sum$ we obtain

$$\widetilde{\sigma}_{1\beta}(\widetilde{x}, t) n_1 = s_1(\widetilde{x}, t) + [A_3^*(t) x_3^2 - A_4^*(t) x_3] n_1,$$

$$\widetilde{\sigma}_{2\beta}(\widetilde{x}, t) n_2 = s_2(\widetilde{x}, t) + [A_3^*(t) x_3^2 - A_4^*(t) x_3] n_2,$$

$$\begin{aligned} p_3(x, t) = \Big\{ M \Big[\frac{d\psi(\widetilde{x}, t)}{dn} &+ 2(A_1(t)(x_2^2 - x_1^2) - a_1(t) x_1^2 - A_3(t) x_1 - \\ - k(t) x_2 \Big] n_1 &+ 2(A_2(t)(x_1^2 - x_2^2) - a_2(t) x_2^2 - A_3(t) x_2 + k(t) x_1) n_2 \Big\} x_3 + \\ &+ M \left| \frac{d\Phi(\widetilde{x}, t)}{dn} + 2(A_4(t) + a_4(t))(x_1 n_1 + x_2 n_2) \right|, \end{aligned} \tag{4.7}$$

for every $x \in \sum$, $t \in [0, \infty)$ where

$$\begin{aligned} s_1(\widetilde{x}, t) = p_1(\widetilde{x}, t) &+ \Big\{ 2M \Big[A_1(t) x_2^2 - \frac{1}{3}(A_1(t) + a_1(t)) x_1^2 \Big] x_1 - \\ &- L[\psi(\widetilde{x}, t)] \Big\} n_1 - M[k(t)](x_2^2 - x_1^2) n_2, \end{aligned}$$

$$\begin{aligned} s_2(\widetilde{x}, t) = p_2(\widetilde{x}, t) - M[k(t)](x_2^2 - x_1^2) n_1 &+ \Big\{ 2M \Big| A_2(t) x_1^2 - \frac{1}{3}(A_2(t) + \\ &+ a_2(t)) x_2^2 \Big| x_2 - L[\psi(\widetilde{x}, t)] n_2. \end{aligned}$$

The conditions (4.7) are satisfied if the conditions

$$A_3^*(t) = A_4^*(t) = 0, \tag{4.9}$$

$$\widetilde{\sigma}_{\alpha\beta}(\widetilde{x}, t) n_\beta = s_\alpha(\widetilde{x}, t), \quad x \in \Gamma, \tag{4.10}$$

$$\begin{aligned} M \Big[\frac{d\psi(\widetilde{x}, t)}{dn} &+ 2(A_1(t)(x_2^2 - x_1^2) - a_1(t) x_1^2 - A_3(t) x_1 - k(t) x_2) n_1 + \\ &+ 2(A_2(t)(x_1^2 - x_2^2) - a_2(t) x_2^2 - A_3(t) x_2 + k(t) x_1) n_2 \Big] = 0, \ \widetilde{x} \in \Gamma, \end{aligned} \tag{4.11}$$

$$M \left[\frac{d\Phi(\widetilde{x}, t)}{dn} + 2(A_4(t) + a_4(t))(x_1 n_1 + x_2 n_2) \right] = p_3(\widetilde{x}, t), \ \widetilde{x} \in \Gamma \tag{4.12}$$

are satisfied for $t \in [0, \infty)$.

Because (4.9) implies $E_j(t) = 2M[A_j(t) + a_j(t)]$, $j = 3, 4$, from (4.5) (4.6) we get

$$M[\Delta\psi(\widetilde{x}, t) + 4a_3(t)] = 0, \tag{4.13}$$

$$M[\Delta\Phi(\widetilde{x}, t) + 2(A_4(t) + a_4(t))] = 0 \tag{4.14}$$

for $t \in [0, \infty)$, $\widetilde{x} \in D$.

In our conditions on the functions $\mu(t)$ and $m(t, \tau)$ (Sec 1) it results, that the integral equation $M[u(t)] = g(t)$ has unique continuous solution on $[0, \infty)$, if $g(t)$ is continuous function of $t \in [0, \infty)$, [5].

Therefore, from (4.13), (4.11) and (4.14), (4.12) we obtain

$$\Delta\psi(\widetilde{x}, t) = -4a_3(t), \quad \widetilde{x} \in D,$$

$$\begin{aligned} \frac{d\psi(\widetilde{x}, t)}{dn} &= 2[A_1(t)(x_1^2 - x_2^2) + a_1(t) x_1^2 + A_3(t) x_1 + k(t) x_2] n_1 + \\ &+ 2[A_2(t)(x_2^2 - x_1^2) + a_2(t) x_2^2 + A_3(t) x_2 - k(t) x_1] n_2, \ \widetilde{x} \in \Gamma \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} &\Delta\Phi(\widetilde{x}, t) = -2(A_4(t) + a_4(t)), \quad \widetilde{x} \in D, \\ &\frac{d\Phi(\widetilde{x}, t)}{dn} = -2(A_4(t) + a_4(t))(x_1 n_1 + x_2 n_2) + P_3(\widetilde{x}, t), \widetilde{x} \in \Gamma, \end{aligned} \tag{4.16}$$

for every $t \in [0, \infty)$.

Here $P_3(\widetilde{x}, t)$ is the solution of the equation (4.12); this equation has continuous solutions on $[0, \infty)$, because $p_3(\widetilde{x}, t)$ is a continuous function of $t \in [0, \infty)$, uniformly with respect to $\widetilde{x} \in \Gamma$.

The boundary value problems (4.15) and (4.16) are Neumann's problems for the determination of the function $\psi(\widetilde{x}, t)$ and $\Phi(\widetilde{x}, t)$; the necessary and sufficient conditions to solve these problems are

$$\int_D \Delta\psi(\widetilde{x}, t)\, d\sigma = \int_\Gamma \frac{d\psi(\widetilde{x}, t)}{dn}\, ds,$$

$$\int_D \Delta\Phi(\widetilde{x}, t)\, d\sigma = \int_\Gamma \frac{d\Phi(\widetilde{x}, t)}{dn}\, ds,$$

for every $t \in [0, \infty)$.

In view of these conditions we have

$$A_3(t) + a_3(t) = -[A_\alpha(t) + a_\alpha(t)]\, x_\alpha^0 , \tag{4.17}$$

$$M[A_4(t) + a_4(t)] = \frac{1}{2S}\int_\Gamma P_3(\widetilde{x}, t)\, ds. \tag{4.18}$$

The integral equation (4.18) has unique continuous solution on $[0, \infty)$ since $\int_\Gamma P_3(\widetilde{x}, t)\, ds$ is continuous function of $t \in [0, \infty)$. Let $q_4(t)$ be the solution of the equation (4.18),

$$A_4(t) + a_4(t) = q_4(t). \tag{4.19}$$

The above relation and $A_4^{\cdot}(t) = 0$ lead to the equation

$$a_4(t) + \int_0^t \frac{3l(t,\tau) + 2m(t,\tau)}{3\lambda(t) + 2\mu(t)}\, a_4(\tau)\, d\tau = \frac{Q_4(t)}{3\lambda(t) + 2\mu(t)}, \tag{4.20}$$

where

$$Q_4(t) = 2(L + M)[q_4(t)].$$

The equation (4.20) has continuous solution on $[0, \infty)$ because $Q_4(t)$ is continuous of $t \in [0, \infty)$ and the functions $\lambda(t), \mu(t), l(t,\tau)$ and $m(t,\tau)$ satisfy the conditions from Sec. **1.** Therefore, the functions $a_4(t)$ and $A_4(t)$ are determined by (4.20) and (4.19).

The function $\Phi(\widetilde{x}, t)$ is determined as solution of the problem (4.16) because $A_4(t)$ and $a_4(t)$ are known function and the necessary and sufficient conditions to solve this problem are satisfied.

Now we consider the problem (4.4), (4.10). This is a viscoelastic plane strain problem for the domain D bounded by Γ. The necessary conditions to solve this problem are

$$\begin{gathered} \int_D F_\alpha(\widetilde{x}, t)\, d\sigma + \int_\Gamma s_\alpha(\widetilde{x}, t)\, ds = 0, \\ \int_D [x_1 F_2(\widetilde{x}, t) - x_2 F_1(\widetilde{x}, t)]\, d\sigma + \int_\Gamma [x_1 s_1(\widetilde{x}, t) - x_2 s_1(\widetilde{x}, t)]\, ds = 0, \end{gathered} \tag{4.21}$$

for every $t \in [0, \infty)$.

Taking into account (3.6), (4.2) and (4.8), the condition $(4.21)_1$ becomes

$$\int_D \sigma_{\alpha 3;3}(\widetilde{x}, t)\, d\sigma + \int_\Gamma p_\alpha(\widetilde{x}, t)\, ds = 0, \qquad t \in [0, \infty). \tag{4.22}$$

Because

$$\int_D \sigma_{\alpha 3;3}(\widetilde{x}, t)\, d\sigma = \frac{\partial}{\partial x_3}\int_D x_\alpha\, \sigma_{33;3}(x, t)\, d\sigma,$$

and taking into account from (3.6) [8] it results $\sigma_{33;3}(x, t) = 2E_\beta(t)(x_\beta - x_\beta^0)\, x_3 - E_4(t)$, it follows that $\int_D \sigma_{\alpha 3;3}(\widetilde{x}, t)\, d\sigma =$

$$= 2I_{\alpha\beta}\, E_\beta(t), \text{ where } I_{\alpha\beta} = \int_D (x_\alpha - x_\alpha^0)(x_\beta - x_\beta^0)\, d\sigma.$$

Thus, in view of the relations (4.3) and (3.4), $j = 1, 2$ from $(4.21)_1$ we obtain

$$I_{\alpha\beta}\, M[A_3(t) + a_\beta(t)] = -\frac{1}{4}\int_\Gamma p_\alpha(\widetilde{x}, t)\, ds. \tag{4.23}$$

Since $\Delta = I_{11} I_{22} - I_{12}^2 > 0$, by (4.23) we have

$$M[A_\alpha(t) + a_\alpha(t)] = f_\alpha(t), \tag{4.24}$$

where

$$f_\alpha(t) = \frac{1}{4\Delta}\, e_{\alpha\beta}\, e_{\gamma\delta}\, I_{\beta\gamma}\int_\Gamma p_\delta(\widetilde{x}, t)\, ds, \tag{4.25}$$

and $e_{\alpha\beta}$ are $e_{11} = e_{22} = 0$, $e_{12} = -e_{22} = 1$.

The function $f_\alpha(t)$ are continuous of $t \in [0, \infty)$ and therefore, the equations (4.24) have unique continuous solutions on $[0, \infty)$. Let $q_\alpha(t)$ be the solutions of these equations

$$A_\alpha(t) + a_\alpha(t) = q_\alpha(t). \tag{4.26}$$

From these relations and (4.3) it follows

$$a_\alpha(t) + \int_0^t \frac{3l(t,\tau) + 2m(t,\tau)}{3\lambda(t) + 2\mu(t)}\, a_\alpha(\tau)\, d\tau = \frac{Q(t)}{3\lambda(t) + 2\mu(t)}, \tag{4.27}$$

where $Q_\alpha(t) = 2(L + M)[q_\alpha(t)]$, $\alpha = 1, 2$. The functions $A_\alpha(t)$ and $a_\alpha(t)$ are determined by (4.26) and (4.27).

By (4.17) and (4.26) it results

$$A_3(t) + a_3(t) = -q_\alpha(t)\, x_\alpha^0 . \tag{4.28}$$

and by $A_3^{\cdot}(t) = 0$ we obtain

$$a_3(t) + \int_0^t \frac{3l(t,\tau) + 2m(t,\tau)}{3\lambda(t) + 2\mu(t)}\, a_3(\tau)\, d\tau = -\frac{Q_\alpha(t)\, x_\alpha^0}{3\lambda(t) + 2\mu(t)}. \tag{4.29}$$

Thus, the functions $A_j(t)$, $a_j(t)$, $j = 1, 2, 3, 4$, are determined.

5. Now we can determine the function $\psi(\widetilde{x}, t)$ as the solution of the problem (4.15). The necessary and sufficient conditions to solve this problem are satisfied. The solution of this problem is

$$\psi(\widetilde{x}, t) = \chi(\widetilde{x}, t) + 2k(t)\, \varphi(x_1, x_2), \tag{5.1}$$

where $\chi(\widetilde{x}, t)$ and $\varphi(x_1, x_2)$, are the solution of the problems

$$\begin{cases} \Delta\varphi(x_1, x_2) = 0, & \widetilde{x} \in D, \\ \dfrac{d\varphi(x_1, x_2)}{dn} = x_2 n_1 - x_1 n_2, & \widetilde{x} \in \Gamma \end{cases} \tag{5.2}$$

and

$$\Delta\chi(\widetilde{x}, t) = -4a_3(t), \ \widetilde{x} \in D,$$

$$\frac{d\chi(\widetilde{x}, t)}{dn} = 2\,[A_1(t)(x^2 - x^2) + a_1(t)\, x_2^2 + A_3(t)\, x_1]\, n_1 = \tag{5.3}$$

$$= 2\,[A_2(t)(x_1^2 - x_1^2) + a_2(t)\, x_1^2 + A_3(t)\, x_2]\, n_2, \ \widetilde{x} \in \Gamma$$

for every $t \in [0, \infty)$. By (4.28) it results that the necessary and sufficient conditions to solve the problem (5.3) are verified.

The function $\psi(\widetilde{x}, t)$ will be known if we determine the function $k(t)$. This function is determined from $(4.21)_2$, by using (4.2), (4.8) and (5.1). Indeed, from these relations we obtain

$$k(t) + \int_0^t \frac{m(t,\tau)}{\mu(t)}\, k(\tau)\, d\tau = \frac{P(t) + Q(t) + R(t)}{2D(t)}, \tag{5.4}$$

where

$$P(t) = \int_\Gamma [x_2 p_1(\widetilde{x}, t) - x_1 p_2(\widetilde{x}, t)]\, ds,$$

$$Q(t) = 2\Big\{ M\,[A_1(t)] \int_D x_2 (x_2^2 - x_1^2)\, d\sigma - M\,[a_1(t)] \int_D x_1^2 x_2\, d\sigma +$$

$$+ M\,[A_2(t)] \int_D x_1 (x_2^2 - x_1^2)\, d\sigma + M\,[a_2(t)] \int_D x_1 x_2^2\, d\sigma \Big\},$$

$$R(t) = 2M\Big[\int_D (x_2 \chi_{,1}(\widetilde{x}, t) - x_1 \chi_{,2}(\widetilde{x}, t))\, d\sigma \Big],$$

$$D(t) = \mu(t) \int_D [x_1^2 + x_2^2 + x_1 \varphi_{,2}(x_1, x_2) - x_2 \varphi_{,1}(x_1, x_2)]\, d\sigma.$$

$D(t)$ is a positive function on $[0, \infty)$ because $\mu(t) > 0$ and $\int_D [x_1^2 + x_2^2 + x_1 \varphi_{,2}(x_1, x_2) - x_2 \varphi_{,1}(x_1, x_2)]\, d\sigma > 0$. It is obvious that $P(t)$ and $Q(t)$ are continuous functions in $[0, \infty)$, since $A_\alpha(t)$, $a_\alpha(t)$ and $p_\alpha(\widetilde{x}, t)$ are continuous functions of $t \in [0, \infty)$.

The function $R(t)$ is continuous in $[0, \infty)$. Indeed, by transforming $R(t)$ and taking into account that $(x_1^2 - x_2^2)\, \chi(\widetilde{x}, t)$ is a single valued function for every $t \in [0, \infty)$, we get

$$\int_D [x_2 \chi_{,1}(\widetilde{x}, t) - x_1 \chi_{,2}(\widetilde{x}, t)]\, d\sigma = \frac{1}{2} \int_\Gamma (x_1^2 + x_2^2) \frac{d\chi(\widetilde{x}, t)}{ds}\, ds$$

so that

$$R(t) = M\Big[\int_\Gamma (x_1^2 + x_2^2) \frac{d\chi(x, t)}{ds}\, ds \Big].$$

Since $\dfrac{d\chi(\widetilde{x}, t)}{dn}$ is continuous of $t \in [0, \infty)$ for every $\widetilde{x} \in \Gamma$ it follows that $\dfrac{d\chi(\widetilde{x}, t)}{ds}$ is a continuous function in $[0, \infty)$ and therefore $R(t)$ is a continuous function on $[0, \infty)$.

It follows that Volterra's equation (5.4) has continuous solution in $[0, \infty)$. The function $k(t)$ is the solution of this equation. Because the functions $\psi(\widetilde{x}, t)$, $k(t)$ and $A_\alpha(t)$, $a_\alpha(t)$, $\alpha = 1, 2$, are determined, it results that $F_\alpha(\widetilde{x}, t)$ and $s_\alpha(\widetilde{x}, t)$ are known functions.

We assume that the plane strain problem (4.4), (4.10) has the solution $v_\alpha(\widetilde{x}, t)$, $\alpha = 1, 2$, which belongs to $C^2(D) \cap C^1(\bar{D})$ for every $t \in [0, \infty)$ continuous of t, uniformly with respect to $\widetilde{x} \in \bar{D}$.

Finally, we can conclude that the Almansi-Michell's problem for visco-elastic beams has the solution (3.1) where: $A_j(t)$, $a_j(t)$, $j = 1, 2, 3, 4$, are given by (4.17) — (4.20), (4.26) — (4.29), $k(t)$ is the solution of the equation (5.4), $\Phi(\widetilde{x}, t)$ is the solution of the problem (4.16), $\psi(\widetilde{x}, t)$ is given by (5.1) and $v_\alpha(\widetilde{x}, t)$, $\alpha = 1, 2$ is the solution of the plane viscoelastic problem (4.4), (4.10).

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ASUPRA PROBLEMEI LUI ALMANSI-MICHELL PENTRU BARE VÎSCOELASTICE

Rezumat

Se consideră problema lui Almansi-Michell pentru bare cilindrice vîsco-elastice, ecuațiile constitutive pentru mediul vîscoelastic fiind date de (1.1). Soluția problemei este dată de (3.1).

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LA DÉFORMATION DES COQUES ÉLASTIQUES MICROPOLAIRES

PAR

CRISTIAN CONSTANDA

Le développement de la théorie des coques élastiques comme une branche spéciale de la théorie de l'élasticité est dû non seulement à son importance pour la pratique, mais encore à la possibilité de réduire le problème tridimensionnel à l'un en deux variables. Les recherches dans ce domaine ont été commencées après 1850, avec les travaux de Kirchhoff [1] sur les plaques élastiques minces et de Love [2] sur les coques élastiques minces. Plus tard, au début de notre siècle, l'attention des auteurs s'est dirigée encore sur les coques épaisses ou d'épaisseur moyenne. Les nombreuses théories développées au XX-ème siècle répondent en grande partie aux nécessités applicatives, parce qu'elles proposent des formules et des schémas approximatives destinées à simplifier les calcules. Mais la réalisation de ce désidérat a influencé d'une manière négative presque chaque fois la rigueur de l'aspect mathématique et mécanique du problème: les théories mentionnées présentent des incompatibilités mathématiques et ne sont pas soumises aux principes mécaniques fondamentaux.

Dans ce qui suit on construit une théorie des coques élastiques micro-polaires qui contribue à élimination de ces inconvénients. Elle peut justifier quelques résultats antérieurs obtenus par des méthodes sans support théorique et, par des particularisations convenables, elle permet de retrouver beaucoup de théories déjà connues dans la littérature de spécialité.

Parmi les auteurs roumains qui ont amélioré la théorie des plaques et des coques élastiques nous citons: M. Haimovici [3], [4], V. Manea [5], M. Mișicu [6], P. P. Teodorescu [7] etc.

On considère un corp élastique occupant un domaine D borné d'une surface S dans l'espace euclidien tridimensionnel. On fait les notations suivantes: