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EXISTENCE AND UNIQUENESS FOR SOME NONLINEAR FUNCTIONAL EQUATIONS IN A BANACH SPACE

BY

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1. Preliminaries. Let X be a Banach space with norm $\| . \|$ and let F be the duality mapping of X into 2^{X^*}, where X^* is the adjoint space of X. Let us denote by $(.)$ the pairing between X and X^*.

Definition 1.1. *A mapping $B : D(B) \subset X \to X$ is said to be dissipative if for any $x, y \in D(B)$, there exists $f \in F(x - y)$ such that* $(Bx - By, f) \leqq \leqq 0$.

It is well known that B is dissipative if and only if for each $\alpha \geqq 0$ and $x, y \in D(B)$, we have

$$\| x - y \| \leqq \| x - \alpha Bx - (y - \alpha By) \|. \tag{1.1}$$

The object of this paper is a result for the global existence of the nonlinear evolution equation

$$\frac{du}{dt} = A(t)u, \; t \in R, \tag{1.2}$$

$$u(0) = x. \tag{i}$$

As an application we shall treat the Hammerstein-Volterra integral equation

$$u(t) = T(t - t_0)x + \int_{t_0}^{t} T(t - s)\, A(s)\, u(s)\, ds, \; x \in X, \tag{1.8}$$

where $T(t)$ is a strongly continuous group of linear isometric operators on X, $-\infty < t < +\infty$, and A is a continuous function from $R \times X$ to X satisfying a condition weaker than dissipativeness.

Our results are closely related to those of [1—3, 5—9]. We will need the following result [1,4 —6].

Theorem 1.1. *Suppose that the following conditions are satisfied*

a) *For every* $x, y \in X$, *there is* $f \in F(x-y)$ *such that*

$$(A(t)x - A(t)y, f) \leqq \gamma(t)\|x-y\|^2 \qquad t \in R_+ \tag{1.4}$$

where $\gamma: R_+ \to R_+$ *is a continuous function and* R_+ *is the positive half — axis*

b) A *is a continuous function from* $R_+ \times X$ *to* X.

Then for each $x \in X$, *there is an unique function* $u: R_+ \to X$ *satisfying* (1.2) *and* (i) *for all* $t \in R_+$.

2. A global existence and uniqueness result. Let us consider the following conditions:

(I) For each $x, y \in X$, there is $f \in F(x-y)$, such that

$$|(A(t)x - A(t)y, f)| \leqq \gamma(t)\|x-y\|^2, \ t \in R, \tag{2.1}$$

where $\gamma: R \to R_+$ is a continuous function, and A is a continuous function form $R \times X$ to X;

(II) For each $(t, x) \in R \times X$ there is an unique function $u: R \to X$ satisfying

$$\frac{du}{dt} = A(t)u, \ t \in R, \tag{2.2}$$

$$u(t_0) = x, \ t_0 \in R. \tag{2.3}$$

In addition, if v is the solution of (2.2) corresponding to y (i. e. $v(t_0) = y$), then

$$\|u(t) - v(t)\| \leqq \{\exp|\int_{t_0}^{t}\gamma(s)ds|\}\|x-y\|, \qquad t \in R. \tag{2.4}$$

Theorem 2.1. *The conditions (I) and (II) are equivalent.*

Proof. We prove that (I) implies (II). Indeed let us consider the following problem

$$\frac{dv}{dt} = -A(-t)v = A_1(t)v, \ t \in R_{t_0} = [t_0, +\infty), \tag{2.5}$$

$$v(t_0) = x, \qquad x \in X. \tag{2.6}$$

As A satisfies (2.1), it follows easily

$$(A_1(t)x - A_1(t)y, f) \leqq \gamma_0(t)\|x-y\|^2, \tag{2.7}$$

where $\gamma_0(t) = \gamma(-t)$.

Taking into account Theorem 1.1, there is an unique function $v: R_{t_0} \to X$, satisfying (2.5) and (2.6). Let us denote by u the solution of (2.2) and (2.3) considered for $t \geqq t_0$ (such a solution exists by Theorem 1.1). Now it is easy to see that the function z defined by

$$z(t) = \begin{cases} u(t), & \text{for } t \geqq 0, \\ v(-t), & \text{for } t \leqq 0, \end{cases} \tag{2.8}$$

is the solution of the problem (2.2) + (2.3), for $t_0 = 0$. If $t_0 \neq 0$, let us consider the following problem

$$\frac{dz}{dt} = A_2(t)z, \quad t \in R, \tag{2.9}$$

$$z(0) = x, \tag{2.10}$$

where $A_2(t)u = A(t + t_0)u$ for all $t \in R$ and $u \in X$. Taking into account the above result, it follows that there is an unique, solution $z: R \to X$ of (2.9) + (2.10). Then the function $u: R \to X$ defined by $u(t) = z(t - t_0)$ is the solution of (2.2) + (2.3). The uniqueness of the solution of (2.2) + (2.3) follows from (2.4); so that we must prove (2.4). For $t \geqq t_0$, (2.4) is known (see [2]). Let us consider the case $t \leqq t_0$. We have succesively

$$\left(\frac{d(u(t) - v(t))}{dt}, f\right) = (A(t)u(t) - A(t)u(t), f) \geqq - \gamma(t)\|u(t) - v(t)\|^2, \quad t \in R. \tag{2.11}$$

$$\frac{d}{dt}\|u(t) - v(t)\| \geqq -\gamma(t)\|u(t) - v(t)\|, \ \text{a. e. on } t \leqq t_0. \tag{2.12}$$

But (2.12) implies

$$\frac{d}{dt}\left\{\|u(t) - v(t)\| \exp\int_{t_0}^{t}\gamma(s)ds\right\} \geqq 0, \ \text{a. e. on } (-\infty, t_0]. \tag{2.13}$$

Integrating (2.13) over $[t_0, t]$, we obtain

$$\|u(t) - v(t)\| \exp\int_{t_0}^{t}\gamma(s)ds - \|x - y\| \leqq 0 \ \text{ for all } t \leqq t_0, \tag{2.14}$$

which implies (2.4). Now we shall prove that (II) implies (I). Let $t_0 \in R$ and x, y in X. For $t \geqq t_0$ the proof is given by L o v e l a d y in [2], so that let us consider the case $t \leqq t_0$. We have

$$(A(t_0)x - A(t_0)y, f) = \left(\frac{d^-}{dt_0}(u(t_0) - v(t_0)), f\right) \tag{2.15}$$

where $\frac{d^-}{dt}$ means the left derivative, and $f \in F(x-y)$. By a standard method (see e. g. K a t o [4]) it follows

$$(2.16)\qquad \left(\frac{d^-}{dt_0}(u(t_0)-v(t_0)),\ f\right) \geq \| u(t_0)-v(t_0)\| \frac{d^-}{dt_0}\| u(t_0)-v(t_0)\|.$$

and therefore

$$(2.17)\qquad (A(t_0)x - A(t_0)y,\ f) \geq \| x-y \| \left\| \frac{d^-}{dt_0} \right\| u(t_0) - v(t_0) \|.$$

Taking into account (2.4) and (2.17) we have succesively

$$(2.18)\qquad (A(t_0) - A(t_0)y, f) \geq \| x-y \| \lim_{h\to 0-} \frac{\| u(t_0+h) - v(t_0+h) \| - \| x-y \|}{h} \geq$$

$$\geq \| x-y \|^2 \lim_{h\to 0-} \frac{\exp\int_{t_0}^{t_0+h} -\gamma(s)ds - 1}{h} = -\gamma(t_0)\| x-y \|^2.$$

Obviously (2.13) and (2.18) implies (2.1), and the theorem is proved.

3. An application. We will aply the result of section 2 to discuss existence and uniqueness of solution to the Hammerstein-Volterra integral equation (1.3).

Theorem 3.1. *Let $T(t)$, $t \in R$ be a strongly continuous group of linear isometric operators on X. Furthermore, if we assume that the condition (I) is true, then for each $x \in X$ and $t_0 \in R$ there is an unique solution $u(t, t_0, x)$ to the integral equation (1.3).*

Proof. As in [10], let us define the mapping $C: R \times X \to X$

$$(3.1)\qquad C(t)u = T(t_0 - t)A(t)T(t-t_0)u,\quad t_0, t \in R,\ u \in X.$$

Obviously, C is a continuous mapping from $R \times X$ to X. We shall prove that C satisfies a condition of type (2.1). Taking into account (1.1) it is sufficient to verify (for each $\alpha > 0$ and t fixed in R) the following inequality

$$(3.2)\qquad \| x_1 - x_2 \| \leq \| [1 + \alpha\gamma(t)](x_1 - x_2) - \alpha[C(t)x_1 - C(t)x_2] \|$$
$$\text{for all } t \in R,\ x_1, x_2 \in X.$$

Since $A(t)$ satisfies the inequality (2.1) it follows

$$(3.3)\qquad \| T(t-t_0)(x_1-x_2) \| \leq \| [1+\alpha\gamma(t)](x_1 - x_2) - \alpha T(t_0 - t) \times$$
$$\times [A(t)\ T(t-t_0)x_1 - A(t)\ T(t-t_0)x_2] \|.$$

In as much as $T(t)$ is isometric and using (3.3) we obtain (3.2) and therefore C satisfies (I). By Theorem 2.1 it follows that there is an unique solution $v: R \to X$ of the problem

$$(3.4)\qquad \frac{dv(t)}{dt} = C(t)v(t),\quad v(t_0) = x.$$

Integrating (3.4) over $[t_0, t]$ we derive

$$(3.5)\qquad v(t, t_0, x) = x + \int_{t_0}^{t} T(t_0 - s)A(s)T(s - t_0)v(s, t_0, x)ds,\ t \in R.$$

Taking into account (3.5) it follows that the function

$$(3.6)\qquad u(t, t_0, x) = T(t-t_0)v(t, t_0, x) \qquad t \in R$$

is a solution of the integral equation (1.3).

To prove the uniqueness of the solution of (1.3) it is sufficient to see that if $u(t, t_0, x)$ is a solution of (1.3), then the function $v(t, t_0, x) = T(t_0 - t)u(t, t_0, x)$, $t \in R$ is the solution of the problem (3.4) which has an unique solution.

The theorem is proved.

Remark 3.1. In the case $T(t)$ is the strongly continuous semigroup of linear transformation on X, and A is a continuous function from $[0, +\infty) \times X$ to X satisfying:

$$(A(t)x - A(t)y, f) \leq \gamma(t)\| x-y \|^2,\ x, y \in X,\ f \in F(x-y),\ t \in [0, +\infty),$$

where $\gamma: R \to R$ is a continuous function, the equation (1.3) has a unique solution on R_+. This is a recent result of L o v e l a d y [2]. In [9] only the local existence of the solution is proved.

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EXISTENȚĂ ȘI UNICITATE PENTRU UNELE ECUAȚII FUNCȚIONALE ÎN SPAȚII BANACH

Rezumat

Se dă o condiție suficientă (mai slabă decît condiția lui Lipschitz) pentru existența și unicitatea soluției problemei (1.2) pe axa reală. Ținînd seama de un rezultat recent al lui L o v e l a d y [2], dacă se cere ca soluțiile problemei (1.2) să fie exponențial mărginite, atunci condiția de mai sus este și necesară. Ca aplicație se dă un rezultat de existență și unicitate a soluției unei ecuații integrale de tip Hammerstein-Volterra.

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DÉPENDENCE DU DOMAIN D'INTÉGRATION DE LA SOLUTION D'UNE ÉQUATION INTÉGRALE SUR UN ESPACE TOPOLOGIQUE MESURÉ

PAR

SENDER SOLOMON

En partant de certaines équations différentielles pour des fonctions d'ensemble de A. H a i m o v i c i (voir par exemple [1]) nous avons étudié, dans [2], l'existence et l'unicité de la solution, d'une équation intégrale de la forme

$$u(x) = u_0(x) + \int_{P_x} f(x, y, u(y))\, dy \tag{1}$$

sur un espace topologique mesuré.

Dans [3] nous avons établi la continuité de l'application $u_0 \to u$. Ici nous nous intéressons à la dépendence $P \to u$, c'est à dire, la dépendence de la solution du domain d'intégration.

Rappellons les éléments de l'équation (1) : u_0 donnée et u cherchée sont des fonctions continues définies sur un espace topologique X et qui prennent les valeurs dans un espace de Banach E; sur X il y a une mesure positive μ telle que chaque compact est mesurable. Soit Γ la classe des parties relativement compactes et sommables; P est une application $X \to \Gamma$ telle que $\lim\limits_{y \to x} \mu(P_x \Delta P_y) = 0$ pour tout $x \in X$; f est une application continue $X \times X \times E \to E$ telle que $|f(x, y, z) - f(x, y, \bar{z})| \leq L\,|z - \bar{z}|$ et $|f(x, y, 0)| \leq a(y)$ où a est une fonction sommable $X \to [0, +\infty)$. Soit $C(X; E)$ l'espace localement convexe des fonctions continues $X \to E$ muni de la topologie de la convergence compacte et $B_\omega(X;E)$ l'espace de Banach des fonctions continues $u : X \to E$ telles que