Il est évident maintenant qu'il éxiste une fonction $\beta \in B_\tau$, $\beta \geq 0$ telle que

$$|v(x) - u(x)| = t \int_{Q_x} |v(y) - u(y)|\, dy + t \int_{R_x \Delta P_x} |u(y)|\, dy +$$

$$+ t \int_{P_x \Delta R_x} a(y)\, dy - \beta(x).$$

D'autre part

$$\int_{R_x \Delta P_x} |u(y)|\, dy \leqq \sup_{y \in Q_x} |u(y)|\, \mu(R_x \Delta P_x) \leqq \sup_{y \in Q_x} |u(y)|\, \mu(R \Delta P)$$

et

$$\sup_{t \in Q_x} \int_{R_t \Delta P_t} |u(y)|\, dy \leqq \sup_{y \in Q_x} |u(y)|\, \mu(R \Delta P).$$

D'après le lemme 1 nous avons

$$|v(x) - u(x)| \leqq \tau(x) \max \{\mu(R\Delta P) \sup_{y \in Q_x} |u(y)|, \sup_{z \in X} \int_{P_z \Delta R_z} a(y) dy\}$$

Le théorème est démontré.

Institutul de matematică
Filiala Iași a Academiei R. S. R.

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DEPENDENȚA DE DOMENIUL DE INTEGRARE A SOLUȚIEI UNEI ECUAȚII INTEGRALE PE UN SPAȚIU TOPOLOGIC CU MĂSURĂ

Rezumat

Se demonstrează dependența continuă a soluției unei ecuații integrale — studiate în [2] și [3] — în raport cu domeniul de integrare.

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ON THE STABILITY OF DIFFERENCE EQUATIONS IN BANACH SPACES

BY

F. S. DE BLASI (*) and J. SCHINAS (*)

1. Introduction. Difference equations occur in many branches of applied mathematics. In a number of practical problems one is required to predict, for a given equation, the asymptotic behavior of its solutions as well as their sensitivity under the effect of small perturbations.

In the present paper we shall be concerned with stability, in particular asymptotic stability properties, of the equilibrium point of a linear time dependent difference equation and we study when these properties are preserved under the effect of a small non-linear perturbation of the forcing term. Similar results on this problem have been obtained by R. Bellman [1] and, more recently, by S. Sugiyama [8].

It is well known that methods in difference equations are, generally speaking, parallel to the corresponding ones, which are valid in the continuous case of differential equations. The discrete analogue of the so-called direct method of Liapunov has been studied by many authors [3], [4], [5], [8]. Here, however, we shall use a technique based on the discrete analogue of the variation of constants formula and Gronwall's inequality; such a technique, which has been widely applied in differential equations (see [2] Ch. III and [7] Ch. IX), was used by R. Bellman also for certain classes of difference equations in E^n with time independent linear part.

The contents of the article are the following. In section 2, notation and definitions are introduced. In the following one, we prove an extension of Gronwall's inequality (see also [6] and [8]) on which depend some of the proofs of the subsequent theorems. Finally, of the last three sections,

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the first refers to stability and uniform stability, the second to various kinds of asymptotic stability and the third to the strong stability of difference equations. Some theorems are followed by examples.

2. Notation and basic definitions. Denote by: $I = N(t_0) = \{t_0, t_0 + 1, \ldots\}$, where t_0 is any natural number or zero; $= \{a, a + 1, \ldots\}$, where a is a fixed integer; $I(t_0) = \{t_0, t_0 + 1, \ldots\}$, $t_0 \in I$; E a Banach space with norm $|\cdot|$; E^n the n-dimensional real Euclidean space with norm $|x| = \sum_{i=1}^{n} |x_i|$; M the space of all linear homeomorphisms from E on E with norm $\|\cdot\|$ induced by $|\cdot|$; M^n the space of all $n \times n$ non-singular matrices $A = (a_{ij})$ with norm induced by the norm in E^n, $\|A\| = \max_k \sum_{i=1}^{n} |a_{ik}|$.

Let $f: I(a + 1) \times E \to E$ and consider the difference equation

$$x(t) = f(t, x(t-1)). \tag{1}$$

Definition 1. *For any $t_0 \in I$, $x_0 \in E$, the function $x: I(t_0) \to E$ $(y: I \setminus I(t_0 + 1) \to E)$ is called right (left) solution of* (1), *if $x(t_0) = x_0$ $(y(t_0) = x_0)$ and $x(t)$ (resp. $y(t)$) satisfies* (1) *for all $t \in I(t_0 + 1)$, $(t \in I(a+1) \setminus I(t_0+1))$; if $x(t)$ and $y(t)$ exist and $x(t_0) = y(t_0) = z_0$, the function $z: I \to E$, $z(t) = x(t)$, $t \in I(t_0)$, $z(t) = y(t)$, $t \in I \setminus I(t_0 + 1)$, will be called solution of* (1) *with $z(t_0) = z_0$.*

For any $t_0 \in I$ and $x_0 \in E$, the right solution $x(t)$ of (1), with $x(t_0) = x_0$, exists and is unique; while the left solution $y(t)$ of (1), with $y(t_0) = x_0$, if exists, can be not unique. It may happen that two right solutions $x_1(t)$, $x_2(t)$ of (1), starting at the same time t_0, from two different points x_1^0, x_2^0, can be equal for some $t_1 \in I(t_0 + 1)$ and therefore for all $t \in I(t_1)$. To exclude such a possibility it is necessary and sufficient that, for any $t \in I(a + 1)$ and $x, y \in E$, $x \neq y$ implies $f(t, x) \neq f(t, y)$.

We give some definitions of stability of the origin for (1). To this end we assume $f(t, 0) \equiv 0$.

Definition 2. *The origin is said to be stable for* (1) *if, for every $\varepsilon > 0$ and any $t_0 \in I$, there exists a $\delta(\varepsilon, t_0) > 0$ such that, for any right solution $x(t)$ of* (1),

$$|x(t_0)| < \delta(\varepsilon, t_0) \text{ implies } |x(t)| < \varepsilon \text{ for all } t \in I(t_0).$$

Definition 3. *The origin is said to be uniformly stable for* (1) *if, for every $\varepsilon > 0$ and any $t_0 \in I$, there exists a $\delta(\varepsilon) > 0$ (independent of t_0) such that, for any right solution $x(t)$ of* (1),

$$|x(t_0)| < \delta(\varepsilon) \text{ implies } |x(t)| < \varepsilon \text{ for all } t \in I(t_0).$$

Definition 4. *The origin is said to be asymptotically stable for* (1) *if it is stable and, for any $t_0 \in I$, there exists a $\delta(t_0) > 0$ such that, for any right solution $x(t)$ of* (1),

$$|x(t_0)| < \delta(t_0) \text{ implies } \lim_{t \to +\infty} |x(t)| = 0.$$

Definition 5. *The origin is said to be uniformly asymptotically stable for* (1) *if: (i) it is uniformly stable and (ii) there is a $\delta_0 > 0$ and, for any $t_0 \in I$ and $\varepsilon > 0$ there exists a $T(\varepsilon) \in N(0)$ such that, for any right solution $x(t)$ of* (1).

$$|x(t_0)| < \delta_0 \text{ implies } |x(t)| < \varepsilon \text{ for all } t \in I(t_0 + T(\varepsilon)).$$

Definition 6. *The origin is said to be strongly stable for* (1) *if (i) there exists a $\delta > 0$ such that, for any $t_0 \in I$ and $|z_0| < \delta$, there exists at least one solution $z(t)$ of* (1) *with $z(t_0) = z_0$; (ii) for any $\varepsilon > 0$ and $t_0 \in I$, there exists a $\delta(\varepsilon)$, $0 < \delta(\varepsilon) \leqslant \delta$ (independent of t_0) such that, for any solution $z(t)$ of* (1),

$$|z(t_0)| < \delta(\varepsilon) \text{ implies } |z(t)| < \varepsilon, \text{ for all } t \in I.$$

For (1) *the origin is said to be unstable if it is not stable.*

3. A discrete analogue of Gronwall's inequality. To prove the discrete analogue of Gronwall's inequality we shall use the following

Lemma 1. *Let $t_0 \in I$ and $b: I(t_0) \to E^+$, $k: I(t_0) \to E^+$, $(E^+ = [0, \infty))$, be given. Then the scalar equation*

$$x(t) = b(t) + \sum_{s=t_0}^{t-1} k(s)\, x(s), \qquad t \in I(t_0) \tag{2}$$

has the unique solution which is expressed by

$$x(t) = b(t) + \sum_{s=t_0}^{t-1} b(s)\, k(s) \prod_{v=s+1}^{t-1} (1 + k(v)), \qquad t \in I(t_0). \tag{3}$$

Proof. Let $x(t)$ and $y(t)$ be solutions of (2). Setting $z(t) = |x(t) - y(t)|$, we have

$$z(t) \leqslant \sum_{s=t_0}^{t-1} k(s)\, z(s), \qquad t \in I(t_0). \tag{4}$$

Since $z(t_0) = 0$, we have $z(t) \equiv 0$, which proves the uniqueness. Let us show that $x(t)$, given by (3), satisfies (2) for any $t \in I(t_0)$. This is certainly true for $t = t_0$. Assume it to be true for $s = t_0, t_0 + 1, \ldots, t$. Then

$$x(t + 1) = b(t + 1) + \sum_{s=t_0}^{t} b(s)\, k(s) \prod_{v=s+1}^{t} (1 + k(v)) =$$

$$= b(t + 1) + \sum_{s=t_0}^{t-1} b(s)\, k(s) \prod_{v=s+1}^{t} (1 + k(v)) + b(t)\, k(t) =$$

$$= b(t + 1) + (1 + k(t)) \sum_{s=t_0}^{t-1} b(s)\, k(s) \prod_{v=s+1}^{t-1} (1 + k(v)) + b(t)\, k(t) =$$

$$= b(t + 1) + (1 + k(t))\,(x(t) - b(t)) + b(t)\, k(t) =$$

$$= b(t + 1) + [x(t) - b(t)] + k(t)\, x(t) = b(t + 1) + \sum_{s=t_0}^{t} k(s)\, x(s),$$

which completes the proof.

The following Lemma is the discrete analogue of Gronwall's Lemma.

Lemma 2. *Let $t_0 \in I$ and $b: I(t_0) \to E^+$, $k: I(t_0) \to E^+$ be given. Then, for any solution $y(t)$ of the scalar inequality*

$$y(t) \leqslant b(t) + \sum_{s=t_0}^{t-1} k(s)\, y(s), \qquad t \in I(t_0), \tag{5}$$

we have

$$y(t) \leqslant b(t) + \sum_{s=t_0}^{t-1} b(s)\, k(s) \prod_{v=s+1}^{t-1} (1 + k(v)), \quad t \in I(t_0), \tag{6}$$

$$y(t) \leqslant b(t) + \sum_{s=t_0}^{t-1} b(s)\, k(s) \exp \sum_{v=s+1}^{t-1} k(v), \qquad t \in I(t_0). \tag{7}$$

In particular, if $b(t) \equiv c$, c a non-negative constant, we have

$$y(t) \leqslant c \exp \sum_{v=t_0}^{t-1} k(v), \quad t \in I(t_0). \tag{8}$$

Proof. Denote by $y(t)$ any solution of (5) and by $x(t)$ the solution of (2) which is given by the formula (3). Subtracting (2) from (5) and setting $z(t) = y(t) - x(t)$, we get inequality (4). From this, it is obvious that $z(t_0) \leqslant 0$ implies $z(t) \leqslant 0$, i. e. $y(t) \leqslant x(t)$, for all $t \in I(t_0)$, which proves (6). Inequality (7) follows from (6) because

$$\prod_{v=s+1}^{t-1} (1 + k(v)) \leqslant \exp \sum_{v=s+1}^{t-1} k(v).$$

Now assume $b(t) \equiv c$ and denote by $y(t)$ any solution of (5). Clearly $y(t) \leqslant$ $\leqslant u(t)$, $u(t) = c\left[1 + \sum_{s=t_0}^{t-1} k(s) \exp \sum_{v=s+1}^{t-1} k(v)\right]$. Let us show that, for any $t \in I(t_0)$,

$$u(t) \leqslant w(t), \quad w(t) = c \exp \sum_{v=t_0}^{t-1} k(v), \; t \in I(t_0). \tag{9}$$

Clearly $u(t_0) = w(t_0)$. Assume $u(s) \leqslant w(s)$ for $s = t_0, t_0 + 1, \ldots, t$. Then

$$w(t+1) = w(t) \exp k(t) \geqslant u(t) \exp k(t) =$$

$$= c\left[\exp k(t) + \sum_{s=t_0}^{t-1} k(s) \exp \sum_{v=s+1}^{t} k(v)\right] \geqslant$$

$$c\left[1 + k(t) + \sum_{s=t_0}^{t-1} k(s) \exp \sum_{v=s+1}^{t} k(v)\right] = u(t+1).$$

So, for any $t \in I(t_0)$, (9) is true and a fortiori (8), since $y(t) \leqslant u(t) \leqslant w(t)$. This completes the proof.

4. Stability and uniform stability. Consider the linear difference equation

$$x(t) = A(t)\, x(t-1) \tag{L}$$

and the perturbed non-linear difference equation

$$x(t) = A(t)\, x(t-1) + f(t, x(t-1)), \tag{P}$$

where $A : I(a+1) \to M$ and $f : I(a+1) \times E \to E$ is such that $f(t, 0) \equiv 0$. Occasionally we shall refer to the time independent linear difference equation

$$x(t) = Ax(t-1), \tag{C}$$

where $A \in M$. For any $t_0 \in I$ and any $x_0 \in E$, the unique right solution $x(t)$ of (P) with $x(t_0) = x_0$ is given by

$$x(t) = X(t) X^{-1}(t_0)\, x(t_0) + \sum_{s=t_0+1}^{t} X(t) X^{-1}(s) f(s, x(s-1)), \; t \in I(t_0) \tag{10}$$

where $X(t)$ is the fundamental operator of (L), i. e. $X(t) = A(t)\, A(t-1) \ldots$ $\ldots A(a+1) J$ and J is the identity in M.

We start with a lemma which connects the stability properties of the origin for equation (L) to the fundamental operator $X(t)$ of (L).

Lemma 3. *For (L) the origin is: stable if and only if $\| X(t) \| \leqslant H$, $H \geqslant 1$, $t \in I$; uniformly stable if and only if*

$$\| X(t) X^{-1}(s) \| \leqslant K, \; K \geqslant 1, \; s \in I, \; t \in I(s);$$

asymptotically stable if and only if $\| X(t) \| \to 0$ as $t \to +\infty$: uniformly asymptotically stable if and only if, for any $s \in I$, we have

$$\| X(t) X^{-1}(s) \| \leqslant K b^{t-s}, \; t \in I(s), \text{ where } K \geqslant 1 \text{ and } 0 < b < 1$$

strongly stable if and only if there exists a constant $K \geqslant 1$ such that

$$\| X(t) \| \leqslant K, \; \| X^{-1}(t) \| \leqslant K, \quad t \in I.$$

Proof. As for differential equations (see [2] Ch. III).

In the particular case of equation (C) we have:

Lemma 4. *For (C) the origin is:*
stable if and only if it is uniformly stable;
asymptotically stable if and only if it is uniformly asymptotically stable.

Theorem 1. *Assume that the fundamental operator $X(t)$ of (L) satisfies the hypotheses:*

(i) $\|X(t)\| \leqslant H, \quad H \geqslant 1, \quad t \in I,$

(ii) *for any* $s \in I, \quad \|X(t)\, X^{-1}(s)\| \leqslant K b(s),\ t \in I(s),$

where $K \geqslant 1$ *and* $b(s) \geqslant 1$. *If the perturbation* $f(t, x)$ *in (P) is such that*

(iii) $|f(t, x)| \leqslant c(t)\,|x|$ *and* $\sum_{s=a+1}^{\infty} b(s)\, c(s) < +\infty,$

then the origin is stable for (P).

Proof. Let $t_0 \in I$. Let $x(t)$ be the right solution of (P) with $x(t_0) = x_0$. From (10), using (i), (ii) and (iii), we have

$$|x(t)| \leqslant H\,\|X^{-1}(t_0)\|\,|x(t_0)| + K \sum_{s=t_0}^{t-1} b(s+1)\, c(s+1)\,|x(s)|, \quad t \in I(t_0).$$

Then, by Gronwall's inequality,

$$|x(t)| \leqslant H\,\|X^{1(-1)}(t_0)\|\,|x(t_0)| \exp K \sum_{s=t_0}^{t-1} b(s+1)\, c(s+1)$$

and so

$$(11)\quad |x(t)| \leqslant M(t_0)\,|x(t_0)|,\ M(t_0) = H\,\|X^{-1}(t_0)\| \exp K \sum_{s=a+1}^{\infty} b(s)\, c(s) < +\infty.$$

This shows that the origin is stable for (P).

Theorem 2. *Assume that the fundamental operator* $X(t)$ *of (L) satisfies the hypothesis:*

(i) *for any* $s \in I$, $\|X(t)\, X^{-1}(s)\| \leqslant K,\ K \geqslant 1,\ t \in I(s).$

Then, if the perturbation $f(t, x)$ *in (P) is such that*

(ii) $|f(t, x)| \leqslant c(t)\,|x|, \quad \sum_{s=a+1}^{\infty} c(s) < +\infty,$

the origin is uniformly stable for (P).

Proof. The proof is as in Theorem 1 with the difference that, instead of (11), we get $|x(t)| \leqslant M\,|x(t_0)|$ where M is a constant independent of t_0.

Example 1. If in (L) $A(t)$ is taken to be

$$A(t) = \begin{pmatrix} 1 & 2^{-1} \\ 0 & 2^{-1} + 2^{-t} \end{pmatrix} \qquad t \in N(1),$$

then

$$\|X(t)\, X^{-1}(s)\| \leqslant \prod_{v=s+1}^{t} \|A(v)\| < e.$$

Condition (i) of Theorem 2 is satisfied and therefore the origin is uniformly stable for (L).

Example 2. For any $t_0 \in N(0)$ the right solution $(x_1(t), x_2(t))$ of

$$(12)\qquad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t-1) \\ x_2(t-1) \end{pmatrix},$$

with $x_1(t_0) = x_1^0,\ x_2(t_0) = x_2^0$, is given by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{t-t_0} \begin{pmatrix} x_1^0(t_0) \\ x_2^0(t_0) \end{pmatrix}, \qquad t \in N(t_0),$$

which shows that the origin is uniformly stable for (12).

Consider the following two perturbed equations of (12):

$$(13)\qquad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1-(t+1)^{-1} \\ 1-(t+1)^{-1} & 0 \end{pmatrix} \begin{pmatrix} x_1(t-1) \\ x_2(t-1) \end{pmatrix}$$

$$(14)\qquad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1+(t+1)^{-1} \\ 1+(t+1)^{-1} & 0 \end{pmatrix} \begin{pmatrix} x_1(t-1) \\ x_2(t-1) \end{pmatrix}.$$

For any $t_0 \in N(0)$ the right solutions of (13) and (14), with the same initial point (x_1^0, x_2^0), are given by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{t_0+1}{t+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{t-t_0} \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix},$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{t+2}{t_0+2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{t-t_0} \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix}, \qquad t \in N(t_0).$$

Therefore for (13) the origin is uniformly stable and also asymptotically stable (but not uniformly asymptotically stable), while the origin is clearly unstable for (14). This example shows that, without hypothesis (ii) of Theorem 2, the uniform stability of the origin for (12) is, in general, not sufficient to guarantee even the stability of the origin for the perturbed equation.

5. Asymptotic and uniform asymptotic stability. In this section we shall prove that, if the origin is asymptotically stable for the linear difference equation (L), a similar property does hold for the perturbed difference equation (P), provided that the forcing term $f(t, x)$ satisfies convenient hypotheses.

Theorem 3. *In addition to the hypotheses (ii) and (iii) of Theorem 1, assume that the fundamental operator* $X(t)$ *of (L) is such that* $\|X(t)\| \to 0$ *as* $t \to +\infty$. *Then the origin is asymptotically stable for (P).*

Proof. Let $t_0 \in I$ and $\varepsilon > 0$ be given. Since also hypothesis (i) of Theorem 1 is satisfied, the origin is stable for (P). This implies the existence of a $\delta(t_0) > 0$ such that

(15) if $|x(t_0)| < \delta(t_0)$ then $|x(t)| < 1, \quad t \in I(t_0)$.

From (10) we have

$$|x(t)| \leqslant \|X(t)\| \, |X^{-1}(t_0)\, x(t_0)| + \|X(t)\| \sum_{s=t_0+1}^{t_1} |X^{-1}(s) f(s, x(s-1))| + \tag{16}$$

$$\sum_{s=t_1+1}^{t} \|X(t)\, X^{-1}(s)\| \, |f(s, x(s-1))|$$

where t_1 is an integer, $t_0 < t_1 < t$. Take $x(t_0)$ satisfying (15). Then, for a fixed and sufficiently large t_1,

$$\sum_{s=t_1+1}^{t} \|X(t)\, X^{-1}(s)\| \, |f(s, x(s-1))| \leqslant K \sum_{s=t_1+1}^{t} b(s)\, c(s)\, |x(s-1)| \leqslant$$

$$K \sum_{s=t_1+1}^{\infty} b(s)\, c(s) < \varepsilon.$$

From the last inequality and (16) we get $\lim_{t \to +\infty} |x(t)| = 0$ if $|x(t_0)| < \delta(t_0)$, because $\|X(t)\| \to 0$ as $t \to \infty$ and $\varepsilon > 0$ was arbitrary. This completes the proof.

Theorem 4. *In addition to the hypotheses (i) and (ii) of Theorem 2, assume that the fundamental operator $X(t)$ of (L) is such that $\|X(t)\| \to 0$ as $t \to +\infty$. Then the origin is uniformly stable and asymptotically stable for* (P).

Proof. From Theorem 2, the origin is uniformly stable for (P). Moreover it is asymptotically stable since all hypotheses of Theorem 3 are satisfied.

From Lemma 3 and Theorems 2 and 4 we have:

Corollary 1. If the origin is uniformly stable for (L) (uniformly stable and asymptotically stable) and the perturbation $f(t, x)$ in (P) satisfies condition (ii) of Theorem 2, then the origin is uniformly stable (uniformly stable and asymptotically stable) also for (P).

Corollary 2. If the origin is uniformly stable (uniformly stable and asymptotically stable) for (L) and the perturbation $f(t, x)$ in (P) is such that

$$f(t, x) = B(t)\, x, \quad B(t) \in M, \quad \sum_{s=a+1}^{\infty} \|B(s)\| < +\infty, \tag{17}$$

then the origin is uniformly stable (uniformly stable and asymptotically stable) also for (P).

From Corollary 1 and Lemma 4 we have:

Corollary 3. If the origin is stable (asymptotically stable) for (C) and the perturbation $f(t, x)$ in (P) satisfies condition (ii) of Theorem 2, then the origin is uniformly stable (uniformly stable and asymptotically stable) also for (P).

Particular cases of Corollary 3 are found in [6] (Ch. I) and in [1].

Example 3. Consider the scalar difference equation

$$x(t) = A(t)\, x(t-1) \tag{18}$$

where $A(2k+1) = k+2$, $k \in N(0)$, $A(2k) = (k+1)^{-2}$, $k \in N(1)$. Let $t_0 \in N(0)$, $x_0 \in E^1$. The right solution $x(t)$ of (18), with initial value $x(t_0) = x_0$, is given, for any $t \in N(t_0)$, by

$$x(t) = X(t)\, X^{-1}(t_0)\, x(t_0), \quad X(2k) = \frac{1}{(k+1)!},$$
$$X(2k+1) = \frac{k+2}{(k+1)!}, \quad k \in N(0). \tag{19}$$

Therefore for (18) the origin is stable and asymptotically stable; but since, by taking in (19) $t_0 = 2h$, $t = 2h+1$, we have $x(2h+1) = (h+2)\, x(2h)$ it cannot be uniformly stable and so is certainly not uniformly asymptotically stable. Now consider the perturbed equation of (18).

$$x(t) = (A(t) + B(t))\, x(t-1), \quad B(2k+1) = 0, \quad k \in N(0), \\ B(2k) = (k+1)^{-1}, \quad k \in N(1). \tag{20}$$

The right solution $x(t)$ of (20) with $x(t_0) = x_0$, is given by $x(t) = 2^{-1}(k+2)^2\, x(t_0)$ if $t = 2k+1$, $k \in N(0)$, $x(t) = 2^{-1}(k+2)\, x(t_0)$ if $t = 2k$, $k \in N(0)$, which shows that, for a perturbation which is not sufficiently small (as required for instance in Theorem 3), the origin may be unstable for (20).

Lemma 5. *Suppose that*

$$\sum_{s=a}^{t} \|X(t)\, X^{-1}(s)\| \leqslant K, \quad K > 1, \quad t \in I,$$

where $X(t) \in M$, $t \in I$.

Then we have

$$\|X(t)\| \leqslant K \|X(a)\| \left(\frac{K-1}{K}\right)^{t-a} \text{ for all } t \in I. \tag{21}$$

Proof. For any $s \in I$, we have $\|X(s)\| > 0$. Then

$$X(t) \sum_{s=a}^{t} \|X(s)\|^{-1} = \sum_{s=a}^{t} (X(t)\, X^{-1}(s))\, X(s)\, \|X(s)\|^{-1}, \quad t \in I$$

from which

$$\|X(t)\| \sum_{s=a}^{t} \|X(s)\|^{-1} \leqslant K. \tag{22}$$

Setting $L(t) = \sum_{s=a}^{t} \| X(s) \|^{-1}$, we have

$$L(t) - L(t-1) = \| X(t) \|^{-1}, \qquad t \in I(a+1)$$

and, after substitution in (22), we get

$$L(t) \geqslant \frac{K}{K-1} L(t-1), \qquad t \in I(a+1),$$

which implies $\quad L(t) \geqslant L(a) \left(\frac{K}{K-1} \right)^{t-a}, \qquad t \in I.$

Using last inequality in (22) we finally obtain (21).

Theorem 5. *Assume that the fundamental operator $X(t)$ of (L) satisfies the condition*

$$\sum_{s=a}^{t} \| X(t) X^{-1}(s) \| \leqslant K, \qquad K > 1, \qquad t \in I.$$

Then, if the perturbation $f(t, x)$ in (P) is such that

$$| f(t, x) | \leqslant (b + c(t)) \, | x |, \text{ where } 0 \leqslant b < K^{-1} \text{ and } \sum_{s=a+1}^{\infty} c(s) = d < +\infty$$

the origin is asymptotically stable for (P).

Proof. First we prove that the origin is stable for (P). From Lemma 5, since $\| X(t) \| \to 0$ as $t \to +\infty$, there exists a positive constant L such that $\| X(t) \| \leqslant L$, $t \in I$. Let $t_0 \in I$ and let $x(t)$ be the right solution of (P) with $x(t_0) = x_0$. From (10) we have, for any $t \in I(t_0)$,

$$| x(t) | \leqslant L \, \| X^{-1}(t_0) \| \, | x(t_0) | + b \sum_{s=t_0+1}^{t} \| X(t) X^{-1}(s) \| \, | x(s-1) | +$$

$$+ \sum_{s=t_0+1}^{t} \| X(t) X^{-1}(s) \| \, c(s) \, | x(s-1) | .$$

Define $p(t) = \max \{ | x(t_0) |, | x(t_0+1) |, \ldots, | x(t) | \}$. Then for a certain integer t_1, $t_0 \leqslant t_1 \leqslant t$, we have

$$p(t) = | x(t_1) | \leqslant L \, \| X^{-1}(t_0) \| \, | x(t_0) | + bp(t) \sum_{s=t_0+1}^{t_1} \| X(t_1) X^{-1}(s) \| +$$

$$+ \sum_{s=t_0+1}^{t_1} Kc(s) \, p(s-1)$$

and therefore, $p(t) \leqslant L \, \| X^{-1}(t_0) \| \, | x(t_0) | + bKp(t) + \sum_{s=t_0+1}^{t} Kc(s) \, p(s-1)$

from which, since $bK < 1$, we have

$$p(t) \leqslant \frac{L \, \| X^{-1}(t_0) \|}{1 - bK} | x(t_0) | + \sum_{s=t_0}^{t-1} \frac{K}{1 - bK} c(s+1) \, p(s).$$

From this, by Gronwall's inequality,

$$| x(t) | \leqslant p(t) \leqslant \left[\frac{L \, \| X^{-1}(t_0) \|}{1 - bK} \exp \frac{Kd}{1 - bK} \right] | x(t_0) | .$$

This inequality implies that the origin is stable for (P).

Consequently, for any $t_0 \in I$ there exists a $\delta(t_0) > 0$ such that any right solution $x(t)$ of (P), with $| x(t_0) | < \delta(t_0)$, satisfies $| x(t) | < 1$ for all $t \in I(t_0)$. To complete the proof we shall show that, if $| x(t_0) | < \delta(t_0)$, we have $| x(t) | \to 0$ as $t \to +\infty$. To this end define

$$M = \overline{\lim_{t \to +\infty}} \, | x(t) | .$$

Obviously, $0 \leqslant M \leqslant 1$. Choose R such that

(23) $$0 \leqslant Kb < R < 1.$$

If $M > 0$, there exists $t_1 \in I(t_0)$ such that

(24) $$| x(t) | < MR^{-1} \text{ for all } t \in I(t_1).$$

From (10),

$$| x(t) | \leqslant \| X(t) \| \left[\| X^{-1}(t_0) \| \, | x(t_0) | + \sum_{s=t_0+1}^{t_1} | X^{-1}(s) \, f(s, x(s-1)) | \right] +$$

$$+ \sum_{s=t_1+1}^{t} \| X(t) X^{-1}(s) \| \, | f(s, x(s-1)) |$$

and, by virtue of (24) and the hypotheses of the theorem, we get

$$| x(t) | \leqslant \| X(t) \| \left[\| X^{-1}(t_0) \| \, | x(t_0) | + \sum_{s=t_0+1}^{t_1} | X^{-1}(s) f(s, x(s-1)) | \right] +$$

$$+ KbMR^{-1} + KMR^{-1} \sum_{s=t_1+1}^{t} c(s).$$

Let $\varepsilon > 0$. Since $\sum_{s=a+1}^{\infty} c(s) < +\infty$, taking if necessary a larger t_1 and keeping it fixed, we have

$$|x(t)| \leqslant \|X(t)\| \left| \|X^{-1}(t_0)\| \, |x(t_0)| + \sum_{s=t_0+1}^{t_1} |X^{-1}(s) f(s, x(s-1))| \right| +$$
$$+ KbMR^{-1} + \varepsilon.$$

By Lemma 5, $\|X(t)\| \to 0$ as $t \to +\infty$. Hence

$$M = \overline{\lim_{t\to\infty}} |x(t)| \leqslant KbMR^{-1} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $M > 0$, this implies $Kb \geqslant R$ in contradiction to (23). Therefore $M = 0$ and the Theorem is proved.

Example 4. Let the matrix $A(t)$ in (L) be given by

$$A(t) = \begin{pmatrix} 0 & (t+1)^{-1} \\ (3t^2+2)^{-1} & (t^4+1)^{-1} \end{pmatrix} \quad t \in N(1).$$

Since $\sum_{s=1}^{t} \|X(t) X^{-1}(s)\| < 3$, the fundamental operator $X(t)$ satisfies the hypotheses of Theorem 5.

Theorem 6. *Assume that the fundamental operator $X(t)$ of (L) is such that, for any $s \in I$,*

$$\|X(t) X^{-1}(s)\| \leqslant Kb^{t-s}, \; K \geqslant 1, \; 0 < b < 1, \; t \in I(s).$$

If the perturbation $f(t, x)$ in (P) satisfies the inequality

$$|f(t, x)| \leqslant (d + c(t)) \, |x|$$

where $d \geqslant 0$, $Kdb^{-1} + \lg b < 0$ and $\sum_{s=a+1}^{\infty} c(s) = L < +\infty$,

hen the origin is uniformly asymptotically stable for (P).

Proof. Let $t_0 \in I$. From (10) we get

$$|x(t)| \leqslant Kb^{t-t_0} |x(t_0)| + \sum_{s=t_0+1}^{t} K(d + c(s)) b^{t-s} |x(s-1)|, \; t \in I(t_0).$$

From this, multiplying both sides by $b^{-(t-t_0)}$ and setting $p(t) = b^{-(t-t_0)} |x(t)|$, we have

$$p(t) \leqslant K |x(t_0)| + \sum_{s=t_0}^{t-1} b^{-1} K(d + c(s+1)) p(s) \, .$$

from which, by Gronwall's inequality,

$$p(t) \leqslant K |x(t_0)| \exp \sum_{s=t_0+1}^{t} b^{-1} K(d + c(s)).$$

Hence

$$|x(t)| \leqslant K |x(t_0)| \exp(t - t_0) [Kdb^{-1} + \lg b + KLb^{-1}(t - t_0)^{-1}],$$

which implies

$$(25) \qquad |x(t)| \leqslant (K \exp KLb^{-1}) |x(t_0)| \, .$$

Let $q > 0$ be such that $-M = Kdb^{-1} + \lg b + q < 0$. Then, for all $t \in I(t_0)$ such that $t - t_0 \geqslant t_1$, where t_1 is a sufficiently large positive integer (independent of t_0), we have

$$|x(t)| \leqslant K |x(t_0)| \exp(-M(t - t_0)).$$

From this and (25) we obtain the uniform asymptotic stability of the origin for (P).

From Lemma 3 and Theorem 6 we have

Corollary 4. *If the origin is uniformly asymptotically stable for* (L) *and the perturbation $f(t, x)$ in (P) satisfies the hypothesis of Theorem 6, then the origin is uniformly asymptotically stable for (P) too.*

Corollary 5. *If the origin is uniformly asymptotically stable for* (L) *and the perturbation $f(t, x)$ in* (P) *is such that*

$$f(t, x) = B(t) x, \; B(t) \in M, \; \sum_{t=a+1}^{\infty} \|B(t)\| < +\infty,$$

then the origin is uniformly asymptotically stable also for (P).

From Corollary 4 and Lemma 4 we have the following Corollary which was firstly proved by R. Bellman [1].

Corollary 6. *If the origin is asymptotically stable for (C) and the perturbation $f(t, x)$ in (P) satisfies the hypothesis of Theorem 6, then the origin is uniformly asymptotically stable for (P).*

Example 5. If in (L)

$$A(t) = \begin{pmatrix} 0 & 2^{-t} \\ -4^{-t} & 3^{-t} \end{pmatrix} \quad t \in N(1)$$

the origin is uniformly asymptotically stable for (L), *because*

$$\|X(t) X^{-1}(s)\| < 2^{1-t+s}.$$

Example 6. If $\|A\| < 1$, the origin is uniformly asymptotically stable for (C) since

$$\|X(t) X^{-1}(s)\| = \|A^{t-s}\| \leqslant \|A\|^{t-s}.$$

Theorem 7. *Assume that the fundamental operator $X(t)$ of (L) satisfies, for any $s \in I$, the inequality*

$$\|X(t) X^{-1}(s)\| \leqslant K(s) b^{t-s}, \quad K(s) \geqslant 1, \; 0 < b < 1, \; t \in I(s),$$

with $K : I(a+1) \to E^+$ *such that, for any* $t_0 \in I$,

$$\overline{\lim_{t\to+\infty}}\,(t-t_0)^{-1}\sum_{s=t_0+1}^{t} K(s) = L(t_0) \leqslant L < +\infty,\ t \in I(t_0+1).$$

If ther perturbation $f(t,x)$ *in* (P) *satisfies the inequality*

$$|f(t,x)| \leqslant (d + c(t))\,|x| \qquad d \geqslant 0,$$

where $c : I(a+1) \to E^+$ *is such that, for any* $t_0 \in I$,

$$\overline{\lim_{t\to+\infty}}\,(t-t_0)^{-1}\sum_{s=t_0+1}^{t} c(s)K(s) = M(t_0) \leqslant M < +\infty,\ t \in I(t_0+1)$$

and if $Lb^{-1}d + Mb^{-1} + \lg b < 0$, *then the origin is asymptotically stable for* (P).

Proof. By the same method as in Theorem 6 we get

$$|x(t)| \leqslant K(t_0)\,|x(t_0)|\exp(t-t_0)\left[b^{-1}d\,(t-t_0)^{-1}\sum_{s=t_0+1}^{t}K(s) + \right.$$

(26)

$$\left. + b^{-1}(t-t_0)^{-1}\sum_{s=t_0+1}^{t}c(s)K(s) + \lg b\right].$$

Let L_1 and M_1 be such that $L_1 > L$, $M_1 > M$ and $-q = L_1 b^{-1}d + M_1 b^{-1} + \lg b < 0$. Then, under our hypotheses, there exists a positive integer t_1 such that,

(27) $\quad |x(t)| \leqslant K(t_0)\,|x(t_0)|\exp(-q(t-t_0))$ for all $t \in I(t_0+t_1)$,

which shows that $|x(t)| \to 0$ as $t \to +\infty$. Finally from (26) and (27) we get

$$|x(t)| \leqslant Q(t_0)\,|x(t_0)|, \qquad t \in I(t_0),$$

where $Q(t_0)$ is a suitable positive constant, which shows the stability of the origin for (P). This completes the proof.

6. Strong stability. In this section we shall prove that, if the origin is strongly stable for (L), the same is true for (P), if $f(t,x)$ satisfies convenient hypotheses.

Theorem 8. *Assume that the origin is strongly stable for* (L) *and let* $K \geqslant 1$ *be the constant of* Lemma 3. *Then, if the perturbation* $f(t,x)$ *in* (P) *is such that*

$$|f(t,x) - f(t,y)| \leqslant b(t)\,|x-y|,\ \sum_{t=a+1}^{\infty} b(t) < +\infty,\ 0 \leqslant b(t) < K^{-2},$$

the origin is strongly stable for (P).

Proof. Firstly we shall prove that, for any $t_0 \in I$ and $z_0 \in E$, (P) has at least one solution $z(t)$ with $z(t_0) = z_0$. To this end it is sufficient to show

that we have $Y(t) = E$, $t \in I$, where $Y(a) = E$, $Y(t) = \{A(t)x + f(t,x) : x \in Y(t-1)\}$, $t \in I(a+1)$. Assume $Y(s) = E$, $s = a, a+1, \dots, t-1$ and let $y \in E$. The equation

$$x = T_y(x) \quad \text{where} \quad T_y(x) = A^{-1}(t)\,y - A^{-1}(t)f(t,x)$$

has, for each $y \in E$, a unique fixed point since $T_y(x)$ is a contraction with Lipschitz constant less than 1. Therefore $Y(t) \equiv E$. We can now show that the origin is strongly stable for (P). For this purpose we establish an useful inequality. Since $X(t) = A(t)X(t-1)$, we have

$$X^{-1}(t)\,z(t) = X^{-1}(t-1)\,z(t-1) + X^{-1}(t)f(t,z(t-1)), \qquad t \in I(a+1).$$

Setting $v(s) = X^{-1}(s)\,z(s)$ we get

$$\big|\,|v(t)| - |v(t-1)|\,\big| \leqslant |v(t) - v(t-1)| \leqslant K^2 b(t)\,|v(t-1)|.$$

Therefore

$$(1 - K^2 b(t))\,|v(t-1)| \leqslant |v(t)| \leqslant (1 + K^2 b(t))\,|v(t-1)|, \qquad t \in I(a+1).$$

Take $t_1 \in I$, $t_2 \in I(t_1)$. Then

$$|v(t_1)|\prod_{s=t_1+1}^{t_2}(1 - K^2 b(s)) \leqslant |v(t_2)| \leqslant v(t_1)\prod_{s=t_1+1}^{t_2}(1 + K^2 b(s)),\ t_2 \in I(t_1),$$

and, setting $m_1 = \prod_{s=a+1}^{\infty}(1 - K^2 b(s))$, $m_2 = \prod_{s=a+1}^{\infty}(1 + K^2 b(s))$, we have

(28) $\qquad m_1\,|v(t_1)| \leqslant |v(t_2)| \leqslant m_2\,|v(t_1)|, \qquad t_2 \in I(t_1),$

where $0 < m_1 \leqslant m_2 < +\infty$. Then, taking $N \geqslant 1$ and sufficiently large, we have from (28).

$$N^{-1}\,|v(t_1)| \leqslant |v(t_2)| \leqslant N\,|v(t_1)|, \qquad t_2 \in I(t_1).$$

Therefore for any $t_0 \in I$ we have

(29) $\qquad |v(t)| \leqslant N\,|v(t_0)| \quad$ for all $t \in I$.

Using this inequality we show that the origin is strongly stable for (P). Let $t_0 \in I$, $z_0 \in E$, and $z(t)$ be any solution of (P) with $z(t_0) = z_0$, whose existence has been proved in the first part of the proof. Then, using (29), we have

$$|z(t)| \leqslant \|X(t)\|\,|v(t)| \leqslant KN\,|v(t_0)| \leqslant K^2N\,|z(t_0)|,\ t \in I.$$

Since this inequality holds for any t_0, $t \in I$, the origin is strongly stable for (P). This concludes the proof.

When the perturbation $f(t,x)$ in (P) is linear in x, from Theorem 8 we have the following

Corollary 7. Let the origin be strongly stable for (L) and let $K \geqslant 1$ be the constant of Lemma 3. Then, if the perturbation $f(t, x)$ in (P) is such that

$$f(t, x) = B(t)\, x, \qquad B(t) \in M, \ \sum_{s=a+1}^{\infty} \| B(s) \| < +\infty, \ 0 \leqslant \| B(s) \| < K^{-2},$$

the origin is strongly stable for (P).

Example 7. The origin is strongly stable for (L), where

$$A(t) = \begin{pmatrix} 0 & 1 + 2^{-t} \\ 1 + 2^{1-t} & 0 \end{pmatrix},$$

because $\| X(t) \| < e^2$, $\| X^{-1}(t) \| \leqslant 1$.

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ASUPRA STABILITĂŢII ECUAŢIILOR CU DIFERENŢE ÎN SPAŢII BANACH

Rezumat

În lucrare sînt considerate ecuaţii cu diferenţe de forma $x(t) = f(t, x(t-1))$. Sînt definite toate tipurile de stabilitate (simplă, uniformă, asimptotică, uniform asimptotică şi forte) analoge celor din teoria ecuaţiilor diferenţiale şi se demonstrează o lemă de tip Gronwall. Utilizîndu-se operatorul $X(t) = A(t)\, A(t-1) \ldots A(a+1)\, J$ (care este analogul metricii fundamentale pentru un sistem de ecuaţii diferenţiale liniare) se stabilesc condiţii necesare şi suficiente (sau numai suficiente) de stabilitate a sistemelor liniare cu diferenţe în sensul definiţiilor introduse. Teoreme asemănătoare sînt demonstrate pentru ecuaţii cu diferenţe, perturbate cu perturbaţii din anumite clase.

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THE CONSISTENCY PROBLEM OF SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS(*)

BY
PATRICK HABETS

1. Introduction. Consider the system

$$\begin{aligned} \dot{x} &= f(t, x, y, \varepsilon) \\ \varepsilon \dot{y} &= g(t, x, y, \varepsilon), \end{aligned} \tag{1}$$

where x and y are respectively n and m dimensional real vectors, and ε is some small positive constant. It is natural to investigate the solutions of this equation with the help of the reduced system

$$\begin{aligned} \dot{x} &= f(t, x, y, 0) \\ 0 &= g(t, x, y, 0). \end{aligned} \tag{2}$$

One is interested in suitable conditions which ensure that a given solution (x^0, y^0) of (2) is in some sense, the limit of solutions of (1) as $\varepsilon \to 0$. This will be called the consistency of (x^0, y^0) with respect to system (1) and expresses the idea that (x^0, y^0) is a good approximation of some solutions of (1). Such concepts have been introduced and investigated in a fairly systematic way in P. Habets [7].

The first important study of this problem has been made by A. N. Tihonov [14], [15], [16]. This author proves the continuity with respect to ε, using some stability assumptions. F. Hoppensteadt [9] has generalized these results using the Lyapunov functions which are deduced from the stability conditions. Notice that the hypothesis of these theorems imply that given (x^0, y^0), any neighbouring solution of (1) tends to (x^0, y^0) as $\varepsilon \to 0$.

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