Corollary 7. Let the origin be strongly stable for (L) and let $K \geqslant 1$ be the constant of Lemma 3. Then, if the perturbation $f(t, x)$ in (P) is such that

$$f(t, x) = B(t)\,x, \qquad B(t) \in M, \ \sum_{s=a+1}^{\infty} \| B(s) \| < +\infty, \ 0 \leqslant \| B(s) \| < K^{-2},$$

the origin is strongly stable for (P).

Example 7. The origin is strongly stable for (L), where

$$A(t) = \begin{pmatrix} 0 & 1 + 2^{-t} \\ 1 + 2^{1-t} & 0 \end{pmatrix},$$

because $\| X(t) \| < e^2$, $\| X^{-1}(t) \| \leqslant 1$.

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ASUPRA STABILITĂȚII ECUAȚIILOR CU DIFERENȚE ÎN SPAȚII BANACH

Rezumat

În lucrare sînt considerate ecuații cu diferențe de forma $x(t) = f(t, x(t-1))$. Sînt definite toate tipurile de stabilitate (simplă, uniformă, asimptotică, uniform asimptotică și forte) analoge celor din teoria ecuațiilor diferențiale și se demonstrează o lemă de tip Gronwall. Utilizîndu-se operatorul $X(t) = A(t)\,A(t-1) \ldots A(a+1)\,J$ (care este analogul metricii fundamentale pentru un sistem de ecuații diferențiale liniare) se stabilesc condiții necesare și suficiente (sau numai suficiente) de stabilitate a sistemelor liniare cu diferențe în sensul definițiilor introduse. Teoreme asemănătoare sînt demonstrate pentru ecuații cu diferențe, perturbate cu perturbații din anumite clase.

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THE CONSISTENCY PROBLEM OF SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS(*)

BY

PATRICK HABETS

1. Introduction. Consider the system

$$\begin{aligned} \dot{x} &= f(t, x, y, \varepsilon) \\ \varepsilon \dot{y} &= g(t, x, y, \varepsilon), \end{aligned} \tag{1}$$

where x and y are respectively n and m dimensional real vectors, and ε is some small positive constant. It is natural to investigate the solutions of this equation with the help of the reduced system

$$\begin{aligned} \dot{x} &= f(t, x, y, 0) \\ 0 &= g(t, x, y, 0). \end{aligned} \tag{2}$$

One is interested in suitable conditions which ensure that a given solution (x^0, y^0) of (2) is in some sense, the limit of solutions of (1) as $\varepsilon \to 0$. This will be called the consistency of (x^0, y^0) with respect to system (1) and expresses the idea that (x^0, y^0) is a good approximation of some solutions of (1). Such concepts have been introduced and investigated in a fairly systematic way in P. Habets [7].

The first important study of this problem has been made by A. N. Tihonov [14], [15], [16]. This author proves the continuity with respect to ε, using some stability assumptions. F. Hoppensteadt [9] has generalized these results using the Lyapunov functions which are deduced from the stability conditions. Notice that the hypothesis of these theorems imply that given (x^0, y^0), any neighbouring solution of (1) tends to (x^0, y^0) as $\varepsilon \to 0$.

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J. J. Levin and N. Levinson [12] investigated the same problem using the properties of the linear approximation of system (1) around the given solution (x^0, y^0) of (2). These results were extended by J. J. Levin [11], V. F. Butuzov [1], [2], and using exponential dichotomies by K. W. Chang [4] and K. W. Chang and W. A. Coppel [5]. A fundamental difference with the Tihonov-Hoppenstead's results is that in most of these theorems, the assumptions imply that, given a solution (x^0, y^0) of (2), only some neighbouring solutions of (1) tend to (x^0, y^0) as $\varepsilon \to 0$. This is sufficient to justify the approximation of system (1) by the reduced system (2). Moreover it enlarges considerably the class of admissible systems (1).

Our purpose in this paper is to generalize the results recalled hereabove, and to extend them to the various consistency definitions introduced in P. Habets [7]. The point of view adopted is similar to the Tihonov-Hoppenstead's one but instead of assuming some stability conditions, we shall suppose the existence of some Lyapunov functions implied by these conditions. In section 3, we introduce general theorems that confine some solutions of (1) in a given domain of $\mathbf{R}^{n+m}$. From this result, we deduce rightaway in section 4, some consistency theorems. On the other hand, considering separately the motions of x and y, we prove in section 5 some extensions of Tihonov-Hoppenstead's theorems.

2. Notations-definitions. Let I denote a finite or infinite interval on the real line $\mathbf{R}$, Ω a domain of $\mathbf{R}^{n+m}$ containing the origin, and E a positive real number. Consider the functions

$$f : I \times \Omega \times [0, E] \to \mathbf{R}^n, \ (t, z, \varepsilon) \to f(t, z, \varepsilon),$$

$$g : I \times \Omega \times [0, E] \to \mathbf{R}^m, \ (t, z, \varepsilon) \to g(t, z, \varepsilon)$$

and suppose $f(t, 0, 0) = 0$, $g(t, 0, 0) = 0$. In the following, we shall write $z = (x, y)$, where $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ and to shorten the notations, we note indifferently $F(z)$ or $F(x, y)$. The system of equations

$$\dot{x} = f(t, z, \varepsilon), \qquad \varepsilon \dot{y} = g(t, z, \varepsilon), \tag{1}$$

where $\dot{} = d/dt$ and ε is different from zero is called the *complete system*. The associate system

$$\dot{x} = f(t, z, 0), \qquad 0 = g(t, z, 0), \tag{2}$$

obtained by setting $\varepsilon = 0$ in (1) is called the *reduced system* and is verified by the application $z^0 : I \to \Omega$, $t \to 0$.

The general solution of (1) considered as a function of t, of its initial value (t_0, z_0) and of ε will be written

$$z : \mathcal{H} \to \mathbf{R}^{n+m}, \ (t, t_0, z_0, \varepsilon) \to z(t; t_0, z_0, \varepsilon).$$

The points of $\mathcal{H}$ are obtained by choosing first a point $(t_0, z_0, \varepsilon) \in I \times \Omega \times [0, E]$ and then $t \in J(t_0, z_0, \varepsilon)$, where $J(t_0, z_0, \varepsilon)$ is the interval of definition of the maximal solution through (t_0, z_0) and for ε. In the sequel, we shall suppose such a solution exists, is unique and depends continuously on the initial conditions (t_0, z_0). Further, when there will be no possible confusion, we shall write $z^\varepsilon(t)$ and J for $z(t; t_0, z_0, \varepsilon)$ and $J(t_0, z_0, \varepsilon)$.

It will be convenient to use the norm

$\|z\| = \max(\|x\|, \|y\|)$ where $z = (x, y)$ and $\|x\|$, $\|y\|$ are respectively the norms of $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$. If η is a real positive number we write

$$B(0, \eta) = \{z \in \mathbf{R}^{n+m}, \|z\| < \eta\}.$$

For any set A, $\bar{A}$ will be its closure and $\operatorname{fr} A$ its boundary.

In the sequel, we shall use several auxiliary functions. Considering a function $u : I \times B(0, \rho) \times]0, E] \to \mathbf{R}$, we define its Dini derivatives

$$D_+ u = \liminf_{h \to 0+} \frac{1}{h} \left[u\left(t + h, x + hf(t, z, \varepsilon), y + \frac{h}{\varepsilon} g(t, z, \varepsilon), \varepsilon\right) - u(t, z, \varepsilon)\right]$$

$$D_- u = \liminf_{h \to 0-} \frac{1}{h} \left[u\left(t + h, x + hf(t, z, \varepsilon), y + \frac{h}{\varepsilon} g(t, z, \varepsilon), \varepsilon\right) - u(t, z, \varepsilon)\right]$$

computed along the solutions of (1). Further we shall use some functions $a : [0, \infty] \to \mathbf{R}$, $r \to a(r)$ continuous, strictly increasing and such that $a(0) = 0$. Such functions will be called *K-functions*.

We now introduce the following consistency concepts.

The origin is *consistent at* $t_0 \in I$ if

$$(\forall \eta > 0)(\exists \varepsilon_0 > 0)(\forall \varepsilon : 0 < \varepsilon \leq \varepsilon_0)(\exists z_0 \in \Omega)(\forall t \geq t_0, t \in J)\|z^\varepsilon(t)\| < \eta.$$

The origin is *consistent* if for any $t_0 \in I$, the origin is consistent at t_0.

The origin is *uniformly consistent* if

$$(\forall \eta > 0)(\exists \varepsilon_0 > 0)(\forall \varepsilon : 0 < \varepsilon \leq \varepsilon_0)(\forall t_0 \in I)(\exists z_0^\varepsilon \in \Omega)(\forall t \geq t_0, t \in J)\|z^\varepsilon(t)\| < \eta.$$

The origin is *completely consistent at* t_0 if

$$(\forall \eta > 0)(\exists \varepsilon_0 > 0)(\exists \delta > 0)(\forall \varepsilon : 0 < \varepsilon \leq \varepsilon_0)(\forall z_0^\varepsilon \in B(0, \delta))(\forall t \geq t_0, t \in J)\|z^\varepsilon(t)\| < \eta.$$

The origin is *completely consistent* if for any $t_0 \in I$, it is completely consistent at t_0.

The origin is *uniformly completely consistent* if

$$(\forall \eta > 0)(\exists \varepsilon_0 > 0)(\exists \delta > 0)(\forall \varepsilon : 0 < \varepsilon \leq \varepsilon_0)(\forall t_0 \in I)(\forall z_0^\varepsilon \in B(0, \delta))$$
$$(\forall t \geq t_0, t \in J)\|z^\varepsilon(t)\| < \eta.$$

These difinitions have been introduced and their properties investigated in P. Habets [7].

3. Preliminary theorems. Consider an open set $T \subset I \times \Omega$ and $E_0 \in]0, E]$.

A point $(t_0, z_0) \in \mathrm{fr}\, T \cap (I \times \Omega)$ is an *egress point* of T with respect to (1), if the solution $z(t; t_0, z_0, \varepsilon)$ is such that $(\exists \tau > 0)(\forall t \in [t_0 - \tau, t_0[)$, $(t, z(t)) \in T$.

A mapping $\pi : S_1 \to S_2 \subset S_1$ is called a *retraction* from S_1 into S_2 if it is continuous and the restriction of π to S_2 is the identity. Let us introduce the following notations:

$$S = \{(t, z) \in \mathrm{fr}\, T : (t, z) \text{ is an egress point}\}$$

$$G = \{(t_0, z_0) \in T : \exists t_1 > t_0, (t_1, z(t_1; t_0, z_0, \varepsilon)) \in S\}$$

and define the mapping $\pi : G \cup S \to S$ by the relationships $\pi(t_0, z_0) = (t_0, z_0)$ if $(t_0, z_0) \in S$ and $\pi(t_0, z_0) = (t_1, z_1)$ where (t_1, z_1) is the first egress point of T along the trajectory $(t, z^\varepsilon(t))$ passing through (t_0, z_0) if $(t_0, z_0) \notin S$. We now state a slight modification of a result due to T. Ważewski [17].

Proposition 1. *Let $u(t, z)$ and $v(t, z)$ be real continuous functions, defined on $I \times \Omega$ and locally lipschitzian with respect to z. Consider, for some $r > 0$, the sets*

$$T = \{(t, z) : u(t, z) < r,\ v(t, z) < r\},$$

$$U = \{(t, z) : u(t, z) = r,\ v(t, z) \leqslant r\},$$

$$V = \{(t, z) : u(t, z) \leqslant r,\ v(t, z) = r\}.$$

and suppose

$$D_+ u(t, z) > 0,\quad (t, z) \in U.$$

$$D_- v(t, z) < 0,\quad (t, z) \in V.$$

Then π is a retraction from $G \cup S$ into $S = U \setminus V$, and therefore is continuous. In case u and v are of class C^1, a proof of this proposition appears in P. Hartman [8].

The results of this section will be based on the following assumptions. Let $t_0 \in I$, $r > 0$ and $E_0 > 0$ be fixed.

Assumptions A_1. *Let Q be a $n + m$ projection matrix and $u(t, z, \varepsilon)$, $v(t, z, \varepsilon)$ some real continuous functions defined on $I \times \Omega \times]0, E_0]$, locally lipschitzian with respect to z. Suppose that for any ε, T, U and V are defined as in proposition 1, and:*

Ia. $u(t, 0, \varepsilon) \leqslant a(\varepsilon) + \alpha r,\ t \geqslant t_0,\ (t, \varepsilon) \in I \times]0, E_0]$;

b. $u(t, z, \varepsilon) = u(t, Qz, \varepsilon) \geqslant b(\| Qz \|)$

$t \geqslant t_0,\ (t, z, \varepsilon) \in I \times \Omega \times]0, E_0]$;

c. $D_+ u(t, z, \varepsilon) > 0,\ t \geqslant t_0,\ (t, z, \varepsilon) \in U \times]0, E_0]$;

IIa. $v(t_0, z, \varepsilon) \leqslant a(\varepsilon + \| z - Qz \|) + \alpha r$

$(z, \varepsilon) \in \Omega \times]0, E_0]$;

b. $D_- v(t, z, \varepsilon) < 0, t \geqslant t_0,\ (t, z, \varepsilon) \in V \times]0, E_0]$;

where a and b are K-functions such that $b^{-1}(r)$ exists and $\alpha \in [0, 1[$.

In the following we shall make use of

Lemma 1. (Non retraction theorem). *Let ω be an open bounded subset of a k-dimensional euclidean space E^k, and $f : \overline{\omega} \to E^k$ a continuous function such that $\forall z \in \mathrm{fr}\, \omega$, $f(z) = z$. Then $f(\omega) \supset \omega$.*

Proof. Let $z_1 \in \omega$, and consider the function $F : \overline{\omega} \times [0,1] \to E^k, (z, \lambda) \to F(z, \lambda)$ where $F(z, \lambda) = z_1 - z + \lambda(z - f(z))$. It is obvious that if $z \in \mathrm{fr}\, \omega$, $F(z, \lambda) = z_1 - z \neq 0$, and the topological degree [13] $d(F(z, \lambda), \omega, 0)$ can be defined. Using the invariance principle with respect to homotopies $d(z_1 - f(z), \omega, 0) = d(F(z, 1), \omega, 0) = d(F(z, 0), \omega, 0) = d(z_1 - z, \omega, 0) = -1$. Thus, there exists $z \in \omega : f(z) = z_1$. Q.E.D.

Now, we are ready to state and prove the main results of this section. Theorem 1 will be helpful in the study of consistency.

Theorem 1. *Suppose $t_0 \in I$, $r > 0$ and $E_0 > 0$ are such that assumptions A_1 are satisfied. Then*

$$(\exists \varepsilon_0 > 0)(\exists \delta > 0)(\forall \varepsilon : 0 < \varepsilon \leqslant \varepsilon_0)(\forall z^* \in B(0, \delta) : Qz^* = 0)$$
$$(\exists z_0^\varepsilon = Qz_0^\varepsilon + z^*)(\forall t \geqslant t_0, t \in J) \quad (t, z^\varepsilon(t)) \in T.$$

Proof. Let us choose $\varepsilon_0 > 0$ and $\delta > 0$ such that $a(\varepsilon_0 + \delta) < (1 - \alpha) r$. Using assumptions A_1—Ia and IIa, it follows $(\forall \varepsilon : 0 < \varepsilon \leqslant \varepsilon_0)(\forall z^* \in B(0, \delta) : Qz^* = 0)(\forall z : Qz + z^* \in \Omega)$ $(\forall t \geqslant t_0, t \in I)$

(i) $\qquad u(t, z^*, \varepsilon) = u(t, 0, \varepsilon) < r$

(ii) $\qquad v(t_0, Qz + z^*, \varepsilon) \leqslant a(\varepsilon + \| z^* \|) + \alpha r < r.$

Let ε and z^* be fixed and suppose ab absurdo $(\forall z_0^\varepsilon = Qz_0^\varepsilon + z^*)(\exists t \geqslant t_0, t \in J)(t, z^\varepsilon(t)) \notin T$. It follows that the application π introduced in proposition 1 is defined at least on $\{t_0\} \times \overline{\omega}$, where $\omega = \{z : z = Qz + z^*, u(t_0, z, \varepsilon) < r\}$, since by (ii), $\forall z \in \omega$, $v(t_0, z, \varepsilon) \leqslant a(\varepsilon + \| z^* \|) + \alpha r < r$. Further ω is an open set with respect to the k-dimensional euclidian space $E^k = \{z \in R^{n+m} : z = Qz + z^*\}$. Let $\pi_t(t, z)$ and $\pi_z(t, z)$ be respectively the time and spatial coordinates of $\pi(t, z)$ and define the continuous mapping

$$f : \overline{\omega} \to E^k,\ z \to Q\pi_z(t_0, z) + z^*$$

We can deduce from (i) that $z^* \in \omega \neq \emptyset$ and from A_1 — Ib that ω is bounded. Further from the definition of ω and (ii), $(\forall z_0 \in \mathrm{fr}\, \omega)$, $(t_0, z_0) \in U \setminus V$.

Therefore $\pi_z(t_0, z_0) = z_0$, $Qz_0 + z^* = z_0$ and $f(z_0) = z_0$. It follows then from lemma 1 that $f(\omega) \supset \omega$. In particular $z^* \in f(\omega)$. That is $(\exists z_0 \in \omega)$, $f(z_0) = Q\pi_z(t_0, z_0) + z^* = z^*$. This implies $Q\pi_z(t_0, z_0) = 0$, $u(\pi_t(t_0, z_0), \pi_z(t_0, z_0)\varepsilon) = u(\pi_t(t_0, z_0), 0, \varepsilon) < r$ and $\pi(t_0, z_0) \notin U \setminus V$ which is impossible. Q. E. D.

As we already pointed out, the proof of this theorem uses techniques very similar to Wazewski's method [17]. Further, let us notice that similar results have been proved by J. A. Yorke [18].

Uniform consistency can be investigated with the help of

Theorem 2. *Let $r > 0$, $E_0 > 0$ be such that for any $t_0 \in J$, assumptions A_1 are satisfied, a, b and α being independent of t_0 then*

$$(\exists \varepsilon_0 > 0)\,(\exists \delta > 0)\,(\forall \varepsilon : 0 < \varepsilon \leqslant \varepsilon_0)\,(\forall t_0 \in I)\,(\forall z^* \in B(0, \delta) : Qz^* = 0)$$
$$(\exists z_0^\varepsilon = Qz_0^\varepsilon + z^*)\,(\forall t \geqslant t_0,\ t \in J)\,(t, z^\varepsilon(t)) \in T.$$

The proof of this theorem is a repetition of the proof of theorem 1, and therefore will be omitted.

4. Consistency theorems. The theorems of section 3 prove that some solutions are confined to a set $T = \{(t, z) : u < r,\ v < r\}$ for any $t \geqslant t_0$, $t \in J$. If this is true for any $r : 0 < r \leqslant r_0$ and if moreover the function $w = \max(u, v)$ is positive definite, then given any ball $B(0, \eta)$ there exists an $r > 0$ such that $T \subset I \times B(0, \eta)$ and thus there exist some solutions confined to $B(0, \eta)$ for any $t \geqslant t_0$, $t \in J$. This argument leads to.

Theorem 3. *Let $t_0 \in I$, $r_0 > 0$ be fixed and suppose there exists a K-function c such that:*

1° $(\forall r : 0 < r \leqslant r_0)\,(\exists E_0 : 0 < E_0 \leqslant E)$ the assumptions A_1 are satisfied;

2° $w(t, z, \varepsilon) = \max(u, v) \geqslant c(\| z \|)$.

Then $(\forall \eta > 0)\,(\exists \varepsilon_0 > 0)\ (\forall \varepsilon : 0 < \varepsilon \leqslant \varepsilon_0)\,(\exists z_0^\varepsilon \in \Omega)\,(\forall t \geqslant t_0,\ t \in J)$ $z(t; t_0, z_0^\varepsilon, \varepsilon) \in B(0, \eta)$ i. e. *the origin is consistent at $t_0 \in I$.*

Proof. Let η be given and choose $r < \min[c(\eta), r_0]$. By assumption, there exists $E_0 > 0$ such that assumptions A_1 are satisfied. Using theorem 1 we obtain then

$$(\exists \varepsilon_0 > 0)\,(\exists \delta > 0)\,(\forall \varepsilon : 0 < \varepsilon \leqslant \varepsilon_0)\,(\forall z^* \in B(0, \delta) : Qz^* = 0)$$
$$(\exists z_0^\varepsilon = Qz_0^\varepsilon + z^*)\,(\forall t \geqslant t_0,\ t \in J)\,(t, z^\varepsilon(t)) \in T.$$

This implies

$$c(\| z^\varepsilon(t) \|) \leqslant w(t, z^\varepsilon(t), \varepsilon) \leqslant r < c(\eta)$$

and since c is a K-function, $\| z^\varepsilon(t) \| < \eta$. The theorem follows by choosing $z^* = 0$. Q. E. D.

Using the same argument, we obtain as follows a slight modification of theorem 1 of [7].

Corollary. *Suppose $t_0 \in I$ and $E_0 > 0$ are fixed. Let $v(t, z, \varepsilon)$ be some real continuous function defined on $I \times B(0, \rho) \times]0, E_0]$ locally lipschitzian with respect to z and such that:*

$$a : v(t_0, z, \varepsilon) \leqslant a(\varepsilon + \| z \|),\ (z, \varepsilon) \in \Omega \times]0, E_0];$$

$$b : (\forall r > 0)\ D\_v(t, z, \varepsilon) < 0,\ t \geqslant t_0,\ (t, z, \varepsilon) \in V \times]0, E_0];$$

$$c : v(t, z, \varepsilon) \geqslant c(\| z \|),\ t \geqslant t_0,\ (t, z, \varepsilon) \in \Omega \times]0, E_0];$$

where a and c are K-functions. Then the origin is completely consistent at t_0.

Proof. Assumption A_1 are satisfied for $Q \equiv 0$. It follows as above

$$(\forall \eta > 0)\,(\exists \varepsilon_0 > 0)\,(\exists \delta > 0)\,(\forall \varepsilon : 0 < \varepsilon \leqslant \varepsilon_0)\,(\forall z^* \in B(0, \delta) : Qz^* = 0)$$
$$(\exists z_0^\varepsilon = Qz_0^\varepsilon + z^*)\ (\forall t \geqslant t_0,\ t \in J)\,\| z^\varepsilon(t) \| < \eta.$$

This proves the proposition since $Q \equiv 0$. Q. E. D.

Similarly, we could deduce an analogous modification of theorem 3 of [7], by imposing that Q be the identity matrix. In this sense theorem 3 gives the link between the simple but opposite situations described in [7].

A uniform consistency theorem is obtained in a similar way from theorem 2.

Theorem 4. *Let $r_0 > 0$ be fixed and suppose there exists a K-function c such that:*

1° $(\forall r : 0 < r \leqslant r_0)\,(\exists E_0 : 0 < E_0 \leqslant E)\ (\forall t_0 \in I)$ the assumptions A_1 are satisfied, a, b and α being independent of t_0;

2° $w(t, z, \varepsilon) = \max(u, v) \geqslant c \| z \|$.

Then the origin is uniformly consistent.

5. The consistency by separation of slow and fast variables. The consistency theorems given so far make use of auxiliary functions u, v whose Dini derivatives must be computed along the solutions of the complete system (1). To find such functions is often difficult because of their awkward dependence in ε. The aim of this section is to simplify the construction of the u, v using the idea that the system can be described by a successive juxtaposition of slow and fast motions. If $g(t, z, \varepsilon) \simeq 0$, the complete system can be approximated by the reduced system

$$\begin{aligned} \dot{x} &= f(t, z, 0), \\ 0 &= g(t, z, 0). \end{aligned} \tag{2}$$

If $\| g(t, z, \varepsilon) \| \simeq \| f(t, z, \varepsilon) \|$, the factor ε is such that $\| y \|$ varies much faster than $\| x \|$. Using the substitution $t = t_0 + \varepsilon s$, the motion is then described by

$$\frac{dx}{ds} = \varepsilon f(t_0 + \varepsilon s, z, \varepsilon),$$

$$\frac{dy}{ds} = g(t_0 + \varepsilon s, z, \varepsilon).$$

It follows that for ε small, we can approximate this motion by the solutions of equation

$$\frac{dy}{ds} = g(t_0, x_0, y, 0), \tag{3}$$

which is called the *boundary layer equation*. The actual motion being a fast change in the y-coordinate only, we shall call y the *fast variables*. By opposition the motion of x, described by (2), is slow and x will be called the *slow variables*. The idea in the following is to build up auxiliary functions u and v for the theorems of section 4, using functions u_r and v_r whose derivatives are computed along the solutions of the reduced system (2), and functions u_b and v_b whose derivatives are computed along the solutions of the boundary layer equation (3). To ease the computation of these derivatives we shall assume in this section that *for every* $(t, x, 0) \in I \times \Omega$

$$g(t, x, 0, 0) = 0$$

and for every $y \neq 0$ *such that* $(t, x, y) \in I \times \Omega$

$$g(t, x, y, 0) \neq 0.$$

It can be shown (F. C. Hoppenstead t [9]) that using changes of variables, these assumptions are satisfied for a large class of systems. Further some additional assumptions on g imply that these changes of variables preserve the consistency properties. The reduced system can then be written

$$\begin{aligned} \dot{x} &= f(t, x, 0, 0) \\ y &= 0, \end{aligned} \tag{4}$$

Considering an auxiliary function $u_r : (t, x) \to u_r(t, x)$ we define its Dini derivatives

$$D^0_+ u_r(t, x) = \liminf_{h \to 0+} \frac{1}{h} [u_r(t + h, x + hf(t, x, 0, 0)) - u_r(t, x)]$$

$$D^0_- u_r(t, x) = \liminf_{h \to 0-} \frac{1}{h} [u_r(t+h, x+hf(t, x, 0, 0)) - u_r(t, x)].$$

Similarly for a function $u_b : (t, x, y) \to u_b(t, x, y)$ we define the Dini derivatives

$$D^*_+ u_b(t, x, u) = \liminf_{h \to 0+} \frac{1}{h} [u_b(t, x, y + hg(t, x, y, 0)) - u_b(t, x, y)]$$

$$D^*_- u_b(t, x, y) = \liminf_{h \to 0-} \frac{1}{h} [u_b(t, x, y + hg(t, x, y, 0)) - u_b(t, x, y)].$$

In the sequel, we shall use the following assumptions written for $t_0 \in I$, $r_r > 0$ and $r_b > 0$ fixed.

Assumption A_2. *Suppose* $\Omega = \Omega_x \times \Omega_y$, $\Omega_x \subset \mathbb{R}^n$ *and* $\Omega_y \subset \mathbb{R}^m$.

Let Q_r *be a* n*-projection matrix and* $u_r(t, x)$, $v_r(t, x)$ *some real continuous functions defined on* $I \times \Omega_x$, *uniformly lipschitzian in* x (i. e. *there exists* $K_r > 0$ *such that*

$| u_r(t, x_1) - u_r(t, x_2) | \leqslant K_r \| x_1 - x_2 \|$ *and*

$| v_r(t, x_1) - v_r(t, x_2) | \leqslant K_r \| x_1 - x_2 \|$). *Suppose that for any* $(t \geqslant t_0, t \in I)$

I a. $u_r(t, 0) \leqslant \alpha r_r$;

b. $u_r(t, x) = u_r(t, Q_r x) \geqslant b_r(\| Q_r x \|)$, $x \in \Omega_x$;

c. $D^0_+ u_r(t, x) \geqslant d_r(r_r)$, $(t, x) \in \{(t, x) : u_r = r_r, v_r \leqslant r_r\}$;

II a. $v_r(t_0, x) \leqslant a_r(\| x - Q_r x \|) + \alpha r_r$, $x \in \Omega_x$;

b. $D^0_- v_r(t, x) \leqslant - d_r(r_r)$, $(t, x) \in \{(t, x) : u_r \leqslant r_r, v_r = r_r\}$:

where a_r, b_r *and* d_r *are* K*-functions such that* $b_r^{-1}(r_r)$ *exists, and* $\alpha \in [0, 1[$.

Let further Q_b *be a* m*-projection matrix and* $u_b(t, z)$, $v_b(t, z)$ *some real lipschitzian functions* (i. e. *there exists* $K_b > 0$ *such that* $| u_b(t_1, z_1) - u_b(t_2, z_2) | \leqslant K_b (| t_1 - t_2 | + \| z_1 - z_2 \|)$, $| v_b(t_1, z_1) - v_b(t_2, z_2) | \leqslant K_b (| t_1 - t_2 | + \| z_1 - z_2 \|)$) *defined on* $I \times \Omega$. *Suppose that for any* $(t \geqslant t_0, t \in I)$

III a. $u_b(t, 0) \leqslant \alpha r_b$;

b. $u_b(t, z) = u_b(t, Q_r x, Q_b y) \geqslant b_b(\| Q_b y \|)$, $z \in \Omega$;

c. $D^*_+ u_b(t, z) \geqslant d_k(r_b)$, $(t, z) \in \{(t, z) : u_b = r_b, v_b \leqslant r_b\}$:

IV a. $v_b(t_0, x, y) \leqslant a_b(\|(x, y) - (Q_r x, Q_b y)\|) + \alpha r_b$, $z \in \Omega$:

b. $v_b(t, z) \geqslant b_b(\| y - Q_b y \|)$, $z \in \Omega$;

c. $D^*_- v_b(t, z) \leqslant - d_b(r_b)$

$$(t, z) \in \{(t, z) : u_b \leqslant r_b, v_b = r_b\}$$

where a_b, b_b *and* d_b *are* K*-functions such that* $b_b^{-1}(r_b)$ *exists.*

Comparing assumptions A_1 and A_2, one sees rightaway that the latter are, with minor modifications, the juxtaposition of assumptions A_1 for the reduced system (2) and for the boundary layer equation (3). Since ε does not appear in these equations and since they are of lower order than (1), we can expect to find more easily the auxiliary functions u, v in A_2 than in A_1. We still need some regularity conditions

Assumptions B_2. *The function* f *is bounded on* $I \times \Omega \times [0, E_0]$ *so that*

$$\| f(t, z, \varepsilon) \| < K_1, \; K_1 \in \mathbb{R}.$$

Moreover the function f is continuous at $\varepsilon = 0$, $y = 0$ uniformly in $(t \geq t_0$, $t \in I)$ and $x \in \Omega_r$ i. e.

$$\| f(t, x, y, \varepsilon) - f(t, x, 0, 0) \| \leq \omega_1 (\| y \| + \varepsilon),$$

where ω_1 is a K-function.

The function g is continuous at $\varepsilon = 0$ uniformly with respect to $(t \geq t_0, t \in I)$ and $z \in \Omega$, i. e.

$$\| g(t, z, \varepsilon) - g(t, z, 0) \| \leq \omega_2(\varepsilon),$$

where ω_2 is a K-function

Lemma 2. *If E_0, r_r and r_b are choosen such that*

$$K_r \omega_1(E_0 + 2b_b^{-1}(r_b)) - d_r(r_r) < 0$$

$$K_b[(1 + K_1)E_0 + \omega_2(E_0)] - d_b(r_b) < 0,$$

then the assumptions A_2 and B_2 imply that the assumptions A_1 are satisfied for $r = r_r$

$$u = \max(u_r, r_r u_b / r_b),$$

$$v = \max(v_r, r_r v_b / r_b).$$

Except for minor modifications the proof of this lemma follows from the proof of theorem 6 in [7], and therefore will be omitted.

Theorem 5. *Let $r_0 > 0$ and $t_0 \in I$ be given and suppose there exists a K-function c such that:*

1° $(\forall r_r : 0 < r_r \leq r_0)\,(\forall r_b : 0 < r_b \leq r_0)$ *the assumptions A_2 and B_2 are satisfied K_r, K_b, b_b, d_b and d_r being independent of r_r and r_b;*

2° $w(t, z) = \max(u_r, u_b, v_r, H\, v_b) \geq c(\| z \|)$.

Then the origin is consistent at t_0.

Let us notice that in this theorem the functions u_r, u_b, v_r and v_b might be functions of r_r and r_b.

Proof. Let η be given and choose $(r_r : 0 < r_r < \min(c(\eta), r_0))$, $(r_b : 0 < r_b < \min(c(\eta), r_0))$ and $(E_0 : 0 < E_0 \leq E)$ such that

$$K_r \omega_1(E_0 + 2b_b^{-1}(r_b)) - d_r(r_r) < 0,$$

$$K_b[(1 + K_1)\, E_0 + \omega_2(E_0)] - d_b(r_b) < 0.$$

Notice that since we can choose r_b as small as desired, the function b_b need not be invertible on the whole semi axis $\mathbb{R}^+$. It follows then from lemma 2 and theorem 1 that,

$$(\exists \varepsilon_0 > 0)\,(\forall \varepsilon : 0 < \varepsilon \leq \varepsilon_0)\,(\exists z_0^\varepsilon \in \Omega)\,(\forall t \geq t_0,\; t \in J)\,(t, z^\varepsilon(t)) \in T$$

i. e. $\max(u_r, v_r) < r_r$ and $\max(u_b, v_b) < r_b$. Thus

$$c(\| z \|) \leq w(t, z) \leq \max(r_r, r_b) < c(\eta)$$

and since c is a K - function $\| z \| < \eta$. Q.E.D.

The same reasoning using theorem 2 instead of theorem 1 leads to

Theorem 6. *If the assumptions of theorem 5 are satisfied for any $t_0 \in I$, a_r, a_b, b_r, b_b and α being independent of t_0 the origin is uniformly consistent.*

Using the matrices $Q_r = 0$, $Q_b = 0$ we can deduce, as we did in the corollary of theorem 3, complete consistency theorems similar to theorem 6 in [7]. In that sense, theorems 5 and 6, extend the complete consistency results of A. N. Tihonov [16], F. Hoppenstead [9] and P. Habets [7]. We must notice that A. N. Tihonov and F. Hoppenstead do not suppose the existence of the auxiliary functions u and v. These authors impose stability conditions on the reduced system and the boundary layer equation. It appears clearly from F. Hoppenstead [9] that these conditions imply the assumptions A_2 with $Q = 0$. Further, we must notice that F. Hoppenstead [10] stated criterions for some of the stability conditions imposed in [16] and [9], without using auxiliary functions. However, it seems that more practical criterions will involve the auxiliary functions we introduced in assumptions A_2. These remarks justify, if needed, the point of view we adopted in this paper.

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PROBLEMA CONSISTENȚEI ECUAȚIILOR DIFERENȚIALE PERTURBATE SINGULAR

Rezumat

Se consideră sistemul (1) $\dot{x} = f(t, x, y, \varepsilon)$, $\varepsilon\dot{y} = g(t, x, y, \varepsilon)$ unde x și y sînt vectori n, respectiv m, dimensionali iar ε un parametru pozitiv constant, studiindu-se soluțiile acestuia cu ajutorul sistemului redus (2) $\dot{x} = f(t, x, y, 0)$, $0 = g(t, x, y, 0)$.

Se dau soluțiile care asigură că o soluție dată (x^0, y^0) a sistemului (2) este limita soluțiilor (sau numai a unei părți de soluții) ale sistemului (1) cu $\varepsilon \to 0$.

Această proprietate este numită consistență a lui $(\dot{x}, \dot{y})$ cu privire la sistemul (1), (x^0, y^0) fiind deci o aproximare a unor soluții ale sistemului (1).

În această Notă sînt generalizate unele rezultate ale altor autori (Tihonov, Hoppensteadt, Butuzov, Chang, Coppel, Levin și Levinson), extinzîndu-se în același timp la diferite tipuri de consistență introduse de autor într-o lucrare anterioară. În locul obișnuitelor condiții de stabilitate utilizate de Tihonov și Hoppensteadt se presupune existența funcțiilor Liapunov, satisfăcînd condiții corespunzătoare.

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ON THE EXISTENCE OF CONVERGENT SOLUTIONS IN THE THEORY OF VOLTERRA INTEGRO-FUNCTIONAL EQUATIONS

BY

GH. BANTAȘ

1. This paper contains some results regarding the existence of the solutions of certain Volterra integro-functional equations in the space $\mathcal{L}$ defined below. The technique used is based on the Massera-Schäffer-Corduneanu method ([11], [3]—[6]), on some fixed point theorems (Schauder-Tychonoff's ([8], pp. 456), Krasnoselskii's ([9], [10]), Dubrovskii's [7] and Nashed-Wong's [12]), on the compactness theorem in $\mathcal{L}$ (Theorem 1) and on some results (Theorems 2 and 3) established in [2].

Let R_a be the semiaxis $[a, \infty)$, R^p the p-dimensional real Euclidean space and $\mathcal{C}$ the locally convex and separable vector space of the continuous functions $x : R_a \to R^p$, with the topology generated by the denumerable and sufficient family of seminorms

$$\| x \|_m = \sup_{a \leq t \leq m} \| x(t) \|,$$

where $m \in N$ and $\| x(t) \|$ is the Euclidean length of the vector $x(t)$. Let g be a continuous and strictly positive scalar function on R_a. In the space $\mathcal{C}$, we consider the Banach space C_g of the functions x, such that $\| x(t) \| / g(t)$ is bounded on R_a, the norm being defined by

$$| x |_g = \sup_{t \geq a} (\| x(t) \| / g(t)).$$

We consider also the Banach space $\mathcal{L}$ of the functions $x \in \mathcal{C}$ with limit at infinity and whose norm is the one from C (the space C_g with $g = 1$). The limit of an element $x \in \mathcal{L}$ will be denoted by x^*: