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PROBLEMA CONSISTENŢEI ECUAŢIILOR DIFERENŢIALE PERTURBATE SINGULAR

Rezumat

Se consideră sistemul (1) $\dot{x} = f(t, x, y, \varepsilon)$, $\varepsilon\dot{y} = g(t, x, y, \varepsilon)$ unde x şi y sînt vectori n, respectiv m, dimensionali iar ε un parametru pozitiv constant, studiindu-se soluţiile acestuia cu ajutorul sistemului redus (2) $\dot{x} = f(t, x, y, 0)$, $0 = g(t, x, y, 0)$.

Se dau soluţiile care asigură că o soluţie dată (x^0, y^0) a sistemului (2) este limita soluţiilor (sau numai a unei părţi de soluţii) ale sistemului (1) cu $\varepsilon \to 0$.

Această proprietate este numită consistenţă a lui $(\dot{x}, \dot{y})$ cu privire la sistemul (1), (x^0, y^0) fiind deci o aproximare a unor soluţii ale sistemului (1).

În această Notă sînt generalizate unele rezultate ale altor autori (Tihonov, Hoppensteadt, Butuzov, Chang, Coppel, Levin şi Levinson), extinzîndu-se în acelaşi timp la diferite tipuri de consistenţă introduse de autor într-o lucrare anterioară. În locul obişnuitelor condiţii de stabilitate utilizate de Tihonov şi Hoppensteadt se presupune existenţa funcţiilor Liapunov, satisfăcînd condiţii corespunzătoare.

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ON THE EXISTENCE OF CONVERGENT SOLUTIONS IN THE THEORY OF VOLTERRA INTEGRO-FUNCTIONAL EQUATIONS

BY

GH. BANTAŞ

1. This paper contains some results regarding the existence of the solutions of certain Volterra integro-functional equations in the space $\mathcal{L}$ defined below. The technique used is based on the Massera-Schäffer-Corduneanu method ([11], [3] — [6]), on some fixed point theorems (Schauder-Tychonoff's ([8], pp. 456). Krasnoselskii's ([9], [10]), Dubrovskii's [7] and Nashed-Wong's [12]), on the compactness theorem in $\mathcal{L}$ (Theorem 1) and on some results (Theorems 2 and 3) established in [2].

Let R_a be the semiaxis $[a, \infty)$, R^p the p-dimensional real Euclidean space and $\mathcal{C}$ the locally convex and separable vector space of the continuous functions $x : R_a \to R^p$, with the topology generated by the denumerable and sufficient family of seminorms

$$\| x \|_m = \sup_{a \leq t \leq m} \| x(t) \|,$$

where $m \in N$ and $\| x(t) \|$ is the Euclidean length of the vector $x(t)$. Let g be a continuous and strictly positive scalar function on R_a. In the space $\mathcal{C}$, we consider the Banach space C_g of the functions x, such that $\| x(t) \| / g(t)$ is bounded on R_a, the norm being defined by

$$| x |_g = \sup_{t \geq a} (\| x(t) \| / g(t)).$$

We consider also the Banach space $\mathcal{L}$ of the functions $x \in \mathcal{C}$ with limit at infinity and whose norm is the one from C (the space C_g with $g = 1$). The limit of an element $x \in \mathcal{L}$ will be denoted by x^* :

$$\lim_{t \to \infty} x(t) = x^* \in R^p.$$

The ball with center in 0 and radius ρ in the space $\mathcal{L}$ will be denoted by S.

In the case $p = 1$, the above spaces will be denoted by $\mathcal{C}^1$, C_g^1, C^1 and $\mathcal{L}^1$ respectively.

Definitions. 1) *A set $A \subset \mathcal{L}$ is said to be an equicontinuous family at $+\infty$ if for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that,*

$$\sup_{t',t'' \geq \delta_\varepsilon} \| x(t') - x(t'') \| < \varepsilon, \quad \forall x \in A.$$

2) *A set $A \subset \mathcal{L}$ is said to be an equicontinuous family on $\widehat{R}_a = [a, \infty]$ if it is equicontinuous on R_a and also at $+\infty$.*

Theorem 1. *The necessary and sufficient condition for the set $A \subset \mathcal{L}$ to be compact is that A be a equibounded and equicontinuous family on $\widehat{R}_a$.*

Proof. From the definition of $\mathcal{L}$ and from the fact that to each function of $\mathcal{L}$ one can assign as value at $+\infty$ the value of its limit at this point, there comes out that the $\mathcal{L}$ space is isometric and isomorphic with continuous $x : [0, 1] \to R^p$ functions, space in which the Arzelà-Ascoli compactness criterium holds. This remark, allows us to draw the conclusion that in the space $\mathcal{L}$ such a criterium holds too ∎.

2. Let us consider the operator $T : \mathcal{C} \to \mathcal{C}$ defined by

$$(Tx)(t) = \int_a^t K(t, s)\, x(s)\, ds, \qquad (t \geq a)$$

where K is a square matrix of order p, continuous on the set $\Delta = \{(t, s) / a \leq s \leq t < \infty\}$. It can be easily shown that T is a continuous linear operator on the space $\mathcal{C}$. Denote by $\mathbf{T}$ the operator obtained from T, if one replaces K by $\| K \|$ ($\| K \|$ denotes a usual norm of K).

Definition. *The pair $(C_g, \mathcal{L})$ is called T-admissible if $TC_g \subset \mathcal{L}$.* ([2], [3]).

Theorem 2. *If the pair $(C_g, \mathcal{L})$ is T-admissible, then T is continuous on C_g. $\mathbf{T}g \in C^1$ and for each $x \in C_g$ we have $| Tx |_1 \leq \mathfrak{C} \, | x |_g$ where $\mathfrak{C} = | \mathbf{T}g |_1$* [2].

Hypothesis. Hereafter, the operator T will be, once for ever, subject to the following assumption:

For $t \to \infty$, we have

$$\text{(H)} \qquad K(t, s) \rightrightarrows k(s) \text{ on every interval } [a, b] \subset R_a.$$

Denote by $(k \circledast .) : \mathcal{C} \to \mathcal{C}$ the operator defined by

$$(k \circledast x)(t) = \int_a^t k(s)\, x(s) ds, \qquad (t \geq a).$$

Similarly one defines the operator $(\| k \| \circledast .) : \mathcal{C}^1 \to \mathcal{C}^1$.

Theorem 3. *The pair $(C_g, \mathcal{L})$ is T-admissible iff, $(\| k \| \circledast g) \in \mathcal{L}^1$ and $(\mathbf{T}g)^* = (\| k \| \circledast g)^*$.*

Under these conditions, $(Tx)^* = (k \circledast x)^*$ for any $x \in C_g$. [2].

3. Let us consider the integro-functional equation

$$\text{(E)} \qquad x = (\mathcal{F}_1 + T\mathcal{F}_2)x,$$

where $\mathcal{F}_1 : S \to \mathcal{L}$ and $\mathcal{F}_2 : S \to C_g$ are two continuous operators on the ball S.

We shall denote in the sequel by r, r_1 and r_2 some positive numbers and by $| . |$ the norm in C or the norm in C_g, if no confusion possible.

Theorem 4. *Under the conditions of Theorem 3 we suppose that*

a) $| \mathcal{F}_i x | \leq r_i \quad \forall x \in S \quad (i = 1, 2)$;

b) *$\mathcal{F}_1 S$ is equicontinuous on $\widehat{R}_a$;*

c) $r_1 + \mathfrak{C} r_2 \leq \rho$.

Then the equation (E) has at least one solution $x \in S$. Moreover, this solution satisfies the functional relation

$$\text{(R)} \qquad x^* = (\mathcal{F}_1 x)^* + (k \circledast \mathcal{F}_2 x)^*.$$

Proof. We apply the Schauder-Tychonoff fixed point theorem to the operator $U = \mathcal{F}_1 + T\mathcal{F}_2$ defined on the closed convex subset S of the space $\mathcal{L}$. From the definition of operators $\mathcal{F}_i$ and from Theorem 3 it follows that $US \subset \mathcal{L}$. According to conditions a), c) and Theorems 2 and 3 we can show that $US \subset S$ and that U is continuous on S. For fulfilling all the conditions of Schauder-Tychonoff theorem it will be enough to show that $\overline{US}$ is a compact set in the topology of $\mathcal{L}$. According to Theorem 1 this means that $\overline{US}$ is a family of equibounded and equicontinuous functions on $\widehat{R}_a$. The first assertion is a consequence of a) and of Theorem 2. The second reduces, because of b), to the equicontinuity on $\widehat{R}_a$ of $\overline{T\mathcal{F}_2(S)}$, which will be proved as follows: Let us consider $[a, b]$ a certain interval of R_a. From a) it follows that for any $t, \tau \in [a, b]$ with $\tau \leq t$, and for any $x \in S$ we have

$$E = \| (T\mathcal{F}_2)(t) - (T\mathcal{F}_2)(\tau) \| \leq r_2 \left(\int_a^\tau \| K(t, s) - K(\tau, s) \| g(s)\, ds + \int_\tau^t \| K(t, s) \| g(s)\, ds \right).$$

Let K_b be the supremum of the function $\| K \|$ on $\Delta_b = \{a \leqslant s \leqslant t \leqslant b\}$ and $\varepsilon > 0$ a certain fixed number. From the uniform continuity of K on Δ_b we derive the existence of $\delta_\varepsilon < \varepsilon$ such that for any $t, \tau \in [a, b]$ with $|t - \tau| < \delta_\varepsilon$ and for every $x \in S$ we have

$$E \leqslant \varepsilon r_2(b - a + K_b).$$

This, together with b), show that $\overline{US}$ is an equicontinuous family on R_a. It remains to prove that $\overline{T(\mathcal{F}_2(S))}$ is an equicontinuous family at $+\infty$. From a) it follows that for any $a \leqslant \tau \leqslant t$ and for any $x \in S$ we have

$$E \leqslant r_2\left(\int_a^t \| H(t, s) \| g(s)\, ds + \int_a^\tau \| H(\tau, s) \| g(s) ds + \right. \tag{1}$$

$$\left. + \int_\tau^t \| k(s) \| g(s)\, ds \right),$$

where $H(t, s) = K(t, s) - k(s)$. On the other hand, for $a \leqslant t' \leqslant t$ we have

$$\int_a^t \| H(t, s) \| g(s)\, ds = \int_a^{t'} \| H(t, s) \| g(s)\, ds + \int_{t'}^t \| H(t, s) \| g(s)\, ds \leqslant$$

$$\leqslant (t' - a) \sup_{a \leqslant s \leqslant t'} \| H(t, s) \| g(s) + \int_{t'}^t \| K(t, s) \| g(s)\, ds + \tag{2}$$

$$+ \int_{t'}^t \| k(s) \| g(s)\, ds.$$

From the conditions of Theorem 3 it follows the existence of a $t_\varepsilon \in R_a$ such that for any $t \geqslant t_\varepsilon$ we have:

$$\begin{cases} \int_a^t \| K(t, s) \| g(s)\, ds < \int_a^\infty \| k(s) \| g(s)\, ds + \varepsilon, \\ \int_{t_\varepsilon}^t \| k(s) \| g(s)\, ds < \varepsilon, \\ \int_a^{t_\varepsilon} \| k(s) \| g(s)\, ds < \int_a^{t_\varepsilon} \| K(t, s) \| g(s)\, ds + \varepsilon. \end{cases} \tag{3}$$

From these inequalities it follows that

$$\int_{t_\varepsilon}^t \| K(t, s) \| g(s)\, ds < 3\varepsilon, \quad (t \geqslant t_\varepsilon). \tag{4}$$

By taking $t' = t_\varepsilon$ in (2), and by using (3) and (4) we get

$$0 \leqslant \overline{\lim_{t \to \infty}} \int_a^t \| H(t, s) \| g(s)\, ds \leqslant 4\varepsilon.$$

As $\varepsilon > 0$ is arbitrary we conclude that

$$\lim_{t \to \infty} \int_a^t \| H(t, s) \| g(s)\, ds = 0.$$

As $(\| k \| \circledast g) \in \mathcal{L}^1$, the last equality together with (i) implies the existence of $\delta_\varepsilon > 0$ such that for every $t, \tau \geqslant \delta_\varepsilon$ and $x \in S$ we have: $E \leqslant \varepsilon r_2$. This fact, together with the assumption b), shows that $\overline{US}$ is an equicontinuous family at $+\infty$. Hence, the operator U is completely continuous on S such that the equation (E) admits at least a solution in S. According to Theorem 3 it satisfies necessarily the relation (R). ■

Remark. In other words, Theorem 4 may be stated as follows: *If the operator T maps C_g in the space $\mathcal{L}$ and each of the continuous operators $\mathcal{F}_1 : S \to \mathcal{L}$ and $\mathcal{F}_2 : S \to C_g$ on S, maps S in a sufficiently small ball such that $\mathcal{F}_1 S$ is equicontinuous on $\hat{R}_a$, then the mapping $U = \mathcal{F}_1 + T\mathcal{F}_2$ maps the ball S in itself and it has at least one fixed point.*

Theorem 5. *Under the assumptions a) and b), with $S = \mathcal{L}$, of Theorem 4, the equation (E) has at least one solution $x \in \mathcal{L}$. Moreover, this solution satisfies the relation (R).*

Proof. Like in Theorem 4 we can show that operator U is completely continuous on $\mathcal{L}$. Then we can prove that U is an asymptotic to zero operator ([7]). Now our theorem comes out from Dubrovski's fixed point theorem and from Theorem 3. ■

Theorem 6. *Under the conditions of Theorem 3 we suppose that*

a) $\| \mathcal{F}_1 x - \mathcal{F}_1 y \|_1 \leqslant r \| x - y \|_1$ *and* $\| \mathcal{F}_2 x \|_g \leqslant r_1$, $(\forall x, y \in S)$;

b) $\| \mathcal{F}_1 0 \|_1 + r_1 C \leqslant (1 - r) \rho$ *and* $r \in [0, 1)$.

Then the equation (E) has at least one solution $x \in S$. Moreover, this solution satisfies the relation (R).

Proof. For this theorem we use Krasnoselskii's fixed point theorem in which we consider $\mathcal{F}_1$ as a contraction and $T\mathcal{F}_2$ as a completely continuous operator. ■

Theorem 7. *Under the hypothesis a) of Theorem 6 where $S = \mathcal{L}$ the equation (E) has at least one solution $x \in \mathcal{L}$. Moreover, this solution satisfies the relation (R).*

Proof. We remark that $\mathcal{F}_1$ is a contraction on $\mathcal{L}$ and that $\mathcal{F}_2$ is an asymptotic to zero and completely continuous operator on $\mathcal{L}$. From this, from the fixed point theorem of Nashed-Wong (Theorem 3, [12]) and from Theorem 3 we can establish the above theorem ∎

Starting from these theorems, we can derive the following results:

Let us consider the integral equation

$$(E_1) \qquad x(t) = h(t) + \int_a^t P(t, s, x(s))ds + \int_a^t K(t, s) F(s, x(s))\, ds,$$

where K is a square matrix of order p, continuous on the set Δ and x, h, P and F are continuous p-vector functions on R_a, $\Delta \times \sum$ and $R_a \times \sum$ respectively, where $\sum$ denotes the ball of center 0 and radius ρ in the space R^p.

We denote by $\mathcal{A}(r)$ the set of all continuous functions $A : \Delta \to R_+$ with the properties

$$\int_a^t A(t, s)\, ds \leqslant r, \quad \left(\int_a^b A(t,s)\, ds\right)^* = 0 \ \ (\forall b \geqslant a)$$

Theorem 8. *Under the conditions of Theorem 3, we suppose that*

a) $\| F(t, x) \| \leqslant r_1 g(t) \quad (\forall t \geqslant a, \ \ \forall x \in \sum)$

b) *For any* $(t, s) \in \Delta$ *and* $x, y \in \sum$, *we have*

$\| P(t, s, x) - P(t, s, y) \| \leqslant A(t, s) \| x - y \|$ *with* $A \in \mathcal{A}(r)$, $r \in [0, 1)$.

c) $h \in \mathcal{L}$ *and* $\left| h(t) + \int_a^t P(t, s, 0)\, ds \right|_1 + \mathfrak{C} r_1 \leqslant (1 - r)\rho$.

Then the equation (E_1) *has at least one solution* $x \in S$. *Moreover, this solution satisfies the relation*

$$x^* = h^* + \left(\int_a^t P(t, s, x^*)\, ds\right)^* + \int_a^\infty k(s)\, F(s, x(s))ds.$$

Let us consider the integral equation

$$(E_2) \qquad x(t) = F_1(t, x(t)) + \int_a^t K(t, s)\, F_2(s, x(s))\, ds,$$

where K is a square matrix of order p, continuous on the set Δ, and x and F_i are continuous p-vector functions on R_a and $R_a \times \sum$, respectively.

Theorem 9. *Under the assumptions of Theorem 3 we suppose that*

a) $\| F_2(t, x) \| \leqslant r_1 g(t) \quad (\forall t \geqslant a, \ \ \forall x \in \sum)$;

b) *For every* $t \geqslant a$ *and* $x, y \in \Sigma$, *we have:*

$\| F_1(t, x) - F_1(t, y) \| \leqslant r \| x - y \|$, $F_1(\,.\,, x) \in \mathcal{L}$ *and* $r \in [0, 1)$;

c) $| F_1(\,.\,, 0) |_1 + \mathfrak{C} r_1 \leqslant (1 - r)\rho$.

Then the equation (E_2) *has at least one solution* $x \in S$. *Moreover, this solution satisfies the relation*

$$x^* = (F_1(\,.\,, x^*))^* + \int_a^\infty k(s)\, F_2(s, x(s))\, ds.$$

Let us consider the differential equation

$$(E_3) \qquad x'(t) = A(t)x + f(t, x),$$

where A is a square matrix of order p, continuous on R_a, and x and f are continuous p — vector functions on R_a and $R_a \times \sum$ respectively. Let X be the fundamental matrix of the homogeneous equation $x' = Ax$ such that $X(a) = I$, where I denotes the unit matrix.

Theorem 10. *Suppose that the following conditions hold:*

a) $\lim\limits_{t \to \infty} X(t) = L$ *and* $\int_a^\infty \| X^{-1}(s) \| g(s)\, ds < \infty$;

b) $\| f(t, x) \| \leqslant r g(t) \quad \forall t \geqslant a$ *and* $\forall x \in \sum$;

c) $| X(t)X^{-1}(a)x_0 |_1 + \mathfrak{C} r \leqslant \rho$.

Then the equation (E_3) *has at least one solution* $x(t\,;\, a, x_0) \in S$ *with* $x_0 \in \sum$ *Moreover, this solution satisfies the relation*

$$x^* = LX^{-1}(a)x_0 + \int_a^\infty LX^{-1}(s)\, f(s, x(s))\, ds.$$

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DESPRE EXISTENŢA SOLUŢIILOR CONVERGENTE ÎN TEORIA ECUAŢIILOR INTEGRO-FUNCŢIONALE VOLTERRA

Rezumat

Această lucrare conţine cîteva rezultate privind existenţa soluţiilor unor ecuaţii integro-funcţionale Volterra în spaţiul $\mathcal{L}$ (spaţiul Banach al funcţiilor continue $x : [a, \infty) \to R^p$, cu limită la infinit şi cu norma $\sup_{t \geq a} \| x(t) \|$). Apartenenţa soluţiei la acest spaţiu îi conferă un anumit gen de comportare asimptotică.

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A FIXED POINT THEOREM ON METRIC SPACES

BY

MIHAI TURINICI

Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. In [4], S. Reich has established the following result:

Theorem 1. *Let a, b, c be functions from $(0, +\infty)$ to $[0, +\infty)$ such that $f = a + b + c$ has the property:*

(P) $\forall t > 0\ \exists e(t) > 0,\ k(t) \in [0, 1)$ *such that* $u \in [t, t + e(t)) \Rightarrow f(u) \leq k(t)$.

If $T : X \to X$ satisfies:

$$d(Tx, Ty) \leq a(d(x, y))\, d(x, Tx) + b(d(x, y))\, d(y, Ty) + c(d(x, y))\, d(x, y), \qquad \forall x, y \in X,\ x \neq y, \tag{1}$$

then T has a unique fixed point.

This result includes the fixed point theorems of Banach [1], Kannan [2], S. Reich [3]. The purpose of the present paper is to extend this theorem.

Definition. *A function $f : [0, +\infty) \times [0, +\infty) \times (0, +\infty) \to [0, +\infty)$ will be said to have the property (Q) if it satisfies:*

(i) $0 \leq f(u, v, w) \leq 1,\ u, v \geq 0,\ w > 0,$

(ii) $\forall t > 0\ \exists e(t) > 0,\ k(t) \in [0, 1)$ *such that*
$(u, v) \in [0, e(t)) \times [0, e(t)) \cup [t, t + e(t)) \times [t, t + e(t)),$
$w \in [t, t + e(t))$ *implies* $f(u, v, w) \leq k(t)$.

Our main result can be stated as follows:

Theorem 2. *Let a, b, c be functions from $[0, +\infty) \times [0, +\infty) \times (0, +\infty) \to [0, +\infty)$ such that:*

(j) $a(u, v, w) = a(v, u, w),\ b(u, v, w) = b(v, u, w),\ c(u, v, w) = c(v, u, w),$